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Submanifolds of Generalized Sasakian-space-forms with Respect to Certain Connections

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Abstract: The present paper deals with some results of submanifolds of generalized Sasakian-space-forms in [3] with respect to semisymmetric metric connection, semisymmetric non-metric connection, Schouten-van Kampen connection and Tanaka-webster connection.

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1. Introduction

As a generalization of Sasakian-space-form, Alegre [2] introduced the notion of generalized Sasakian-space-form as that an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ whose curvature tensor $\bar{R}$ of $\bar{M}$ satisfies

$$\begin{aligned} \bar{R}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{1}$$

for all vector fields X, Y, Z on $\bar{M}$ and f_1, f_2, f_3 are certain smooth functions on $\bar{M}$. Such a manifold of dimension $(2n+1)$, $n > 1$ (the condition $n > 1$ is assumed throughout the paper), is denoted by $\bar{M}^{2n+1}(f_1, f_2, f_3)$ [2]. Many authors studied this space form with different aspects. For this, we may refer ([11–15, 17, 18, 23]). It reduces to Sasakian-space-form if $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ [2]. After introducing the semisymmetric linear connection by Friedman and Schouten [7], Hayden [9] gave the idea of metric connection with torsion on a Riemannian manifold. Later, Yano [29] and many others (see, [21, 22, 24] and references therein) studied semisymmetric metric connection in different context. The idea of semisymmetric non-metric connection was introduced by Agashe and Chafle [1]. The Schouten-van Kampen connection introduced for the study of non-holomorphic manifolds ([20, 27]). In 2006, Bejancu [6] studied Schouten-van Kampen connection on foliated manifolds. Recently Olszak [19] studied Schouten-van Kampen connection on almost(para) contact metric structure. The Tanaka-Webster connection [25, 28] is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold. Tanno [26] defined the Tanaka-Webster connection for contact metric manifolds. The submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ are

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studied in [3, 10, 16]. In [3], Alegre and Carriazo studied submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to Levi-Civita connection $\bar{\nabla}$. The present paper deals with study of such submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to semisymmetric metric connection, semisymmetric non-metric connection, Schouten-van Kampen connection and Tanaka-webster connection respectively.

2. Preliminaries

In an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$, we have [4]

$$\phi^2(X) = -X + \eta(X)\xi, \ \ \phi\xi = 0, \tag{2}$$

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{4}$$

$$g(\phi X, Y) = -g(X, \phi Y). \tag{5}$$

In $\bar{M}^{2n+1}(f_1, f_2, f_3)$, we have [2]

$$(\bar{\nabla}_X \phi)(Y) = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \tag{6}$$

$$\bar{\nabla}_X \xi = -(f_1 - f_3)\phi X, \tag{7}$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}^{2n+1}(f_1, f_2, f_3)$. Let M be a submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$. If ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively then the Gauss and Weingarten formulae are given by [30]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are second fundamental form and shape operator (corresponding to the normal vector field V), respectively and they are related by $g(h(X, Y), V) = g(A_V X, Y)$. For any $X \in \Gamma(TM)$, we may write

$$\phi X = TX + FX, \tag{9}$$

where TX is the tangential component and FX is the normal component of ϕX. In particular, if $F = 0$ then M is invariant [5] and here $\phi(TM) \subset TM$. Also if $T = 0$ then M is anti-invariant [5] and here $\phi(TM) \subset T^{\perp}M$. Also here we assume that ξ is tangent to M. The semisymmetric metric connection $\tilde{\bar{\nabla}}$ and the Riemannian connection $\bar{\nabla}$ on $\bar{M}^{2n+1}(f_1, f_2, f_3)$ are related by [29]

$$\tilde{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi. \tag{10}$$

The Riemannian curvature tensor $\tilde{\bar{R}}$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\bar{\nabla}}$ is

$$\begin{aligned} \tilde{\bar{R}}(X, Y)Z \quad &= (f_1 - 1)\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ &+2g(X, \phi Y)\phi Z\} + (f_3 - 1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &-g(Y, Z)\eta(X)\xi\} + (f_1 - f_3)\{g(X, \phi Z)Y - g(Y, \phi Z)X + g(Y, Z)\phi X - g(X, Z)\phi Y\}. \end{aligned} \tag{11}$$

