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Some Special Analysation on a Pythagorean Triangle which Satisfies $\alpha\left(\left(Hypotonuse \times Perimeter\right) - 4\left(Area\right)\right) = \lambda^2\left(Perimeter\right)$ for Some Particular Different Values of α

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Abstract: We obtain non-trivial integer solutions for the sides of the Pythagorean triangle, for some particular values of α which satisfies $(\alpha((Hypotonuse \times Perimeter) - 4\,(Area))) = \lambda^2\,(Perimeter)$. A few interesting relations between the sides of the Pythagorean triangle are presented.

Keywords: Integral Solutions, Pythagorean Triangles.

1. Introduction

One well-known set of solutions of the Pythagorean equation $x^2 + y^2 = z^2$ are $x = 2uv$, $y = u^2 - v^2$ and $z = u^2 + v^2$. many mathematicians have been used this set of solutions to obtain the non-zero integral values for x, y, and z [1-3]. As a new approach, in this paper we introduce another set of solutions $x = 2U + 1$, $y = 2U^2 + 2U$ and $z = 2U^2 + 2U + 1$ for the equation $x^2 + y^2 = z^2$. By using this solution we obtain three non-zero integers x, y, and z under certain relations satisfying the equation $x^2 + y^2 = z^2$ [4-6]. In this communication, we present yet another interest Pythagorean Triangles, wherein each of which, $\alpha\left(\left(Hypotonuse \times Perimeter\right) - 4\left(Area\right)\right) = \lambda^2\left(Perimeter\right)$. A few interesting relations are also given. In addition, the recurrence relations for the sides of the triangle are presented.

2. Methods and Material

Taking $U > 0$, to be the generators of the Pythagorean triangle (x, y, z), the assumption that $\alpha\left(\left(Hypotonuse \times Perimeter\right) - 4\left(Area\right)\right) = \lambda^2\left(Perimeter\right)$ becomes

$$\alpha\left(\left(\left(2U^2+2U+1\right) \times \left(2U+1+2U^2+2U+2U^2+2U+1\right)\right) - 4\left(\tfrac{1}{2}\left(2U+1\right)\left(2U^2+2U\right)\right)\right) = \lambda^2(2U+1+2U^2+2U+2U^2+2U+1)$$

and which leads to the Pellian equation

$$X^2 = DY^2 + \alpha \tag{1}$$

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where $D = 2\alpha$, not a perfect square, $\lambda = X$, and $U = Y$. For the understanding, we consider the cases, $\alpha > 4$.

Choice I

Suppose that α is a perfect square, say $\alpha = a^2$. So that equation (1) becomes

$$X^2 = DY^2 + a^2 \tag{2}$$

It is well known that the general form of integral solutions $(\overline{x_n}, \overline{y_n})$, $n = 0, 1, 2, 3, \ldots$ of the Pellian equation

$$X^2 = DY^2 + 1 \tag{3}$$

is represented by $\overline{x}_n + \sqrt{D}\overline{y}_n = \left(\overline{x_0} + \sqrt{D}\overline{y_0}\right)^{n+1}$, $n = 0, 1, 2, 3, \ldots$ where $(\overline{x_0}, \overline{y_0})$ is the smallest positive integer solution $(\overline{x_n}, \overline{y_n})$ of (3) and (X_0, Y_0) is the fundamental solution of equation (2). We may find the solution of equation (1) by using the following two methods. They are,

Method I

$$X_n = a\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{4}$$

$$Y_n = a\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{5}$$

Where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

Method II

$$X_{n+1} = X_0\overline{x}_n + DY_0\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{6}$$

$$Y_{n+1} = X_0\overline{y}_n + Y_0\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{7}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

For the sake of clear understanding, we present below forms of integral solutions of (6) and (7) and thus the following choice of α

(i). $\alpha = 9$ (ii). $\alpha = 16$ (iii). $\alpha = 25$ (iv). $\alpha = 36$ (v). $\alpha = 49$ (vi). $\alpha = 64$ (vii). $\alpha = 100$ (viii). $\alpha = 144$ (ix). $\alpha = 196$

Case 1: Setting $\alpha = 9$, so that $a = 3$ and $D = 18$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 18Y^2 + 9 \tag{8}$$

$(X_0, Y_0) = (51, 12)$ be the initial solution of (8). Consider the Pellian

$$X^2 = 18Y^2 + 1 \tag{9}$$

Let $(\overline{x_0}, \overline{y_0}) = (17, 4)$ be the initial solution of (9). In this case, the corresponding integral solutions of (8) are given by the following two methods.

