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Fuzzy Dominating Set

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Abstract: In this paper, we define effective vertex, vertex effective fuzzy graph, fuzzy dominating set of a fuzzy graph. Some properties of minimum fuzzy dominating set are discussed. Bounds of fuzzy dominating number is also studied.

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1. Preliminaries

We summarise some basic definitions in fuzzy graphs. A fuzzy subset of a non-empty set V is a mapping σ: $V \to [0, 1]$. A fuzzy relation on V is a fuzzy subset of $V \times V$. If μ and λ are fuzzy relations, then $(\mu \circ \lambda)(u, w) = \sup\{\mu(u, v)\Lambda\lambda(v, w) : v \in V\}$ where Λ means "inf". The powers of fuzzy relations are defined as $\mu^1 = \mu$, $\mu^2 = \mu o \mu$, $\mu^3 = \mu o \mu^2$ and so on. It is also defined that $\mu^\infty = \sup\limits_{k=1,2,3,\ldots} \mu^k$. A fuzzy graph $G(\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where for all u, v in V we have $\mu(u, v) = \sigma(u)\Lambda\sigma(v)$. The fuzzy graph $H(\tau, \eta)$ is called a fuzzy subgraph of $G(\sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ for all $u \in V$ and $\eta(u, v) \leq \mu(u, v)$ for all $u, v \in V$. A path ρ in a fuzzy graph is a sequence of distinct vertexs $u_0, u_1, u_2, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $1 \leq i \leq n$; here $n \geq 0$ is called the length of the path ρ. The consecutive pairs (u_{i-1}, u_i) are called the arcs of the path. The strength of a path is defined as $\Lambda_{i=1}^{n}\mu(u_{i-1}, u_i)$.

In other words, the strength of a path is defined to be the weight of the weakest arc of the path. Two vertexs are joined by a path are said to be connected. Clearly u and v are connected if and only if $\mu^\infty(u, v) > 0$. A strongest path joining any two vertexs u and v has strength $\mu^\infty(u, v)$; we shall sometimes refer to this as the strength of connectedness between the vertexs. The underlying crisp graph of the fuzzy graph $G(\sigma, \mu)$ is denoted by the pair of sets $G^* : (\sigma^*, \mu^*)$, where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. In a fuzzy graph G, u dominates v [3] if (u, v) is a strong arc. A subset D of V is called a dominating set of G if for every $v \in D$, there exists $u \in D$ such that u dominates v. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The smallest number of vertexs in any dominating set of G is called its domination number and is denoted by $\gamma(G)$. A dominating set D of a fuzzy graph G such that $|D| = \gamma(G)$ is called a minimum dominating set.

In this article, fuzzy domination in a fuzzy graph is defined. Some properties of fuzzy dominating set are also discussed. Throughout, we assume that $G(\sigma, \mu)$ is a finite fuzzy graph. *i.e.* σ^* is finite and by G, we mean the fuzzy graph $G(\sigma, \mu)$.

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2. Fuzzy Domination in Fuzzy Graph

Let us recollect the definition of fuzzy subset of a fuzzy set. The function α is said to be a fuzzy subset of σ if $\alpha, \sigma : V \to [0, 1]$ and $\alpha(x) \leq \sigma(x) \quad \forall \quad x \in V$. Further, α is said to be a proper fuzzy subset of σ if $\alpha(x) < \sigma(x) \forall x \in V$ when $\sigma(x) > 0$. Also, the cardinality of a fuzzy set σ is defined as $|\sigma| = \Sigma\sigma(u)$ for all $u \in V$. This definition gives rise to a new approach of domination in fuzzy graph. The fuzzy dominating set of a fuzzy graph $G(\sigma, \mu)$ naturally should be a fuzzy sub set of σ. The minimum fuzzy dominating set should possess minimum cardinality. Before we define fuzzy domination, the basic definitions required are presented here.

Definition 2.1 ([1]). *An arc (u, v) is said to be strong if $\mu(u, v) \geq \mu^\infty(u, v)$.*

Definition 2.2 ([3]). *The strong neighborhood of u is $N_s(u) = \{v \in \sigma^* \colon (u, v) \text{ is a strong arc }\}$.*

Notation 2.3. *Let σ be a fuzzy set on V. By $u \in \sigma$, we mean that $\sigma(u) > 0$ and otherwise $u \notin \sigma$.*

Definition 2.4. *Let u and v be two fuzzy vertexs of σ of a fuzzy graph G. Then we say that either u dominates v or vice versa if*

(1). the membership values of u and v are $\geq \mu(u, v)$ in the reference set.