The semisymmetric non-metric connection $\bar{\nabla}'$ and the Riemannian connection $\bar{\nabla}$ on $\bar{M}^{2n+1}(f_1, f_2, f_3)$ are related by [1]

$$\bar{\nabla}'_X Y = \bar{\nabla}_X Y + \eta(Y)X. \tag{12}$$

The Riemannian curvature tensor $\bar{R}'$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\bar{\nabla}'$ is

$$\begin{aligned}\bar{R}'(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} + (f_1 - f_3)[g(X,\phi Z)Y - g(Y,\phi Z)X] \\ & + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y.\end{aligned} \tag{13}$$

The Schouten-van Kampen connection $\hat{\bar{\nabla}}$ and the Riemannian connection $\bar{\nabla}$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ are related by [19]

$$\hat{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + (f_1 - f_3)\eta(Y)\phi X - (f_1 - f_3)g(\phi X, Y)\xi. \tag{14}$$

The Riemannian curvature tensor $\hat{\bar{R}}$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\bar{\nabla}}$ is

$$\begin{aligned}\hat{\bar{R}}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ & - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + \{f_3 + (f_1 - f_3)^2\}\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} + (f_1 - f_3)^2\big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\big],\end{aligned} \tag{15}$$

where $(f_1 - f_3)$ is constant function. The Tanaka-Webster connection $\overset{*}{\bar{\nabla}}$ and the Riemannian connection $\bar{\nabla}$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ are related by [8]

$$\overset{*}{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \eta(X)\phi Y + (f_1 - f_3)\eta(Y)\phi X - (f_1 - f_3)g(\phi X, Y)\xi. \tag{16}$$

The Riemannian curvature tensor $\overset{*}{\bar{R}}$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$ is

$$\begin{aligned}\overset{*}{\bar{R}}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + \{f_3 + (f_1 - f_3)^2\}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ & + (f_1 - f_3)^2\big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\big] + 2(f_1 - f_3)g(X,\phi Y)\phi Z,\end{aligned} \tag{17}$$

where $(f_1 - f_3)$ is constant function.

3. Submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with $\tilde{\tilde{\bar{\nabla}}}$

Lemma 3.1. *If M is invariant submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\bar{\nabla}}}$, then $\tilde{\tilde{\bar{R}}}(X,Y)Z$ is tangent to M, for any $X, Y, Z \in \Gamma(TM)$.*

Proof. If M is invariant then from (11) we say that $\tilde{\tilde{\bar{R}}}(X,Y)Z$ is tangent to M because ϕX and ϕY are tangent to M. This proves the lemma. $\square$

Lemma 3.2. *If M is anti-invariant submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$, then*

$$\begin{aligned} tan(\tilde{\tilde{R}}(X,Y)Z) = &(f_1-1)\{g(Y,Z)X - g(X,Z)Y\} + (f_3-1)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{18}$$

$$nor(\tilde{\tilde{R}}(X,Y)Z) = (f_1-f_3)\{g(Y,Z)\phi X - g(X,Z)\phi Y\} \tag{19}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. Since M is anti-invariant, we have $\phi X, \phi Y \in \Gamma(T^{\perp}M)$. Then equating tangent and normal component of (11) we get the result. □

Lemma 3.3. *If $f_1(p) = f_3(p)$ and M is either invariant or anti-invariant submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$, then $\tilde{\tilde{R}}(X,Y)Z$ is tangent to M for any $X, Y, Z \in \Gamma(TM)$.*

Proof. Using Lemma 3.1 and Lemma 3.2 we get the result. □

Lemma 3.4. *If M is invariant or anti-invariant submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$, then $\tilde{\tilde{R}}(X,Y)V$ is normal to M, for any $X, Y, \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.*

Proof. If M is invariant from (11) we have $\tilde{\tilde{R}}(X,Y)V$ normal to M, and if M is anti-invariant then $\tilde{\tilde{R}}(X,Y)V = 0$ i.e. $\tilde{\tilde{R}}(X,Y)V$ normal to M for any $X, Y, \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. □

Lemma 3.5. *let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$. If $f_2(p) \neq 0$, $f_1(p) = f_3(p)$ and TM is invariant under the action of $\tilde{\tilde{R}}(X,Y)$, $X, Y \in \Gamma(TM)$, then M is either invariant or anti-invariant.*