Method I

$$X_n = 3\overline{x}_n, \quad n = 0, 1, 2, 3, \dots \tag{10}$$

$$Y_n = 3\overline{y}_n, \quad n = 0, 1, 2, 3, \dots \tag{11}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \dots$$
$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \dots$$

On substituting $(\overline{x_0}, \overline{y_0})$ and D, we get,

$$X_n = \frac{3}{2}\left[\left(17 + 4\sqrt{18}\right)^{n+1} + \left(17 - 4\sqrt{18}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \dots \tag{12}$$
$$Y_n = \frac{3}{2\sqrt{18}}\left[\left(17 + 4\sqrt{18}\right)^{n+1} - \left(17 - 4\sqrt{18}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \dots \tag{13}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	51	12	25	312	313
1	1731	408	817	333744	333745
2	58803	13860	27721	384226920	384226921
3	1997571	470832	941665	4433523612	4433523613

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 34X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 34Y_{n+1} + Y_n = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 34)$ and $Y_{n+2} + Y_n \equiv 0\,(mod\ 34)$.

4. For all values of n, X is divisible by 3 and Y is divisible by 2 and 4.

Method II

$$X_{n+1} = 51\overline{x}_n + 216\overline{y}_n, \quad n = 0, 1, 2, 3, \dots \tag{14}$$

$$Y_{n+1} = 51\overline{y}_n + 12\overline{x}_n, \quad n = 0, 1, 2, 3, \dots \tag{15}$$

Where,

$$\overline{x}_n = \frac{1}{2}\left[\left(17 + 4\sqrt{18}\right)^{n+1} + \left(17 - 4\sqrt{18}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \dots \tag{16}$$
$$\overline{y}_n = \frac{1}{2\sqrt{18}}\left[\left(17 + 4\sqrt{18}\right)^{n+1} - \left(17 - 4\sqrt{18}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \dots \tag{17}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	1731	408	817	333744	333745
1	58803	13860	27721	384226920	384226921
2	1097885	470832	941665	4433523612	4433523613

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 34X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 34Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 34)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 34)$.

5. For all values of n, X is divisible by 3 and Y is divisible by 2 and 4.

Case 2: Setting $\alpha = 16$, so that $a = 4$ and $D = 32$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 32Y^2 + 16 \tag{18}$$

$(X_0, Y_0) = (12, 2)$ be the initial solution of (18). Consider the Pellian

$$X^2 = 32Y^2 + 1 \tag{19}$$

Let $(\overline{x_0}, \overline{y_0}) = (17, 3)$ be the initial solution of (19). Using Brahmagupta lemma, the integral solutions of (18) are given by the following two methods.

Method I

$$X_n = 4\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{20}$$

$$Y_n = 4\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{21}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

On substituting $(\overline{x_0}, \overline{y_0})$, a and D, we get,

$$X_n = 2\left[\left(17 + 3\sqrt{32}\right)^{n+1} + \left(17 - 3\sqrt{32}\right)^{n+1}\right], \qquad n = -1, 0, 1, 2, 3, \ldots \tag{22}$$

$$Y_n = \frac{2}{\sqrt{32}}\left[\left(17 + 3\sqrt{3}\right)^{n+1} - \left(17 - 3\sqrt{32}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{23}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	68	12	25	312	313
1	2308	408	817	333744	333745
2	78408	13860	27721	384226920	384226921
3	2663428	470832	941665	4433523612	4433523613

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 34X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 34Y_{n+1} + Y_n = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 34)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 34)$.

5. For all values of n, X is divisible by 3 and Y is divisible by 2 and 4.

Method II

$$X_{n+1} = 12\overline{x}_n + 64\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{24}$$

$$Y_{n+1} = 12\overline{y}_n + 2\overline{x}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{25}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(17 + 3\sqrt{32}\right)^{n+1} + \left(17 - 3\sqrt{32}\right)^{n+1}\right], \qquad n = -1, 0, 1, 2, 3, \ldots \tag{26}$$

$$\overline{y}_n = \frac{1}{2\sqrt{32}}\left[\left(17 + 3\sqrt{32}\right)^{n+1} + \left(17 - 3\sqrt{32}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{27}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	12	2	5	12	13
0	396	70	141	9940	9941
1	13452	2378	4757	11314524	11314525
2	456972	80782	161565	13051624612	13051624613

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 34X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 34Y_{n+1} + Y_n = 0$$

2. For all values of n, both X and Y are even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 34)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 34)$.

5. For all values of n, X is divisible by 4 and Y is divisible by 2.

Case 3: Setting $\alpha = 25$, so that $a = 5$ and $D = 50$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 50Y^2 + 25 \tag{28}$$

$(X_0, Y_0) = (15, 2)$ be the initial solution of (28). Consider the Pellian

$$X^2 = 50Y^2 + 1 \tag{29}$$

Let $(\overline{x_0}, \overline{y_0}) = (99, 14)$ be the initial solution of (29). Using Brahmagupta lemma, the integral solutions of (28) are given by the following two methods.

Method I

$$X_n = 5\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{30}$$

$$Y_n = 5\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{31}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

On substituting $(\overline{x_0}, \overline{y_0})$, a and D, we get,

$$X_n = \frac{5}{2}\left[\left(99 + 14\sqrt{50}\right)^{n+1} + \left(99 - 14\sqrt{50}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots \tag{32}$$

$$Y_n = \frac{5}{2\sqrt{8}}\left[\left(99 + 14\sqrt{50}\right)^{n+1} - \left(99 - 14\sqrt{50}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots \tag{33}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	495	70	141	9940	9941
1	98005	13860	27721	384226920	384226921
2	19404495	2744210	5488421	15061382536620	15061382536621
3	3841992005	543339720	1086679441	295218052416357840	295218052416357841

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 198X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 198Y_{n+1} + Y_n = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 198)$.

