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Generalized Ulam - Hyers - Rassias Stability of Additive - Quadratic Functional Equation in $(\beta;p)-$ Banach Space

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Abstract: The main aim of this research article is to analyze the generalized Ulam-Hyers-Rassias stability of additive-quadratic functional equation in $(\beta;p)-$ Banach spaces.

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1. Introduction

A classical issue in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem admits a solution, we say that the equation is stable. The primary stability problem concerning group homomorphisms was raised by S.M.Ulam [34] in 1940. In the subsequent year, D.H.Hyers [17] gave a constructive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Th.M.Rassias [26] proved a generalization of Hyers theorem for additive mappings for sum of powers of norms. Also, In 1982, J.M. Rassias [23] followed the innovative approach of the Th.M. Rassias theorem [26] in which he replaced the factor sum of powers of norms by product of powers of norms. The result of Rassias's provided a lot of influences during the last seven decades in the development of a generalization of the Hyers-Ulam stability concept, this new concept is known as the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations (see [1, 11, 18, 20]). Furthermore, in 1994, a generalization of Rassias theorem was obtained in Gavruta [12] by replacing the bound by a general control function. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [30] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations were extensively investigated by a number of authors and there were many interesting results concerning this problem [21, 24, 30]) and reference cited there in. The general solution and generalized Ulam - Hyers stability of several mixed type additive quadratic functional equation were analyzed

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in [3–9, 25] In this research paper, the authors establish the generalized Ulam-Hyers-Rassias stability of additive-quadratic functional equation

$$g\left(x+2y+3z\right)-g\left(x-2y+3z\right)+g\left(x+2y-3z\right)-g\left(x-2y-3z\right)=8g\left(x\right)-8g\left(x-y\right)+4\left[g(y)+g(-y)\right] \quad (1)$$

in (β,p)- Banach spaces. It is easy that $g(x)=ax+bx^2$ is the solution of the above functional equation. In Section 2, we present some definitions and notations about (β,p)- Banach spaces. In Sections 3, 4, 5 the generalized Ulam-Hyers-Rassias stability of (1) when g is even, odd and mixed cases are analyzed respectively.

2. Basic Definitions, Notations about (β,p)- Banach Spaces

In this section, we present some basic facts concerning (β,p)- Banach spaces and some preliminary results. We fix a real number β with $0<\beta\leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. *Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:*

$(QNS1)$ $\|x\|\geq 0$ *for all* $x\in X$ *and* $\|x\|=0$ *if and only if* $x=0$.

$(QNS2)$ $\|\rho x\| =|\rho|^{\beta}.\|x\|$ *for all* $\rho\in\mathbb{K}$ *and all* $x\in X$.

$(QNS3)$ *There is a constant* $K\geq 1$ *such that* $\|x+y\|\leq K(\|x\|+\|y\|)$ *for all* $x,y\in X$.

The pair $(X,\|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.2. *A quasi-β-Banach space is a complete quasi-β-normed space.*

Definition 2.3. *A quasi-β-norm $\|\cdot\|$ is called a (β,p)-norm $(0<p\leq 1)$ if $\|x+y\|^p\leq\|x\|^p+\|y\|^p$ for all $x,y\in X$. In this case, a quasi-β-Banach space is called a (β,p)-Banach space.*

More details, one can refer [10, 31, 32] for the concepts of quasi-normed spaces and p-Banach space. Given a $p-$norm, the formula $d(x,y):=||x-y||^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [31], each quasi-norm is equivalent to some $p-$norm. Since it is much easier to work with $p-$norms than quasi-norms, henceforth we restrict our attention mainly to $p-$norms. In [32], J. Tabor has investigated a version of the Hyers-Rassias-Gajda theorem in quasi-Banach spaces. Hereafter, throughout this research paper, let $\mathcal{X}$ be a quasi $\beta-$ normed space and $\mathcal{Y}$ be a quasi $\beta-$ Banach space respectively. Define a mapping $g:\mathcal{X}\to\mathcal{Y}$ by

$$D\ g(x,y,z)=g\left(x+2y+3z\right)-g\left(x-2y+3z\right)+g\left(x+2y-3z\right)-g\left(x-2y-3z\right)-8g\left(x\right)+8g\left(x-y\right)-4\left[g(y)+g(-y)\right].$$

for all $x,y,z\in\mathcal{X}$.