Proof. For $X, Y \in \Gamma(TM)$, we have from (11) that

$$\begin{aligned} \tilde{\tilde{R}}(X,Y)X = &(f_1-1)\{g(Y,X)X - g(X,X)Y\} + f_2\{g(X,\phi X)\phi Y - g(Y,\phi X)\phi X + 2g(X,\phi Y)\phi X\} \\ &+ (f_3-1)\{\eta(X)\eta(X)Y - \eta(Y)\eta(X)X + g(X,X)\eta(Y)\xi - g(Y,X)\eta(X)\xi\} \\ &+ (f_1-f_3)\{g(\phi Y,X)X - g(\phi X,X)Y + g(Y,X)\phi X - g(X,X)\phi Y\}. \end{aligned} \tag{20}$$

Note that $\tilde{\tilde{R}}(X,Y)X$ should be tangent if $[-3f_2 g(Y,\phi X)\phi X + (f_1-f_3)\{g(Y,X)\phi X - g(X,X)\phi Y\}]$ is tangent. Since $f_2(p) \neq 0$, $f_1(p) = f_3(p)$ at any point p then by similar way of proof of Lemma 3.2 of [3], we can prove that either M is invariant or anti-invariant. This proves the Lemma. □

Remark 3.6. *let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$. If $f_1(p) \neq f_3(p)$ and TM is invariant under the action of $\tilde{\tilde{R}}(X,Y)$, $X, Y \in \Gamma(TM)$, then M is invariant.*

Theorem 3.7. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$. If $f_2(p) \neq 0$, $f_1(p) = f_3(p)$ then M is either invariant or anti-invariant if and only if TM is invariant under the action of $\tilde{\tilde{R}}(X,Y)$ for all $X, Y \in \Gamma(TM)$.*

Proof. It follows from Lemma 3.3 and Lemma 3.5. □

Proposition 3.8. *Let M be a submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$. If M is invariant, then TM is invariant under the action of $\tilde{\tilde{R}}(U,V)$ for any $U, V \in \Gamma(T^{\perp}M)$.*

Proof. Replacing X, Y, Z by U, V, X in (11), we get

$$\tilde{\tilde{R}}(U,V)X = (f_1-1)\{g(V,X)U - g(U,X)V\} + f_2\{g(U,\phi X)\phi V - g(V,\phi X)\phi U + 2g(U,\phi V)\phi X\} \tag{21}$$
$$+ (f_3-1)\{\eta(U)\eta(X)V - \eta(V)\eta(X)U + g(U,X)\eta(V)\xi - g(V,X)\eta(U)\xi\}$$
$$+ (f_1-f_3)\{g(\phi V,X)U - g(\phi U,X)V + g(V,X)\phi U - g(U,X)\phi V\}.$$

As M is invariant, $U, V \in \Gamma(T^\perp M)$, we have

$$g(X,\phi U) = -g(\phi X,U) = g(\phi V,X) = 0 \tag{22}$$

for any $X \in \Gamma(TM)$. Using (22) in (21), we have

$$\tilde{\tilde{R}}(U,V)X = 2f_2 g(U,\phi V)\phi X, \tag{23}$$

which is tangent as ϕX is tangent. This proves the proposition. □

Proposition 3.9. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\tilde{\tilde{\nabla}}$. If $f_2(p) \neq 0$, $f_1(p) = f_3(p)$ for each $p \in M$ and $T^\perp M$ is invariant under the action of $\tilde{\tilde{R}}(U,V)$, $U,V \in \Gamma(T^\perp M)$, then M is either invariant or anti-invariant.*

Proof. The proof is similar as it is an Lemma 3.4, just assuming that $\tilde{\tilde{R}}(U,V)U$ is normal for any $U, V \in \Gamma(T^\perp M)$. □

4. Submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with $\bar{\nabla}'$

Lemma 4.1. *If M is either invariant or anti-invariant submanifold of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\bar{\nabla}'$, then $\bar{R}'(X,Y)Z$ is tangent to M and $\bar{R}'(X,Y)V$ normal to M for any $X,Y,Z \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.*

Proof. If M is invariant then from (13) we say that $\bar{R}'(X,Y)Z$ is tangent to M because ϕX and ϕY are tangent to M. If M is anti-invariant then

$$g(X,\phi Z) = g(Y,\phi Z) = g(\phi X,Z) = g(\phi Y,Z) = 0. \tag{24}$$

From (13) and (24) we have

$$\bar{R}'(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \tag{25}$$
$$+ [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y],$$

which is tangent. If M is invariant then from (13), it follows that $\bar{R}'(X,Y)V$ is normal to M, and if M is anti-invariant then $\bar{R}'(X,Y)V = 0$ i.e. $\bar{R}'(X,Y)V$ is normal to M for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$. This proves the Lemma. □