5. For all values of n, X is divisible by 5 and Y is divisible by 5 and 10.

Method II

$$X_{n+1} = 15\overline{x}_n + 100\overline{y}_n,\ \ n = -1, 0, 1, 2, 3, \ldots \tag{34}$$

$$Y_{n+1} = 15\overline{y}_n + 2\overline{x}_n, \qquad n = -1, 0, 1, 2, 3, \ldots \tag{35}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(99 + 14\sqrt{50}\right)^{n+1} + \left(99 - 14\sqrt{50}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots \tag{36}$$

$$\overline{y}_n = \frac{1}{2\sqrt{32}}\left[\left(99 + 14\sqrt{50}\right)^{n+1} - \left(99 - 14\sqrt{50}\right)^{n+1}\right],\ \ n =, 0, 1, 2, 3, \ldots \tag{37}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	15	2	5	12	13
0	2885	408	817	333744	333745
1	571215	80782	161565	13051624612	13051624613
2	113097685	15994428	31988857	511643486083224	511643486083225

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 198X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 198Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 198)$.

5. For all values of n, X is divisible by 5 and Y is divisible by 2.

Case 4: Setting $\alpha = 36$, so that $a = 6$ and $D = 72$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 72Y^2 + 36 \tag{38}$$

$(X_0, Y_0) = (18, 2)$ be the initial solution of (37). Consider the Pellian

$$X^2 = 72Y^2 + 1 \tag{39}$$

Let $(\overline{x_0}, \overline{y_0}) = (17, 2)$ be the initial solution of (38). Using Brahmagupta lemma, the integral solutions of (37) are given by the following two methods.

Method I

$$X_n = 6\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{40}$$

$$Y_n = 6\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{41}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$
$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

On substituting $(\overline{x_0}, \overline{y_0})$, a and D, we get,

$$X_n = 3\left[\left(17 + 2\sqrt{72}\right)^{n+1} + \left(17 - 2\sqrt{72}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots \tag{42}$$
$$Y_n = \frac{3}{\sqrt{8}}\left[\left(17 + 2\sqrt{72}\right)^{n+1} - \left(17 - 2\sqrt{72}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots \tag{43}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	102	12	25	312	313
1	3462	408	817	333744	333745
2	117606	13860	27721	384226920	384226921
3	3995142	470832	941665	4433523612	4433523613

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 34X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 34Y_{n+1} + Y_n = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 34)$ and $Y_{n+2} + Y_n \equiv 0\,(mod\ 34)$.

4. For all values of n, X is divisible by 2,3 and 6 and Y is divisible by 2, 3, and 6.

Method II

$$X_{n+1} = 18\overline{x}_n + 144\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{44}$$

$$Y_{n+1} = 18\overline{y}_n + 2\overline{x}_n, \qquad n = -1, 0, 1, 2, 3, \ldots \tag{45}$$

Where,

$$\overline{x}_n = \frac{1}{2}\left[\left(17 + 2\sqrt{72}\right)^{n+1} + \left(17 - 2\sqrt{72}\right)^{n+1}\right], \qquad n = -1, 0, 1, 2, 3, \ldots \tag{46}$$
$$\overline{y}_n = \frac{1}{2\sqrt{72}}\left[\left[\left(17 + 2\sqrt{72}\right)^{n+1} + \left(17 - 2\sqrt{72}\right)^{n+1}\right]\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{47}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	18	2	5	12	13
0	594	70	141	9940	9941
1	20178	2378	4757	11314524	11314525
2	383130	45150	90301	4077135300	4077135300

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 34X_{n+2} + X_{n+1} = 0,$$
$$Y_{n+3} - 34Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, both X and Y are even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 34)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 34)$.

5. For all values of n, X is divisible by 18 and Y is divisible by 2.

Case 5: Setting $\alpha = 49$, so that $a = 7$ and $D = 98$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 98Y^2 + 49 \tag{48}$$

$(X_0, Y_0) = (21, 2)$ be the initial solution of (47). Consider the Pellian

$$X^2 = 98Y^2 + 1 \tag{49}$$

Let $(\overline{x_0}, \overline{y_0}) = (99, 10)$ be the initial solution of (48). Using Brahmagupta lemma, the integral solutions of (49) are given by the following two methods.

Method I

$$X_n = 7\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{50}$$
$$Y_n = 7\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{51}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$
$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

On substituting $(\overline{x_0}, \overline{y_0})$, and D, we get,

$$X_n = \frac{7}{2}\left[\left(99 + 10\sqrt{98}\right)^{n+1} + \left(99 - 10\sqrt{98}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots \tag{52}$$
$$Y_n = \frac{7}{2\sqrt{98}}\left[\left(99 + 10\sqrt{98}\right)^{n+1} - \left(99 - 10\sqrt{98}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots \tag{53}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	693	70	141	9940	9941
1	137207	13860	27721	384226920	384226921
2	27166293	2744210	5488421	15061382536620	15061382536621
3	5378788807	543339720	1086679441	590436103746036240	590436103746036241

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 198X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 198Y_{n+1} + Y_n = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 198)$.