(2). (u, v) is a strong arc in G.

This definition is well explained with the help of the following example.

Example 2.5. *Let $V = \{u_1, u_2, u_3, u_4, u_5\}$, σ on V is defined as $\sigma(u_1) = 0.8$, $\sigma(u_2) = 0.8$, $\sigma(u_3) = 0.3$, $\sigma(u_4) = 0.6$, $\sigma(u_5) = 0.9$ and μ on $V \times V$ is defined as $\mu(u_1, u_2) = 0.6$, $\mu(u_2, u_3) = 0.2$, $\mu(u_1, u_4) = 0.6$, $\mu(u_2, u_5) = 0.7$ and $\mu(u_4, u_5) = 0.5$. Then $G(\sigma, \mu)$ is a fuzzy graph on V. The fuzzy vertex $u_3 \in \sigma$ dominates the fuzzy vertex u_2. Also $u_3(.2)$ also dominates the fuzzy vertex u_2 where the .2 represents the membership value of u_3 in the reference set. If we take $\alpha \subseteq \sigma$ defined as $\alpha = \{u_1(.6), u_2(.7)\}$ then $u_1, u_2 \in \alpha$ dominate each other in G.*

Definition 2.6. *A fuzzy subset α of σ is called a fuzzy dominating set of G if for every $v \in \sigma - \alpha^*$, there exists $u \in \alpha$ such that u dominates v. A fuzzy dominating set α is called a minimal fuzzy dominating set if no proper subset of α is a fuzzy dominating set. The cardinality of any minimum fuzzy dominating subset of σ is called its fuzzy domination number and is denoted by $\gamma_f(G)$.*

Example 2.7. *In the fuzzy graph defined in the example 2.5, the fuzzy subset $\{u_1(.8),\ u_2(.8),\ u_3(.3)\}$ is a fuzzy dominating set. The fuzzy subset $\{u_1(.6),\ u_2(.7),\ u_3(.2)\}$ is a minimal fuzzy dominating set and the fuzzy subset $\{u_2(.7), u_4(.6)\}$ is a minimum fuzzy dominating set. Also $\gamma_f(G) = 1.3$.*

Definition 2.8. *A fuzzy vertex of $u \in G(\sigma, \mu)$ is said to be an effective vertex if $\sigma(u) = \max_{v \in N_s(u)} \mu(u, v)$.*

Definition 2.9 ([2]). *A fuzzy edge of $(u, v) \in G(\sigma, \mu)$ is said to be an effective edge if $\sigma(u) \min \sigma(v) = \mu(u, v)$.*

Definition 2.10 ([2]). *A fuzzy graph G having every edge as effective edge is called an effective fuzzy graph.*

Definition 2.11. *A fuzzy graph G having every vertex as effective vertex is called vertex effective fuzzy graph.*

Example 2.12. *Let $V = \{u_1, u_2, u_3, u_4, u_5\}$, σ on V is defined as $\sigma(u_1) = 0.6$, $\sigma(u_2) = 0.7$, $\sigma(u_3) = 0.2$, $\sigma(u_4) = 0.6$, $\sigma(u_5) = 0.7$ and μ on $V \times V$ is defined as $\mu(u_1, u_2) = 0.6$, $\mu(u_2, u_3) = 0.2$, $\mu(u_1, u_4) = 0.6$, $\mu(u_2, u_5) = 0.7$ and $\mu(u_4, u_5) = 0.5$. Then $G(\sigma, \mu)$ is a vertex effective fuzzy graph.*

Definition 2.13. *Two fuzzy vertexs of a fuzzy graph are said to be fuzzy independent if there is no strong arc between them. A subset ζ of σ is said to be a fuzzy independent set of G if any two fuzzy vertices of ζ are fuzzy independent. A fuzzy independent set ζ is called a maximal fuzzy independent set if no superset of ζ is a fuzzy independent set. The cardinality of any maximal fuzzy independent subset of σ is called its fuzzy independent number and is denoted by $\beta_f(G)$.*

Definition 2.14. *A fuzzy graph G is said to be fuzzy bipartite if the vertex set σ can be partitioned into two nonempty sets σ_1 and σ_2 such that σ_1 and σ_2 are fuzzy independent sets. These σ_1 and σ_2 are called fuzzy bipartition of σ.*

Thus every strong arc of G has one end in σ_1 and the other end in σ_2.