Lemma 4.2. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\bar{\nabla}'$. If $f_2(p) \neq 0$ for each $p \in M$ and TM is invariant under the action of $\bar{R}'(X,Y)$, $X,Y \in \Gamma(TM)$, then M is either invariant or anti-invariant.*

Proof. For $X, Y \in \Gamma(TM)$, we have from (13) that

$$\bar{R}'(X,Y)X = f_1\{g(Y,X)X - g(X,X)Y\} + f_2\{g(X,\phi X)\phi Y - g(Y,\phi X)\phi X + 2g(X,\phi Y)\phi X\} \tag{26}$$

$$+ f_3\{\eta(X)\eta(X)Y - \eta(Y)\eta(X)X + g(X,X)\eta(Y)\xi - g(Y,X)\eta(X)\xi\}$$
$$- (f_1 - f_3)g(\phi X, Y)X + \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}.$$

Note that $\bar{R}^{'}(X,Y)X$ should be tangent if $3f_2(p)g(Y,\phi X)\phi X$ is tangent. Since $f_2(p) \neq 0$ for each $p \in M$, as similar as proof of Lemma 3.2 of [3], we may conclude that either M is invariant or anti-invariant. This proves the Lemma. □

Theorem 4.3. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\bar{\nabla}^{'}$. If $f_2(p) \neq 0$ for each $p \in M$, then M is either invariant or anti-invariant if and only if TM is invariant under the action of $\bar{R}^{'}(X,Y)$ for all $X, Y \in \Gamma(TM)$.*

Proof. It is obvious from Lemma 4.1 and Lemma 4.2. □

Proposition 4.4. *Let M be a submanifold of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\bar{\nabla}^{'}$. If M is invariant, then TM is invariant under the action of $\bar{R}^{'}(U,V)$ for any $U, V \in \Gamma(T^{\perp}M)$.*

Proof. Replacing X, Y, Z by U, V, X in (13), we get

$$\begin{aligned}\bar{R}^{'}(U,V)X &= f_1\{g(V,X)U - g(U,X)V\} + f_2\{g(U,\phi X)\phi V - g(V,\phi X)\phi U + 2g(U,\phi V)\phi X\} \\ &+ f_3\{\eta(U)\eta(X)V - \eta(V)\eta(X)U + g(U,X)\eta(V)\xi - g(V,X)\eta(U)\xi\} \\ &+ (f_1 - f_3)\{g(U,\phi X)V - g(V,\phi X)U\} + \{\eta(V)\eta(X)U - \eta(U)\eta(X)V\}.\end{aligned} \tag{27}$$

As M is invariant, $U \in \Gamma(T^{\perp}M)$, we have

$$g(X,\phi U) = -g(\phi X, U) = g(\phi V, X) = 0 \tag{28}$$

for any $X \in \Gamma(TM)$. Using (28) in (27), we have

$$\bar{R}^{'}(U,V)X = 2f_2 g(U,\phi V)\phi X, \tag{29}$$

which is tangent as ϕX is tangent. This proves the proposition. □

Proposition 4.5. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\bar{\nabla}^{'}$. If $f_2(p) \neq 0$ for each $p \in M$ and $T^{\perp}M$ is invariant under the action of $\bar{R}(U,V)$, $U, V \in \Gamma(TM)$, then M is either invariant or anti-invariant.*

Proof. The proof is similar as the proof of Lemma 4.2, just imposing that $\bar{R}^{'}(U,V)U$ is normal for any $U, V \in \Gamma(TM)$. □

5. Submanifolds of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with $\hat{\bar{\nabla}}$

Lemma 5.1. *If M is either invariant or anti-invariant submanifold of $\bar{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\hat{\bar{\nabla}}$, then $\hat{\bar{R}}(X,Y)Z$ is tangent to M and $\hat{\bar{R}}(X,Y)V$ is normal to M for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.*

Proof. If M is invariant then from (15) we say that $\hat{\bar{R}}(X,Y)Z$ is tangent to M because ϕX and ϕY are tangent to M. If M is anti-invariant then

$$g(X,\phi Z) = g(Y,\phi Z) = g(\phi X, Z) = g(\phi Y, Z) = 0. \tag{30}$$

From (15) and (30) we have

$$\hat{\bar{R}}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + \{f_3 + (f_1 - f_3)^2\}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \tag{31}$$

which is tangent. If M is invariant from (15) we have $\hat{\bar{R}}(X,Y)V$ is normal to M, and if M is anti-invariant then $\hat{\bar{R}}(X,Y)V = 0$ i.e. $\hat{\bar{R}}(X,Y)V$ is normal to M for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. This proves the Lemma. □