3. Even Case Stability Analysis

In this section, we analyze the generalized Ulam - Hyers stability of the functional equation (1) when g is even.

Theorem 3.1. *Suppose that $\Phi : \mathcal{X}^3 \to [0,\infty)$ is a function satisfies the condition*

$$\lim_{a\to\infty} \frac{\Phi\left(4^{aj}x, 4^{aj}y, 4^{aj}z\right)}{16^{aj}} < \infty \tag{2}$$

for all $x, y, z \in \mathcal{X}$. Assume $g_e : \mathcal{X} \to \mathcal{Y}$ be an even function satisfies the inequality

$$\|D\ g_e(x,y,z)\|_{\mathcal{Y}} \leq \Phi\left(x,y,z\right) \tag{3}$$

for all $x, y, z \in \mathcal{X}$. Then there exists one and only quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g_e(x) - Q(x)\|_{\mathcal{Y}}^p \leq \left(\frac{K^{a-1}}{16^\beta} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{1}{16^{ij}} \Psi_E\left(4^{ij}x\right)\right)^p \tag{4}$$

where

$$\Psi_E(4^{ij}x) = \Phi\left(4^{ij}x, 4^{ij}x, \left(\frac{4^{ij}}{3}\right)x\right) \tag{5}$$

for all $x \in \mathcal{X}$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{a\to\infty} \frac{g_e(4^{aj}x)}{16^{aj}} \tag{6}$$

for all $x \in \mathcal{X}$, where $j \in \{-1, 1\}$.

Proof. First, we give proof for $j = 1$. Changing (x,y,z) by $\left(x, x, \frac{x}{3}\right)$ in (3) and using evenness of g_e, we arrive

$$\|g_e\left(4x\right) - 16g_e(x)\|_{\mathcal{Y}} \leq \Phi\left(x, x, \frac{x}{3}\right) \tag{7}$$

for all $x \in \mathcal{X}$. Using $(QNS2)$ in (7), we reach

$$\left\|\frac{g_e(4x)}{16} - g_e(x)\right\|_{\mathcal{Y}} \leq \frac{1}{16^\beta}\Phi\left(x, x, \frac{x}{3}\right) = \frac{1}{16^\beta}\ \Psi_E(x) \tag{8}$$

for all $x \in \mathcal{X}$. Replacing x by $4x$ in (8) and multiply by $\frac{1}{16}$, we have

$$\left\|\frac{g_e(4^2x)}{16^2} - \frac{g_e(4x)}{16}\right\|_{\mathcal{Y}} \leq \frac{1}{16^{\beta+1}}\Psi_E(4x) \tag{9}$$

for all $x \in \mathcal{X}$. Combining (8), (9) with the help of $(QNS3)$, we find

$$\left\|\frac{g_e(4^2x)}{16^2} - g_e(x)\right\|_{\mathcal{Y}} \leq \frac{K}{16^\beta}\left[\Psi_E\left(x\right) + \frac{1}{16}\Psi_E\left(4x\right)\right] \tag{10}$$

for all $x_1 \in \mathcal{X}$. Using induction on a positive integer a , we obtain that

$$\left\|\frac{g_e(4^ax)}{16^a} - g_e(x)\right\|_{\mathcal{Y}} \leq \frac{K^{a-1}}{16^\beta} \sum_{i=0}^{a-1} \frac{1}{16^i}\Psi_E\left(4^ix\right) \tag{11}$$

for all $x \in \mathcal{X}$. It is easy to verify that $\left\{\frac{g_e(4^ax)}{16^a}\right\}$ is a Cauchy sequence, by interchanging x by 4^bx and divided by 16^b in (11), for any $a, b > 0$ and letting b tends to infinity. Since $\mathcal{Y}$ is complete, there exists a mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$Q(x) = \lim_{a\to\infty} \frac{g_e(4^ax)}{16^a}, \quad \forall\ x \in \mathcal{X}.$$