Definition 2.15. *A fuzzy bipartite graph G with fuzzy bipartition σ_1 and σ_2 is said to be a complete fuzzy bipartite if for each vertex of σ_1, every vertex of σ_2 is a strong neighbour.*

Example 2.16. *The fuzzy subset $\{u_4(.5), u_5(.5)\}$ of σ is a fuzzy independent set of the fuzzy graph given in 2.5.*

From the above it is clear that a fuzzy dominating set of a fuzzy graph is a subset of the vertex set of a vertex effective fuzzy graph. Hence, in what follows, we consider only the vertex effective fuzzy graph.

Definition 2.17. *A fuzzy subgraph $H(\tau, \rho)$ of G is said to be a spanning subgraph of G, if $\sigma(u) = \tau(u)$ for all $u \in V$.*

Definition 2.18. *Let F be a spanning subgraph of G. If $(x, y) \in G$ but not in F, then there is a path in F between x and y whose strength is greater than $\mu(x, y)$, then G is a fuzzy forest. A connected fuzzy forest is a fuzzy tree.*

3. Some Results

By the definition of a minimum fuzzy dominating set, the following result follows easily.

Proposition 3.1. *If $\sigma(v_i) > \sigma(v)$ for every i where $v_i \in N_s(v)$, then v is a member of any minimum fuzzy dominating set.*

The characterization of a minimal fuzzy dominating set is given in the following theorem.

Theorem 3.2. *A fuzzy dominating set α of G is a minimal fuzzy dominating set if and only if for each $u \in \alpha$ one of the following two conditions holds.*

(1). u is an isolated fuzzy vertex of G

(2). There is a fuzzy vertex $v \notin \alpha$ such that $N_s(v) \cap \alpha = \{u(\alpha(u))\}$.

Proof. Suppose α is a minimal fuzzy dominating set of G. Then for each fuzzy vertex $u \in \alpha$, the fuzzy set $\alpha' = \alpha - \{u(1)\}$ is not a fuzzy dominating set. Thus, there is a fuzzy vertex $v \in \sigma - \alpha'^*$ which is not dominated by any fuzzy vertex in α'. Now either $u = v$ or $v \in \sigma - \alpha^*$. If $u = v$, then u is an isolated vertex. If $v \in \sigma - \alpha^*$ and v is not dominated by α', but is dominated by α, then u is the only strong neighbor of v in α. Hence condition 2 follows.
Conversely, suppose α is a fuzzy dominating set and each fuzzy vertex $u \in \alpha$, one of the two stated conditions holds. Now we prove that α is a fuzzy minimal dominating set. Suppose α is not a minimal fuzzy dominating set. Then there exists a fuzzy vertex $u \in \alpha$ such that $\alpha - \{u(1)\}$ is a fuzzy dominating set. Thus u is a strong neighbor to at least one fuzzy vertex in $\alpha - \{u(1)\}$. Therefore condition 1 does not hold. Also if $\alpha - \{u(1)\}$ is a fuzzy dominating set, then every vertex in $\sigma - \alpha^*$ is a strong neighbor to at least one vertex in $\alpha - \{u(1)\}$. Therefore condition 2 does not hold. Hence neither condition 1 nor 2 holds, which is a contradiction. □

Theorem 3.3. *Every non-trivial connected fuzzy graph G has a fuzzy dominating set α with another fuzzy dominating set α' such that $\alpha \cap \alpha' = \phi$.*

Proof. Let u be a fuzzy vertex in G and T be a spanning fuzzy tree of G. Then the fuzzy vertexs in T fall into two disjoint fuzzy sets α and α' consisting of the vertexs with an even and odd distances from u in T. Obviously α and α' are fuzzy dominating sets of G. □

Theorem 3.4. *Let G be a fuzzy graph without isolated vertexs. If α is a minimal fuzzy dominating set, then there exists a fuzzy subset α' of σ such that α' is a fuzzy dominating set, where $\alpha \cap \alpha' = \phi$ and $\alpha \cup \alpha' = \sigma$.*

Proof. Let α be a minimal fuzzy dominating set of G. Suppose α' is not a fuzzy dominating set. Then there exists a fuzzy vertex $u \in \alpha$ such that u is not dominated by any vertex in α'. Since G has no isolated fuzzy vertexs, u is a strong neighbor of at least one vertex in $\alpha - \{u(1)\}$. Then $\alpha - \{u(1)\}$ is a fuzzy dominating set, which contradicts the minimality of α. Thus every fuzzy vertex in α is a strong neighbor of at least one fuzzy vertex in α'. Hence α' is a fuzzy dominating set. □

Theorem 3.5. *Every minimum fuzzy dominating set of a fuzzy graph is a minimal fuzzy dominating set.*

The converse of this result need not be true.

