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# Stability of Finite Difference Formula of Implicit Type for IBVP of Heat Equation

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**Abstract:** In the paper entitled "A Finite Difference Formula of Implicit Type for IBVP of Heat Equation" [2] we have derived a Finite Difference Formula to get numerical solutions of Initial-Boundary Value Problem of Heat Equation. As, we all know that only stable Formulas can give Numerical Solutions which are almost close to the Exact Solution. Here in this paper the stability of the Finite Difference Formula is checked with the help of Eigen Values and using the concept of Diagonalization of matrix.

**MSC:** 35K05, 35K20, 65N06, 65N12.

**Keywords:** Heat Equation, Solution of First Order Linear PDE, Eigen values, Diagonalization of Matrix, Tridiagonal Matrix.



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## 1. Introduction

In our paper entitled "A Finite Difference Formula of Implicit Type for IBVP of Heat Equation" [2], A Finite Difference Formula of Implicit Type for the numerical solution of Initial-Boundary Value Problem of Heat Equation is derived. In Numerical Methods for Differential Equations, the growth of round-off errors or small change in initial data which may cause a large change in final answer from the exact solution becomes responsible for the instability of the formula. In this paper, in section 2, a brief introduction about the stability of A Finite Difference Formula is carried out. And also an Initial-Boundary Value Problem of Heat Equation is stated [1, 3]. Moreover, in this section a Finite Difference Formula is given to calculate the round-off error at each mesh point. In section 3 with the help of Eigen Values and applying the concept of Diagonalization of a matrix, stability of The Finite Difference Formula derived in [2] is checked and the criteria for the same is derived. In section 4 an IBVP of Heat Equation is Ilustrated. Also its Numerical Solutions for different values of $r$ obtained from the Finite Difference Formula developed in [2] are compared with Mathematica Exact values of its Analytical Solution.

## 2. Stability

When we solve any finite difference formula and carry out each calculation to a finite number of decimal places, the solution we actually obtain is not the exact solution but the numerical solution. For example, let us denote the exact solution by

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$u$ and the numerical solution by $N$, where $u = \{u_{i,j}\}$ is exactly as given by the Finite Difference Formula which we are considering and $u_{i,j}$ is the value of $u$ at the mesh point $P_{i,j}$. Moreover, $N = \{N_{i,j}\}$ is the truncated series obtained from the Finite Difference Formula and $N_{i,j}$ is the value of $N$ at the mesh point $P_{i,j}$. Generally a round-off error is calculated at each mesh point. That is, the round-off error at a point $P_{i,j}$ is $e_{i,j} = u_{i,j} - N_{i,j}$. More specifically, if errors $e_{1,1}$, $e_{1,2}$, ..., $e_{i,j}$, ... are introduced at the mesh points $P_{1,1}$, $P_{1,2}$, ..., $P_{i,j}$, ... respectively, and $|e_{1,1}|$, $|e_{1,2}|$, ..., $|e_{i,j}|$, ... are each less than $\delta$, then the finite difference formula is stable when the maximum value of $u - N$ tends to zero as $\delta$ tends to zero [3]. Here, we consider the Initial-Boundary Value Problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le l,\ 0 \le t \le T \tag{1}$$

with initial condition,

$$u(x,0) = f(x), \qquad 0 \le x \le l$$

and boundary conditions,

$$u(0,t) = g_0(t), \qquad 0 \le t \le T$$
$$u(l,t) = g_l(t), \qquad 0 \le t \le T$$

Here, to get finite difference formula, Taylor Series Expansion is used and the scheme forward in time and centred in space [1, 3] is considered. Also we use the notations as in [2],

$$\begin{aligned} u(x,t) &= u(ih, jk) \\ &= u_{i,j} \\ u(x+h,t) &= u((i+1)h, jk) \\ &= u_{i+1,j} \\ u(x,t+k) &= u(ih,(j+1)k) \\ &= u_{i,j+1}. \end{aligned}$$