5. For all values of n, Y is divisible by 2, 5, and 10.

Method II

$$X_{n+1} = 21\overline{x}_n + 196\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{54}$$

$$Y_{n+1} = 21\overline{y}_n + 2\overline{x}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{55}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(99 + 10\sqrt{98}\right)^{n+1} + \left(99 - 10\sqrt{98}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots \tag{56}$$

$$\overline{y}_n = \frac{1}{2\sqrt{96}}\left[\left[\left(99 + 10\sqrt{98}\right)^{n+1} - \left(99 - 10\sqrt{98}\right)^{n+1}\right]\right], \quad n = 0, 1, 2, 3, \ldots \tag{57}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	21	2	5	12	13
0	4039	408	817	333744	333745
1	614481	63142	126285	7973950612	797950613
2	122712779	12395668	24791337	3073051951784	3073051951785
3	18876793340	1897406282	3794812565	7200301201740939612	7200301201740939613

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 198X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 198Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 198)$.

5. For all values of n, X is divisible by 7 and Y is divisible by 2.

Case 6: Setting $\alpha = 64$, so that $a = 8$ and $D = 128$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 128Y^2 + 64 \tag{58}$$

$(X_0, Y_0) = (24, 2)$ be the initial solution of (58). Consider the Pellian

$$X^2 = 128Y^2 + 1 \tag{59}$$

Let $(\overline{x_0}, \overline{y_0}) = (577, 51)$ be the initial solution of (59). Using Brahmagupta lemma, the integral solutions of (58) are given by the following two methods.

Method I

$$X_n = 8\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{60}$$

$$Y_n = 8\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{61}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$
$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

On substituting $(\overline{x_0}, \overline{y_0})$, a and D, we get,

$$X_n = \frac{8}{2}\left[\left(577 + 51\sqrt{128}\right)^{n+1} + \left(577 - 51\sqrt{128}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots \tag{62}$$

$$Y_n = \frac{8}{2\sqrt{128}}\left[\left[\left(577 + 51\sqrt{128}\right)^{n+1} - \left(577 - 51\sqrt{128}\right)^{n+1}\right]\right], \quad n = 0, 1, 2, 3, \ldots \tag{63}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	4616	408	817	333744	333745
1	5326856	470832	941665	4433523612	4433523613
2	614787208	543339720	1086679441	590436103746036240	590436103746036241

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 1154X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 1154Y_{n+1} + Y_n = 0$$

2. For all values of n, X and Y are even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 1154)$.

4. For all values of n,$Y_{n+2} + Y_n \equiv 0\,(mod\ 1154)$.

5. For all values of n, X and Y are divisible by 2 and 4.

Method II

$$X_{n+1} = 24\overline{x}_n + 256\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{64}$$

$$Y_{n+1} = 24\overline{y}_n + 2\overline{x}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{65}$$

Where,

$$\overline{x}_n = \frac{1}{2}\left[\left(577 + 51\sqrt{128}\right)^{n+1} + \left(577 - 51\sqrt{128}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{66}$$

$$\overline{y}_n = \frac{1}{2\sqrt{128}}\left[\left(577 + 51\sqrt{128}\right)^{n+1} + \left(577 - 51\sqrt{128}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{67}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	24	2	5	12	13
0	26904	2378	4757	11314524	11314525
1	31047192	2744210	62094385	15061382536620	15061382536621
2	35828432664	3166815962	6333631925	2024517209602812	2024517209602813

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 1154X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 1154Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, both X and Y are even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 1154)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 1154)$.

5. For all values of n, X is divisible by 8 and Y is divisible by 2.

Case 7: Setting $\alpha = 100$, so that $a = 10$ and $D = 200$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 200Y^2 + 100 \tag{68}$$

$(X_0, Y_0) = (30, 2)$ be the initial solution of (68). Consider the Pellian

$$X^2 = 200Y^2 + 1 \tag{69}$$

Let $(\overline{x_0}, \overline{y_0}) = (99, 7)$ be the initial solution of (69). Using Brahmagupta lemma, the integral solutions of (68) are given by the following two methods.