Lemma 5.2. *let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\hat{\nabla}}$. If $3f_2 \neq (f_1 - f_3)^2$ on M and TM is invariant under the action of $\hat{\hat{R}}(X, Y)$, $X, Y \in \Gamma(TM)$, then M is either invariant or anti-invariant.*

Proof. For $X, Y \in \Gamma(TM)$, we have from (15) that

$$\hat{\hat{R}}(X,Y)X = f_1\{g(Y,X)X - g(X,X)Y\} + f_2\{g(X,\phi X)\phi Y - g(Y,\phi X)\phi X + 2g(X,\phi Y)\phi X\} \\ + \{f_3 + (f_1 - f_3)^2\}\{\eta(X)\eta(X)Y - \eta(Y)\eta(X)X + g(X,X)\eta(Y)\xi - g(Y,X)\eta(X)\xi\} \\ + (f_1 - f_3)^2\{g(X,\phi X)\phi Y - g(Y,\phi X)\phi X\}. \tag{32}$$

Now, we see that $\hat{\hat{R}}(X, Y)X$ should be tangent if $\{3f_2 + (f_1 - f_3)^2\}g(Y, \phi X)\phi X$ is tangent. Since $3f_2 \neq -(f_1 - f_3)^2$ then in similar way of proof of Lemma 3.2 of [3] we may conclude that either M is invariant or anti-invariant. This proves the Lemma. □

Theorem 5.3. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\hat{\nabla}}$. If $3f_2 \neq -(f_1 - f_3)^2$, then M is either invariant or anti-invariant if and only if TM is invariant under the action of $\hat{\hat{R}}(X, Y)$ for all $X, Y \in \Gamma(TM)$.*

Proof. Using Lemma 5.1 and Lemma 5.2, we get the result. □

Proposition 5.4. *Let M be a submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\hat{\nabla}}$. If M is invariant, then TM is invariant under the action of $\hat{\hat{R}}(U, V)$ for any $U, V \in \Gamma(T^\perp M)$.*

Proof. Replacing X, Y, Z by U, V, X in (15), we get

$$\hat{\hat{R}}(U,V)X = f_1\{g(V,X)U - g(U,X)V\} + f_2\{g(U,\phi X)\phi V - g(V,\phi X)\phi U + 2g(U,\phi V)\phi X\} \\ + \{f_3 + (f_1 - f_3)^2\}\{\eta(U)\eta(X)V - \eta(V)\eta(X)U + g(U,X)\eta(V)\xi - g(V,X)\eta(U)\xi\} \\ + (f_1 - f_3)^2\{g(U,\phi X)\phi V - g(V,\phi X)\phi U\}. \tag{33}$$

As M is invariant, $U \in \Gamma(T^\perp M)$, we have

$$g(X, \phi U) = -g(\phi X, U) = g(\phi V, X) = 0 \tag{34}$$

for any $X \in \Gamma(TM)$. Using (34) in (33), we have

$$\hat{\hat{R}}(U,V)X = 2f_2 g(U,\phi V)\phi X, \tag{35}$$

which is tangent as ϕX is tangent. This proves the proposition. □

Proposition 5.5. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\hat{\nabla}}$. If $3f_2 \neq -(f_1 - f_3)^2$ on M and $T^\perp M$ is invariant under the action of $\hat{\hat{R}}(U, V)$, $U, V \in \Gamma(T^\perp M)$, then M is either invariant or anti-invariant.*

Proof. The proof is similar as the proof of Lemma 5.2, just imposing that $\hat{\hat{R}}(U, V)U$ is normal for any $U, V \in \Gamma(T^\perp M)$. □

6. Submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with $\overset{*}{\bar{\nabla}}$

Lemma 6.1. *If M is either invariant or anti-invariant submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$, then $\overset{*}{\bar{R}}(X,Y)Z$ is tangent to M and $\overset{*}{\bar{R}}(X,Y)V$ is normal to M for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.*