Where, $x = ih$ and $t = jk$, $i = 0,1,2,\ldots,m$, $\quad j = 0,1,2,\ldots,n$. Here, we must note that $h$ is the difference between two consecutive values of $x$ and $k$ is the difference between two consecutive values of $t$. So, we must fix the values of $h$ and $k$ before starting the calculations to get numerical solutions. Therefore, for fixed $h$ and $0 \le x \le l$,

$$l = mh \quad \Rightarrow \quad m = \frac{l}{h}.$$

Similarly, for fixed $k$ and $0 \le t \le T$,

$$T = nk \quad \Rightarrow \quad n = \frac{T}{k}.$$

A Finite Difference Formula of Implicit Type for above IBVP of Heat equation, which is derived in [2] is,

$$[2I + r(M_1 + M_2)][u_{i,j+1}] = [2I - r(M_1 + M_2)][u_{i,j}] + (b_1 + b_2)$$

Here, for each $i$, $u_{i,0}$ is known from the initial condition and in above Finite Difference Formula taking $j = 0$, $u_{i,1}$ is calculated for each $i$. In Similar way, once $u_{i,1}$'s are known, using the Finite Difference Formula we may calculate $u_{i,2}$ for each $i$ by taking $j = 1$ and simultaneously $u_{i,j}$ are obtained for other values of $j$. Moreover, in above Finite Difference Formula for simplicity, we use the notations, $u_{i,j} = u_j$ and $u_{i,j+1} = u_{j+1}$. Therefore, the above Finite Difference Formula is rewritten as,

$$[2I + r(M_1 + M_2)][u_{j+1}] = [2I - r(M_1 + M_2)][u_j] + (b_1 + b_2) \tag{2}$$

In the above formula (2), $I$ is the Identity Matrix, $r = \frac{k}{h^2}$, $M_1 = [M^1_{ij}]$ and $M_2 = [M^2_{ij}]$. Where,

$$M^1_{ij} = \begin{cases} 1 & \text{if } \ i = j \\ -1 & \text{if } \ i = 2q, \ j = 2q - 1 \\ -1 & \text{if } \ i = 2q - 1, \ j = 2q \end{cases}$$

$$M^2_{ij} = \begin{cases} 1 & \text{if } \ i = j \\ -1 & \text{if } \ i = 2q + 1, \ j = 2q \\ -1 & \text{if } \ i = 2q, \ j = 2q + 1 \end{cases}$$

$q = 1, 2, \ldots, \frac{m-2}{2}$; $b_1 = r[b^1_{ij}]$, where,

$$b^1_{ij} = \begin{cases} u_{0,j} & \text{if } \ i = 1, \ j = 1 \\ u_{m,j+1} & \text{if } \ i = m - 1, \ j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

and $b_2 = r[b^2_{ij}]$, where,

$$b^2_{ij} = \begin{cases} u_{0,j+1} & \text{if } \ i = 1, \ j = 1 \\ u_{m,j} & \text{if } \ i = m - 1, \ j = 1 \\ 0 & \text{otherwise} \end{cases}$$

**Remark 2.1.** *Here we must note that the terms containing higher orders of $h$ and $k$ are ignored while deriving (2). Hence (2) represents the truncated series and therefore gives the numerical solution $N$.*

Now by writing $e_{i,j} = u_{i,j} - N_{i,j}$, we can see that $e_{i,j}$ satisfies the Finite Difference Formula,

$$[2I + r(M_1 + M_2)][e_{j+1}] = [2I - r(M_1 + M_2)][e_j]$$

which is same as (2), but the term $b_1 + b_2$ is dropped. The above formula is rewritten as,

$$[e_{j+1}] = \frac{[2I - r(M_1 + M_2)]}{[2I + r(M_1 + M_2)]}[e_j]. \tag{3}$$

which gives the round-off error at each mesh point. In the next section we will use above formula (3) to obtain the criteria for the stability of the Finite Difference Formula (2).