Theorem 3.6. *For a fuzzy bipartite graph G, $\gamma_f(G) \leq min\{|\sigma_1|, |\sigma_2|\}$ where σ_1 and σ_2 are fuzzy bipartition of G.*

Proof. Since G is a fuzzy bipartite graph with partitions σ_1 and σ_2, both are fuzzy dominating sets. Hence the proof follows. □

Theorem 3.7. *For a complete fuzzy bipartite graph G, $\gamma_f(G) = \min_{u \in \sigma_1} \sigma_1(u) + \min_{v \in \sigma_2} \sigma_2(v)$, where σ_1 and σ_2 are fuzzy bipartition of G.*

Theorem 3.8. *A fuzzy independent set is a maximal fuzzy independent set if and only if it is fuzzy independent and fuzzy dominating.*

Proof. Suppose a fuzzy independent set ζ is a maximal fuzzy independent, then for every fuzzy vertex $u \in \sigma - \zeta^*$, the set $\zeta \cup \{u(\sigma(u)\}$ is not fuzzy independent. That is for every vertex $u \in \sigma - \zeta$, there is a vertex $v \in S$ such that (u, v) is strong. Thus ζ is a dominating set. Hence ζ is both fuzzy independent and dominating. Conversely, suppose a set ζ is both fuzzy independent and dominating. We show that it is maximal fuzzy independent. Suppose ζ is not maximal fuzzy independent. Then there exists a vertex $u \in \sigma - \zeta$ such that ζ is fuzzy independent. But if $\zeta \cup \{u(\sigma(u))\}$ is fuzzy independent then no fuzzy vertex in ζ is a strong neighbor of u. Hence ζ is not a dominating set, which is a contradiction. Hence ζ is maximal fuzzy independent □

Theorem 3.9. *Every maximal fuzzy independent set in a fuzzy graph G is a minimal dominating set in G.*

Proof. Let ζ be a maximal fuzzy independent set in a graph G. By previous theorem ζ is a fuzzy dominating set. We now show that ζ is a minimal fuzzy dominating set. Suppose ζ is not a minimal dominating set then there exists a vertex $u \in \zeta$ such that $\zeta' = \zeta - \{u(1)\}$ is a fuzzy dominating set. Then at least one vertex in ζ' is a strong neighbor of u. This contradicts that ζ is a fuzzy independent set in G. Hence ζ is a minimal fuzzy dominating set. □

A relationship between the domination number and independence numbers of a fuzzy graph is given.

Theorem 3.10. *For any graph G, $\gamma_f(G) \leq \beta_f(G)$.*

Proof. Let ζ be a fuzzy independent set of vertexs in G such that $\zeta = \beta(G)$. Then G contains no larger fuzzy independent set. This means that every vertex in $\sigma - \zeta$ has at least one strong neighbor in ζ. Therefore ζ is a dominating set. Thus $\gamma_f(G) \leq \zeta = \beta_f(G)$. It is easy to observe that for any complete fuzzy graph G, $\gamma_f(G) = \beta_f(G) = 1$. □

4. Bounds

We are interested in finding a minimum dominating set in a fuzzy graph. In this section we obtain some bounds for $\gamma_f(G)$ of a fuzzy graph G. One can observe that if a fuzzy graph has a vertex which is a strong neighbor to every other vertexs of G then $\gamma_f(G) = 1$ and conversely. Also one can easily see that if G is a totally disconnected fuzzy graph, then $\gamma_f(G) = |\sigma|$ and conversely. Thus we have $\gamma_f(G) \leq |\sigma|$. Clearly every isolated vertex is a member of any fuzzy dominating set.

Theorem 4.1. *For any fuzzy graph G without isolated vertexs $\gamma_f(G) \leq \frac{|\sigma|}{2}$.*

Proof. The proof of the theorem is the direct consequence of the Theorem 3.3. □

Theorem 4.2. *For any fuzzy graph G, $\gamma_f(G) \leq |\sigma| - \Delta(G)$.*

Proof. Let u be a fuzzy vertex of G such that $N_s(u)$ be its strong neighbours. Clearly u dominates $N_s(u)$. Then $\sigma - N_s(u)$ is a dominating set. Therefore $\gamma_f(G) \leq |\sigma - N_s(u)| = |\sigma| - \Delta(G)$. □

Theorem 4.3. *For any fuzzy graph G, $\max\limits_{u,v\in\sigma} \mu(u,v) \leq \gamma_f(G)$.*

Proof. The proof follows from the definition of effective vertex and the fact that a fuzzy dominating set is non empty. □

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