Method I

$$X_n = 10\overline{x}_n, \quad n = 0, 1, 2, 3, \dots \tag{70}$$

$$Y_n = 10\overline{y}_n, \quad n = 0, 1, 2, 3, \dots \tag{71}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \dots$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \dots$$

On substituting $(\overline{x_0}, \overline{y_0})$, a and D, we get,

$$X_n = \frac{10}{2}\left[\left(99 + 7\sqrt{200}\right)^{n+1} + \left(99 - 7\sqrt{200}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \dots \tag{72}$$

$$Y_n = \frac{10}{2\sqrt{128}}\left[\left[\left(99 + 7\sqrt{200}\right)^{n+1} - \left(99 - 7\sqrt{200}\right)^{n+1}\right]\right], \quad n = 0, 1, 2, 3, \dots \tag{73}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	990	70	141	9940	9941
1	196010	13860	27721	384226920	384226921
2	38808990	2744210	5488421	15061382536620	15061382536621
3	7683984010	543339720	1086679441	590436103746036240	590436103746036241

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 198X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 198Y_{n+1} + Y_n = 0$$

2. For all values of n, X and Y are even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 198)$.

5. For all values of n, X and Y are divisible by 2, 5, and 10.

Method II

$$X_{n+1} = 30\overline{x}_n + 400\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \dots \tag{74}$$

$$Y_{n+1} = 30\overline{y}_n + 2\overline{x}_n, \qquad n = -1, 0, 1, 2, 3, \dots \tag{75}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(99 + 7\sqrt{200}\right)^{n+1} + \left(99 - 7\sqrt{200}\right)^{n+1}\right], \qquad n = -1, 0, 1, 2, 3, \dots \tag{76}$$

$$\overline{y}_n = \frac{1}{2\sqrt{200}}\left[\left(99 + 7\sqrt{200}\right)^{n+1} + \left(99 - 7\sqrt{200}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \dots \tag{77}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	30	2	5	12	13
0	5770	408	817	333744	333745
1	1142430	80702	161405	13025787012	13025787013
2	226195370	15994428	31988857	511643486083224	511643486083225

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 198X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 198Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, both X and Y are even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 198)$.

5. For all values of n, X is divisible by 10 and Y is divisible by 2.

Case 8: Setting $\alpha = 144$, so that $a = 12$ and $D = 242$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 288Y^2 + 144 \tag{78}$$

$(X_0, Y_0) = (36, 2)$ be the initial solution of (78). Consider the Pellian

$$X^2 = 200Y^2 + 1 \tag{79}$$

Let $(\overline{x_0}, \overline{y_0}) = (17, 1)$ be the initial solution of (79). Using Brahmagupta lemma, the integral solutions of (78) are given by the following two methods.

Method I

$$X_n = 12\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{80}$$

$$Y_n = 12\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{81}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

On substituting $(\overline{x_0}, \overline{y_0})$, a and D, we get,

$$X_n = \frac{12}{2}\left[\left(17 + \sqrt{288}\right)^{n+1} + \left(17 - \sqrt{288}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots \tag{82}$$

$$Y_n = \frac{12}{2\sqrt{288}}\left[\left[\left(17 + \sqrt{288}\right)^{n+1} - \left(17 - \sqrt{288}\right)^{n+1}\right]\right], \quad n = 0, 1, 2, 3, \ldots \tag{83}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	204	12	25	312	313
1	6924	408	817	333744	333745
2	235212	13860	27721	384226920	384226921
3	7990284	470832	941665	4433523612	4433523613

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 34X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 34Y_{n+1} + Y_n = 0$$

2. For all values of n, X and Y are even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 34)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 34)$.

5. For all values of n, X and Y are divisible by 2, 3, 6, and 12.

Method II

$$X_{n+1} = 36\overline{x}_n + 576\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{84}$$

$$Y_{n+1} = 36\overline{y}_n + 2\overline{x}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{85}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(17 + \sqrt{288}\right)^{n+1} + \left(17 - \sqrt{288}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{86}$$

$$\overline{y}_n = \frac{1}{2\sqrt{288}}\left[\left(17 + \sqrt{288}\right)^{n+1} + \left(17 - \sqrt{288}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{87}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	36	2	5	12	13
0	1188	70	141	9940	9941
1	40356	2378	4757	11314524	11314525
2	1370916	80782	161565	13051624612	13051624613
3	46570788	2744210	5488421	15061382536620	15061382536621

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 34X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 34Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, both X and Y are even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 34)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 34)$.

5. For all values of n, X is divisible by 36 and Y is divisible by 2.

Case 9: Setting $\alpha = 196$, so that $a = 14$ and $D = 492$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 392Y^2 + 196 \tag{88}$$

$(X_0, Y_0) = (46, 2)$ be the initial solution of (88). Consider the Pellian

$$X^2 = 392Y^2 + 1 \tag{89}$$

Let $(\overline{x_0}, \overline{y_0}) = (99, 5)$ be the initial solution of (89). Using Brahmagupta lemma, the integral solutions of (88) are given by the following two methods.