## 3. Stability of the Finite Difference Formula

In (3) let us take

$$A = \frac{P}{Q}$$

where,

$$P = [2I - r(M_1 + M_2)]$$

and

$$Q = [2I + r(M_1 + M_2)].$$

Then,

$$\lambda_A = \frac{\lambda_P}{\lambda_Q}$$

where $\lambda_A$, $\lambda_P$ and $\lambda_Q$ are eigen values of $A$, $P$ and $Q$ respectively. Now, if we diagonalize $A$ then the diagonal values in the diagonal matrix of $A$ are the eigen values. Hence, we can see from (3), $[e_{j+1}] = [D_A][e_j]$, where, $D_A$ is diagonal matrix of order $m-1$, whose diagonal elements are eigen values of $A$. Hence, it is observed that round-off error decreases if each eigen value of $A$ is small. Also according to [3], if $max|\lambda_A| \leq 1$, then the truncation error decreases and (2) gets stable. In $P$ and $Q$, $I$ is identity matrix of order $m-1$ and $M_1$ and $M_2$ are square matrices of order $m-1$. Here, if we take $M = M_1 + M_2 = [M_{ij}]$ then,

$$M_{ij} = \begin{cases} 2 & \text{if } \ i = j \\ -1 & \text{if } \ i = q \ \& \ j = q+1 \\ -1 & \text{if } \ i = q+1 \ \& \ j = q \end{cases}$$

$q = 1, 2, \ldots, m-1$. Which is a Tridiagonal Matrix. According to [3] the eigen values of Tridiagonal Matrix whose elements in principle diagonal are 'a' and elements in the upper and lower diagonal are 'b' and 'c' respectively are given by,

$$\lambda^k = \mathrm{a} + 2\sqrt{\mathrm{bc}}cos\left(\frac{k\pi}{n+1}\right)$$

$k = 1, 2, \ldots, n$. Hence, eigen values of $M$ are,

$$\begin{aligned} \lambda_M^s &= 2 + 2\cos\left(\frac{\pi s}{m}\right) \\ &= 4\cos^2\left(\frac{\pi s}{2m}\right) \end{aligned}$$

$s = 1, 2, \ldots, m-1$. This implies,

$$\lambda_P^s = 2 - 4r\cos^2\left(\frac{\pi s}{2m}\right)$$

and,

$$\lambda_Q^s = 2 + 4r\cos^2\left(\frac{\pi s}{2m}\right)$$

$s = 1, 2, \ldots, m-1$

$$\Rightarrow \quad \lambda_A^s = \frac{2 - 4r\cos^2\left(\frac{\pi s}{2m}\right)}{2 + 4r\cos^2\left(\frac{\pi s}{2m}\right)}$$

$$\Rightarrow \quad \lambda_A^s = \frac{1 - 2r\cos^2\left(\frac{\pi s}{2m}\right)}{1 + 2r\cos^2\left(\frac{\pi s}{2m}\right)}$$

$s = 1, 2, \ldots, m-1$. Now, as $r = \frac{k}{h^2}$, $r$ is always positive i.e., $r > 0$. Multiplying both the sides by $4\cos^2\left(\frac{\pi s}{2m}\right)$, we get,

$$0 < 4r\cos^2\left(\frac{\pi s}{2m}\right)$$

Also, as $-2 < 0$, we may write,

$$-2 < 0 < 4r\cos^2\left(\frac{\pi s}{2m}\right)$$

Moreover, on adding $1 - 2r\cos^2\left(\frac{\pi s}{2m}\right)$ to all the terms, we get,

$$-\left(1 + 2r\cos^2\left(\frac{\pi s}{2m}\right)\right) < 1 - 2r\cos^2\left(\frac{\pi s}{2m}\right) < 1 + 2r\cos^2\left(\frac{\pi s}{2m}\right)$$

And on dividing all the terms by $\left(1 + 2r\cos^2\left(\frac{\pi s}{2m}\right)\right)$, we get,

$$-1 < \frac{1 - 2r\cos^2\left(\frac{\pi s}{2m}\right)}{1 + 2r\cos^2\left(\frac{\pi s}{2m}\right)} < 1$$

$s = 1, 2, \ldots, m-1$. Hence,

$$-1 < \lambda_A^s < 1$$

Therefore we may write,

$$-1 < \max[\lambda_A^s] < 1.$$

Hence we may write,

$$-1 \leq \max[\lambda_A^s] \leq 1.$$

i.e.

$$\max|\lambda_A^s| \leq 1$$

Now, as we have discussed in the beginning of this section, the Finite Difference Formula is stable if $\max|\lambda_A^s|$ is less than or equal to 1. Hence, according to above discussions, the Finite Difference Formula (2) is stable for all positive values of $r$.