Proof. If M is invariant then from (17) we say that $\overset{*}{\bar{R}}(X,Y)Z$ is tangent to M because ϕX and ϕY are tangent to M. If M is anti-invariant then

$$g(X, \phi Z) = g(Y, \phi Z) = g(\phi X, Z) = g(\phi Y, Z) = 0. \tag{36}$$

From (17) and (36) we have

$$\overset{*}{\bar{R}}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + \{f_3 + (f_1 - f_3)^2\}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \tag{37}$$

which is tangent. If M is invariant from (17) we have $\overset{*}{\bar{R}}(X,Y)V$ normal to M and if M is anti-invariant then $\overset{*}{\bar{R}}(X,Y)V = 0$ i.e. $\overset{*}{\bar{R}}(X,Y)V$ normal to M for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. This proves the Lemma. □

Lemma 6.2. *let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$. If $\{3f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2\}(p) \neq 0$ for each $p \in M$ and TM is invariant under the action of $\overset{*}{\bar{R}}(X,Y)$, $X, Y \in \Gamma(TM)$, then M is either invariant or anti-invariant.*

Proof. For $X, Y \in \Gamma(TM)$, we have from (17) that

$$\begin{aligned}\overset{*}{\bar{R}}(X,Y)X = {} & f_1\{g(Y,X)X - g(X,X)Y\} + f_2\{g(X,\phi X)\phi Y - g(Y,\phi X)\phi X + 2g(X,\phi Y)\phi X\} \\ & + \{f_3 + (f_1 - f_3)^2\}\{\eta(X)\eta(X)Y - \eta(Y)\eta(X)X + g(X,X)\eta(Y)\xi - g(Y,X)\eta(X)\xi\} \\ & + (f_1 - f_3)^2\{g(X,\phi X)\phi Y - g(Y,\phi X)\phi X\} + 2(f_1 - f_3)g(X,\phi Y)\phi X.\end{aligned} \tag{38}$$

Now we see that $\overset{*}{\bar{R}}(X,Y)X$ should be tangent if $\{3f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2\}(p)g(Y,\phi X)\phi X$ is tangent. Since $\{3f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2\}(p) \neq 0$ then by similar way of proof of Lemma 3.2 of [3] we can proved that either M is invariant or anti-invariant. This proves the Lemma. □

Theorem 6.3. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$. If $\{3f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2\}(p) \neq 0$, then M is either invariant or anti-invariant if and only if TM is invariant under the action of $\overset{*}{\bar{R}}(X,Y)$ for all $X, Y \in \Gamma(TM)$.*

Proof. It follows from Lemma 6.1 and Lemma 6.2. □

Proposition 6.4. *Let M be a submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$. If M is invariant, then TM is invariant under the action of $\overset{*}{\bar{R}}(U,V)$ for any $U, V \in \Gamma(T^{\perp}M)$.*

Proof. Replacing X, Y, Z by U, V, X in (17), we get

$$\begin{aligned}\overset{*}{\bar{R}}(U,V)X = {} & f_1\{g(V,X)U - g(U,X)V\} + f_2\{g(U,\phi X)\phi V - g(V,\phi X)\phi U + 2g(U,\phi V)\phi X\} \\ & + \{f_3 + (f_1 - f_3)^2\}\{\eta(U)\eta(X)V - \eta(V)\eta(X)U + g(U,X)\eta(V)\xi - g(V,X)\eta(U)\xi\} \\ & + (f_1 - f_3)^2\{g(U,\phi X)\phi V - g(V,\phi X)\phi U\} + 2(f_1 - f_3)g(U,\phi V)\phi X.\end{aligned} \tag{39}$$

As M is invariant, $U \in \Gamma(T^{\perp}M)$, we have

$$g(X, \phi U) = -g(\phi X, U) = g(\phi V, X) = 0 \tag{40}$$

for any $X \in \Gamma(TM)$. Using (40) in (39), we have

$$\overset{*}{\bar{R}}(U, V)X = \{2f_2 + 2(f_1 - f_3)\}g(U, \phi V)\phi X, \tag{41}$$

which is tangent as ϕX is tangent. This proves the proposition. □

Proposition 6.5. *Let M be a connected submanifold of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$. If $\{3f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2\}(p) \neq 0$ for each $p \in M$ and $T^{\perp}M$ is invariant under the action of $\overset{*}{\bar{R}}(U, V)$, $U, V \in \Gamma(T^{\perp}M)$, then M is either invariant or anti-invariant.*

Proof. The proof is similar as the proof of Lemma 6.2, just considering that $\overset{*}{\bar{R}}(U, V)U$ is normal for any $U, V \in \Gamma(T^{\perp}M)$. □

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