Method I

$$X_n = 14\overline{x}_n, \quad n = 0, 1, 2, 3, \ldots \tag{90}$$

$$Y_n = 14\overline{y}_n, \quad n = 0, 1, 2, 3, \ldots \tag{91}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

On substituting $(\overline{x_0}, \overline{y_0})$, a and D, we get,

$$X_n = \frac{14}{2}\left[\left(99 + 5\sqrt{288}\right)^{n+1} + \left(99 - 5\sqrt{288}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots \tag{92}$$

$$Y_n = \frac{14}{2\sqrt{392}}\left[\left[\left(99 + 5\sqrt{288}\right)^{n+1} - \left(99 - 5\sqrt{288}\right)^{n+1}\right]\right], \quad n = 0, 1, 2, 3, \ldots \tag{93}$$

Numerical examples

n	X_n	$U_n = Y_n$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
0	204	12	25	312	313
1	6924	408	817	333744	333745
2	235212	13860	27721	384226920	384226921
3	7990284	470832	941665	4433523612	4433523613

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 198X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 198Y_{n+1} + Y_n = 0$$

2. For all values of n, X and Y are even.

3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 198)$.

5. For all values of n, X and Y are divisible by 2, 3, 6, and 12.

Method II

$$X_{n+1} = 42\overline{x}_n + 784\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{94}$$

$$Y_{n+1} = 42\overline{y}_n + 2\overline{x}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{95}$$

Where,

$$\overline{x}_n = \frac{1}{2}\left[\left(99 + 5\sqrt{392}\right)^{n+1} + \left(99 - 5\sqrt{392}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{96}$$

$$\overline{y}_n = \frac{1}{2\sqrt{392}}\left[\left(99 + 5\sqrt{392}\right)^{n+1} - \left(99 - 5\sqrt{392}\right)^{n+1}\right], \quad n = -1, 0, 1, 2, 3, \ldots \tag{97}$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	42	2	5	12	13
0	8078	408	817	333744	333745
1	1599402	80782	161565	13051624612	13051624613
2	316673518	15994428	31988857	511643486083224	511643486083225

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 198X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 198Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, both X and Y are even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 198)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 198)$.

5. For all values of n, X is divisible by 7 and Y is divisible by 2.

Choice II

Suppose that α is not a perfect square. In this case, we use the Method II of Choice I. So that,

$$X_{n+1} = X_0\overline{x}_n + DY_0\overline{y}_n, \quad n = -1,\ 0, 1, 2, 3, \ldots \tag{4a}$$

$$Y_{n+1} = X_0\overline{y}_n + Y_0\overline{x}_n, \quad n = -1,\ 0, 1, 2, 3, \ldots \tag{5a}$$

Where, (X_0, Y_0) be the fundamental solution of the equation (1) and

$$\overline{x}_n = \frac{1}{2}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} + \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots \tag{6a}$$

$$\overline{y}_n = \frac{1}{2\sqrt{D}}\left[\left(\overline{x_0} + \overline{y_0}\sqrt{D}\right)^{n+1} - \left(\overline{x_0} - \overline{y_0}\sqrt{D}\right)^{n+1}\right],\ n = 0, 1, 2, 3, \ldots \tag{7a}$$

For the sake of clear understanding, we present below forms of integral solutions of (6a) and (7a) and thus the following choice of α

(i). $\alpha = 12$ (ii). $\alpha = 27$ (iii). $\alpha = 33$ (iv). $\alpha = 44$ (v). $\alpha = 48$

Case 1: Setting $\alpha = 12$, so that $D = 24$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 24Y^2 + 12 \tag{98}$$

$(X_0, Y_0) = (6, 1)$ be the initial solution of (98). Consider the Pellian

$$X^2 = 24Y^2 + 1 \tag{99}$$

Let $(\overline{x_0}, \overline{y_0}) = (5, 1)$ be the initial solution of (99). In this case, the corresponding integral solutions of (98) are seen to be

$$X_{n+1} = 6\overline{x}_n + 24\overline{y}_n,\ n = -1, 0, 1, 2, 3, \ldots \tag{100}$$

$$Y_{n+1} = 6\overline{y}_n + \overline{x}_n,\quad n = -1, 0, 1, 2, 3, \ldots \tag{101}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(5 + \sqrt{24}\right)^{n+1} + \left(5 - \sqrt{24}\right)^{n+1}\right],\qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{24}}\left[\left(5 + \sqrt{24}\right)^{n+1} - \left(5 - \sqrt{24}\right)^{n+1}\right],\ n = 0, 1, 2, 3, \ldots$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	6	1	3	4	5
0	54	11	23	264	265
1	534	109	219	23980	23981
2	5286	1079	2159	2330640	2330641
3	52326	10681	21363	228188884	228188885

Properties

1. Recurrence relations for X and Y are

$$X_{n+2} - 10X_{n+1} + X_n = 0,$$

$$Y_{n+2} - 10Y_{n+1} + Y_n = 0$$

2. For all values of n, X is even and Y is odd.
3. For all values of n, $X_{n+2} + X_n \equiv 0\,(mod\ 10)$.
4. For all values of n, $Y_{n+2} + Y_n \equiv 0\,(mod\ 10)$.
5. For all values of n, X is divisible by 2, 3, and 6.