## 4. Comparison of Solutions for Different Values of $r$

Let us take an example of IBVP of Heat Equation and solve it for different values of $r$ using the Finite Difference Formula (2.2). and we also compare it with Mathematica Exact Values of Analytical Solution.

**Solve the heat equation** [2]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with initial condition

$$u(x, 0) = 4x(1-x), \qquad 0 \leq x \leq 1$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t \geq 0$$

*Solution.* First of all we note that the analytical solution of this IBVP of Heat Equation is,

$$u(x, t) = \frac{32}{\pi^3}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^3}e^{-(2p+1)^2\pi^2 t}\sin((2p+1)\pi x). \tag{4}$$

which is derived using known methods [4].

**Note:** In following tables,

- For The Analytical Solution (4.1),

$$u(1-x,t)=\frac{32}{\pi^3}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^3}e^{-(2p+1)^2\pi^2 t}\sin((2p+1)\pi(1-x))$$

Also,

$$\begin{aligned}\sin((2p+1)\pi(1-x)) &= \sin((2p+1)\pi-(2p+1)\pi x))\\ &= \sin((2p+1)\pi x), \qquad p=0,1,2,\ldots,\infty.\end{aligned}$$

Therefore,

$$\begin{aligned}u(1-x,t) &= \frac{32}{\pi^3}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^3}e^{-(2p+1)^2\pi^2 t}\sin((2p+1)\pi(1-x))\\ &= \frac{32}{\pi^3}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^3}e^{-(2p+1)^2\pi^2 t}\sin((2p+1)\pi x)\\ &= u(x,t)\end{aligned}$$

Hence,

$$u(0.1,t)=u(0.9,t)$$
$$u(0.2,t)=u(0.8,t)$$
$$u(0.3,t)=u(0.7,t)$$
$$u(0.4,t)=u(0.6,t)$$

- We shall calculate the time row $j+1$ from the values in time row $j$. That is, The first time row $j=1$ is calculated from the values in $j=0$ which is obtained from the Initial Condition. Consequently $j=2$ is obtained from $j=1$, $j=3$ is obtained from $j=3$ and this process continues for all the values of $j$.

- Here, as $0\le x\le 1$, smaller values of $h$ gives system with larger number of linear equations. Hence, in such cases we are comparing the solutions only at few mesh points.

- Numerical Solutions obtained by (2) implementing on Mathematica are denoted by NS.

- Mathematica Exact Values of The Analytical Solution are denoted by AS.

- Absolute Difference between numerical solution and mathematica exact value of analytical solution is denoted by AD.

(1). We consider $r=1$, with $h=0.01$ and $k=0.0001$. Hence for $t=0.01$, $j=100$ (i.e. the 100th time row).

| | $x=0.01$ | $x=0.02$ | $x=0.03$ | $x=0.04$ | $x=0.05$ |
|---|---|---|---|---|---|
| NS | 0.0309673 | 0.0618895 | 0.0927217 | 0.123419 | 0.1539390 |
| AS | 0.0309654 | 0.0618858 | 0.0927162 | 0.123412 | 0.1539303 |
| AD | 0.0000019 | 0.0000037 | 0.0000155 | 0.000007 | 0.0000900 |

**Table 1.** $u[x,0.01]$ **and** $r=1$

(2). We consider $r=1$, with $h=0.01$ and $k=0.0001$. Hence for $t=0.02$, $j=200$ (i.e. the 200th time row).

| | $x = 0.01$ | $x = 0.02$ | $x = 0.03$ | $x = 0.04$ | $x = 0.05$ |
|---|---|---|---|---|---|
| NS | 0.0272299 | 0.0544278 | 0.081562 | 0.108601 | 0.135513 |
| AS | 0.0272285 | 0.0544252 | 0.0815581 | 0.108596 | 0.135506 |
| AD | 0.0000014 | 0.0000026 | 0.0.0000049 | 0.000005 | 0.000007 |

**Table 2.** $u[x, 0.02]$ **and** $r = 1$

(3). We consider $r = 20$, with $h = 0.01$ and $k = \frac{1}{500}$. Hence for $t = 0.01$, $j = 5$ (i.e. the 5th time row).

| | $x = 0.01$ | $x = 0.02$ | $x = 0.03$ | $x = 0.04$ | $x = 0.05$ |
|---|---|---|---|---|---|
| NS | 0.0307990 | 0.0617761 | 0.0926787 | 0.1234100 | 0.1539295 |
| AS | 0.0309654 | 0.0618858 | 0.0927162 | 0.1234123 | 0.1539303 |
| AD | 0.0001664 | 0.0001097 | 0.0000305 | 0.0000023 | 0.0000008 |

**Table 3.** $u[x,0.01]$ **and** $r = 20$

(4). We consider $r = 20$, with $h = 0.01$ and $k = \frac{1}{500}$. Hence for $t = 0.02$, $j = 10$ (i.e. the 10th time row).

| | $x = 0.01$ | $x = 0.02$ | $x = 0.03$ | $x = 0.04$ | $x = 0.05$ |
|---|---|---|---|---|---|
| NS | 0.0272921 | 0.0544321 | 0.0815343 | 0.1085693 | 0.1354870 |
| AS | 0.0272285 | 0.0544251 | 0.0815580 | 0.1085956 | 0.1355064 |
| AD | 0.0000636 | 0.0000070 | 0.0000237 | 0.0000263 | 0.0000194 |

**Table 4.** $u[x, 0.02]$ **and** $r = 20$

(5). We consider $r = 50$, with $h = 0.01$ and $k = \frac{1}{200}$. Hence for $t = 0.01$, $j = 2$ (i.e. the 2nd time row).

| | $x = 0.01$ | $x = 0.02$ | $x = 0.03$ | $x = 0.04$ | $x = 0.05$ |
|---|---|---|---|---|---|
| NS | 0.0316397 | 0.0627401 | 0.0934678 | 0.1239202 | 0.1541456 |
| AS | 0.0309654 | 0.0618858 | 0.0927162 | 0.1234123 | 0.1539303 |
| AD | 0.0006742 | 0.0008543 | 0.0007515 | 0.0005078 | 0.0002153 |

**Table 5.** $u[x, 0.01]$ **and** $r = 50$

(6). We consider $r = 50$, with $h = 0.01$ and $k = \frac{1}{200}$. Hence for $t = 0.02$, $j = 4$ (i.e. the 4th time row).

| | $x = 0.01$ | $x = 0.02$ | $x = 0.03$ | $x = 0.04$ | $x = 0.05$ |
|---|---|---|---|---|---|
| NS | 0.0276592 | 0.0548839 | 0.0818736 | 0.1087232 | 0.1354632 |
| AS | 0.0272285 | 0.0544251 | 0.0815580 | 0.1085956 | 0.1355064 |
| AD | 0.0004307 | 0.0004558 | 0.0003156 | 0.0001276 | 0.0000432 |

**Table 6.** $u[x, 0.02]$ **and** $r = 50$

(7). We consider $r = 200$, with $h = 0.01$ and $k = \frac{1}{50}$. Hence for $t = 0.02$, $j = 1$ (i.e. the 1st time row).

| | $x = 0.01$ | $x = 0.02$ | $x = 0.03$ | $x = 0.04$ | $x = 0.05$ |
|---|---|---|---|---|---|
| NS | 0.0243814 | 0.0494107 | 0.0749501 | 0.1008750 | 0.1270727 |
| AS | 0.0272285 | 0.0544251 | 0.0815580 | 0.1085956 | 0.1355064 |
| AD | 0.0028471 | 0.0050144 | 0.0066079 | 0.0077206 | 0.0084337 |

**Table 7.** $u[x, 0.02]$ **and** $r = 200$

## 5. Conclusion

We have given brief explanation of truncation error in section 2 and in section 3 stability criteria for A Finite Difference Formula of Implicit Type for IBVP of Heat Equation is derived. In section 4 we have taken an example of IBVP of Heat equation and its numerical solutions are compared with the Mathematica Exact Values of its Analytical solution. From the discussions in sections 3 & 4 it is observed that the Finite Difference Formula is stable for all positive values of $r$.

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