Case 2: Setting $\alpha = 27$, so that $D = 54$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 54Y^2 + 27 \tag{102}$$

$(X_0, Y_0) = (9, 1)$ be the initial solution of (102). Consider the Pellian

$$X^2 = 54Y^2 + 1 \tag{103}$$

Let $(\overline{x_0}, \overline{y_0}) = (485, 66)$ be the initial solution of (103). In this case, the corresponding integral solutions of (102) are seen to be

$$X_{n+1} = 9\overline{x}_n + 54\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{104}$$

$$Y_{n+1} = 9\overline{y}_n + \overline{x}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{105}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(485 + 66\sqrt{54}\right)^{n+1} + \left(485 - 66\sqrt{54}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{54}}\left[\left(485 + 66\sqrt{54}\right)^{n+1} - \left(485 - 66\sqrt{54}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

So that the equation (104) and (105) becomes,

$$X_{n+1} = \frac{1}{2}\left[\left(485 + 66\sqrt{54}\right)^{n+1}\left(9 + \sqrt{54}\right) + \left(485 - 66\sqrt{54}\right)^{n+1}\left(9 - \sqrt{54}\right)\right], \qquad n = -1, 0, 1, 2, 3, \ldots$$

$$Y_{n+1} = \frac{1}{2\sqrt{54}}\left[\left(485 + 66\sqrt{54}\right)^{n+1}\left(9 + \sqrt{54}\right) - \left(485 - 66\sqrt{54}\right)^{n+1}\left(9 - \sqrt{54}\right)\right], \quad n = -1, 0, 1, 2, 3, \ldots$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	9	1	3	4	5
0	7929	1079	2159	2330640	2330641
1	7691121	1046629	2093259	2190866620540	2190866620541
2	7460379441	1015229051	2030458103	2061380054019179304	2061380054019179305

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 970X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 970Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, both X and Y are odd.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 970)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 970)$.

5. For all values of n, X is divisible by 9.

Case 3: Setting $\alpha = 33$, so that $D = 66$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 66Y^2 + 33 \tag{106}$$

$(X_0, Y_0) = (33, 4)$ be the initial solution of (106). Consider the Pellian

$$X^2 = 66Y^2 + 1 \tag{107}$$

Let $(\overline{x_0}, \overline{y_0}) = (65, 8)$ be the initial solution of (107). In this case, the corresponding integral solutions of (106) are seen to be

$$X_{n+1} = 9\overline{x}_n + 54\overline{y}_n, \;\; n = -1, 0, 1, 2, 3, \ldots \tag{108}$$

$$Y_{n+1} = 9\overline{y}_n + \overline{x}_n, \quad\;\; n = -1, 0, 1, 2, 3, \ldots \tag{109}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(65 + 8\sqrt{66}\right)^{n+1} + \left(65 - 8\sqrt{66}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$
$$\overline{y}_n = \frac{1}{2\sqrt{66}}\left[\left[\left(65 + 8\sqrt{66}\right)^{n+1} - \left(65 - 8\sqrt{66}\right)^{n+1}\right]\right], \;\; n = 0, 1, 2, 3, \ldots$$

So that the equation (108) and (109) becomes,

$$X_{n+1} = \frac{1}{2}\left[\left(65 + 8\sqrt{66}\right)^{n+1}\left(33 + 4\sqrt{66}\right) + \left(65 - 8\sqrt{66}\right)^{n+1}\left(33 - 4\sqrt{66}\right)\right], \qquad n = -1, 0, 1, 2, 3, \ldots$$
$$Y_{n+1} = \frac{1}{2\sqrt{66}}\left[\left(65 + 8\sqrt{66}\right)^{n+1}\left(33 + 4\sqrt{66}\right) - \left(65 - 8\sqrt{66}\right)^{n+1}\left(33 - 4\sqrt{66}\right)\right], \;\; n = -1, 0, 1, 2, 3, \ldots$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	33	4	9	40	41
0	4257	524	1049	550200	550201
1	553377	68116	136233	9279715144	9279715145
2	71934753	8854556	17709113	156806341623384	156806341623385
3	93509645513	1151024164	23020448329	2649713254525846120	2649713254525846121

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 130X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 130 + Y_{n+1} = 0$$

2. For all values of n, X is odd and Y is even.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 130)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 130)$.

5. For all values of n, X is divisible by 33 and Y is divisible by 2.

Case 4: Setting $\alpha = 44$, so that $D = 88$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 88Y^2 + 44 \tag{110}$$

$(X_0, Y_0) = (66, 7)$ be the initial solution of (110). Consider the Pellian

$$X^2 = 88Y^2 + 1 \tag{111}$$

Let $(\overline{x_0}, \overline{y_0}) = (197, 21)$ be the initial solution of (111). In this case, the corresponding integral solutions of (110) are seen to be

$$X_{n+1} = 66\overline{x}_n + 616\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{112}$$

$$Y_{n+1} = 66\overline{y}_n + 7\overline{x}_n, \qquad n = -1, 0, 1, 2, 3, \ldots \tag{113}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(197 + 21\sqrt{88}\right)^{n+1} + \left(197 - 21\sqrt{88}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{88}}\left[\left(197 + 21\sqrt{88}\right)^{n+1} - \left(197 - 21\sqrt{88}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

So that the equation (112) and (113) becomes,

$$X_{n+1} = \frac{1}{2}\left[\left(197 + 21\sqrt{88}\right)^{n+1}\left(66 + 7\sqrt{88}\right) + \left(197 - 21\sqrt{88}\right)^{n+1}\left(66 - 7\sqrt{88}\right)^{n+1}\right], \qquad n = -1, 0, 1, 2, 3, \ldots$$

$$Y_{n+1} = \frac{1}{2\sqrt{88}}\left[\left(197 + 21\sqrt{88}\right)^{n+1}\left(66 + 7\sqrt{88}\right) - \left(197 - 21\sqrt{88}\right)^{n+1}\left(66 - 7\sqrt{88}\right)^{n+1}\right], \quad n = -1, 0, 1, \ldots$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	66	7	15	112	113
0	25938	2765	5531	15295980	15295981
1	10219506	1089403	2178807	2373599971624	2373599971625
2	4026459426	429222017	858444035	368463080613540612	368463080613540613

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 394X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 394Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, X is even and Y is odd.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 394)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 394)$.

5. For all values of n, X is divisible by 11 and Y is divisible by 7.

Case 5: Setting $\alpha = 48$, so that $D = 96$.

The equation $X^2 = DY^2 + \alpha$ becomes

$$X^2 = 96Y^2 + 48 \tag{114}$$

$(X_0, Y_0) = (12, 1)$ be the initial solution of (114). Consider the Pellian

$$X^2 = 96Y^2 + 1 \tag{115}$$

Let $(\overline{x_0}, \overline{y_0}) = (49, 5)$ be the initial solution of (115). In this case, the corresponding integral solutions of (114) are seen to be

$$X_{n+1} = 12\overline{x}_n + 96\overline{y}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{116}$$

$$Y_{n+1} = 12\overline{y}_n + \overline{x}_n, \quad n = -1, 0, 1, 2, 3, \ldots \tag{117}$$

where,

$$\overline{x}_n = \frac{1}{2}\left[\left(49 + 5\sqrt{96}\right)^{n+1} + \left(49 - 5\sqrt{96}\right)^{n+1}\right], \qquad n = 0, 1, 2, 3, \ldots$$

$$\overline{y}_n = \frac{1}{2\sqrt{96}}\left[\left(49 + 5\sqrt{96}\right)^{n+1} - \left(49 - 5\sqrt{96}\right)^{n+1}\right], \quad n = 0, 1, 2, 3, \ldots$$

So that the equation (116) and (117) becomes,

$$X_{n+1} = \frac{1}{2}\left[\left(49 + 5\sqrt{96}\right)^{n+1}\left(12 + \sqrt{96}\right) + \left(49 - 5\sqrt{96}\right)^{n+1}\left(12 - \sqrt{96}\right)\right], \qquad n = -1, 0, 1, 2, 3, \ldots$$

$$Y_{n+1} = \frac{1}{2\sqrt{96}}\left[\left(49 + 5\sqrt{96}\right)^{n+1}\left(12 + \sqrt{96}\right) - \left(49 - 5\sqrt{96}\right)^{n+1}\left(12 - \sqrt{96}\right)\right], \quad n = -1, 0, 1, 2, 3, \ldots$$

Numerical examples

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U+1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	12	1	3	4	5
0	1068	109	219	23980	23981
1	104652	10681	21363	228188884	228188885
2	10254828	1046629	2093259	2190866620540	2190866620541
3	1004868492	102558961	205117923	21036681167916964	21036681167916965

Properties

1. Recurrence relations for X and Y are

$$X_{n+3} - 98X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 98Y_{n+2} + Y_{n+1} = 0$$

2. For all values of n, X is even and Y is odd.

3. For all values of n, $X_{n+3} + X_{n+1} \equiv 0\,(mod\ 98)$.

4. For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0\,(mod\ 98)$.

5. For all values of n, X is divisible by 12.

III General Observations

1. If α is an odd Perfect Square, then the $X_i's$ and $Y_i's$ are odd and even respectively.

2. If α is an even Perfect Square, then the $X_i's$ and $Y_i's$ are even.

3. The only α which is a cubic integer 27 having both the values $X_i's$ and $Y_i's$ are odd.

4. If α is a non-perfect Square and non-cubic integer, then

   (i). $X_i's$ and $Y_i's$ are odd and even respectively when α is odd.

   (ii). $X_i's$ and $Y_i's$ are even and odd respectively when α is even.

5. The equation $x^2 = Dy^2 + 1$ has no positive integer solution if $D \not\equiv 0, 3, 8, 15\,(mod\ 16)$ and there is no factorization $D = pq$, where $p > 1$ is odd, $gcd\,(p, q) = 1$, and either $p \equiv \pm 1\ \ (mod\ 16)$, $p \equiv q \pm 1\ \ (mod\ 16)$, or $p \equiv 4q \pm 1\ \ (mod\ 16)$.

3. Conclusion

We get integer solutions for $\alpha = 9, 12, 16, 25, 27, 33, 36, 44, 48, 64, 100, 144,$ and 196. We may search for other patterns of solutions.

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