Letting $a \to \infty$ in (11), we see that (4) holds for all $x \in \mathcal{X}$. Now, we need to prove Q satisfies (1), replacing (x, y, z) by $(4^k x, 4^k y, 4^k z)$ and divided by 16^a in (3), we get

$$\|DQ\left(4^a x, 4^a y, 4^a z\right)\|_{\mathcal{Y}} = \frac{1}{16^a}\|Dg_e\left(4^a x, 4^a y, 4^a z\right)\|_{\mathcal{Y}} \leq \frac{\Phi\left(4^a x, 4^a y, 4^a z\right)}{16^a} \to \quad 0 \quad as \quad a \quad \to \infty$$

for all $x, y, z \in \mathcal{X}$. Thus Q satisfies (1) for all $x, y, z \in \mathcal{X}$. In order to prove $Q(x)$ is unique, let $R(x)$ be another quadratic mapping satisfying (4) and (1). Then

$$\begin{aligned}
\|Q(x) - R(x)\|_{\mathcal{Y}} &= \frac{1}{16^a}\|Q(4^a x) - R(4^a x)\|_{\mathcal{Y}} \\
&\leq \frac{K}{16^a}\left\{\|Q(4^a x) - g_e(4^a x)\|_{\mathcal{Y}} + \|g_e(4^a x) - R(4^a x)\|_{\mathcal{Y}}\right\} \\
&\leq \frac{2K^a}{16^\beta}\sum_{i=0}^{\infty}\frac{1}{16^{(a+i)}}\Psi_E(4^{a+i}x) \quad \to 0 \quad as \ \ a \to \infty
\end{aligned}$$

for all $x \in \mathcal{X}$. Hence Q is unique. Thus the theorem is holds for $j = 1$. Second, we give proof for $j = -1$. Changing x by $x/4$ in (8), we have

$$\left\|g_e\left(x\right) - 16 g_e\left(\frac{x}{4}\right)\right\|_{\mathcal{Y}} \leq \Psi_E\left(\frac{x}{4}\right) \tag{12}$$

for all $x \in \mathcal{X}$. The rest of the proof is similar to that of case $j = 1$. Thus the theorem holds for $j = -1$. This completes the proof of the theorem. □

The following corollary is a immediate consequence of Theorem 3.1 concerning the stabilities of Ulam - Hyers [17], Ulam - TRassias [26], Ulam - GRassias [23], Ulam - JRassias [30] of (1).

Corollary 3.2. *Let σ and α be nonnegative real numbers. If a even function $g_e : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|Dg_e(x, y, z)\|_{\mathcal{Y}} \leq \begin{cases} \sigma, \\ \sigma\left(||x||^\alpha + ||y||^\alpha + ||z||^\alpha\right), \\ \sigma||x||^\alpha||y||^\alpha||z||^\alpha, \\ \sigma\left\{||x||^\alpha||y||^\alpha||z||^\alpha + \left(||x||^{3\alpha} + ||y||^{3\alpha} + ||z||^{3\alpha}\right)\right\} \end{cases} \tag{13}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g_e(x) - Q(x)\|_{\mathcal{Y}}^p \leq \begin{cases} \left(\dfrac{K^{a-1}}{16^\beta}\dfrac{\sigma\ 16}{|15|}\right)^p, \\ \left(\dfrac{K^{a-1}}{16^\beta}\sigma\left(2 + \dfrac{1}{3^{\beta\alpha}}\right)\dfrac{16}{|16 - 4^{\beta\alpha}|}||x||^\alpha\right)^p, & \alpha \neq 2; \\ \left(\dfrac{K^{a-1}}{16^\beta}\sigma\left(\dfrac{1}{3^{\beta\alpha}}\right)\dfrac{16}{|16 - 4^{3\beta\alpha}|}||x||^{3\alpha}\right)^p, & \alpha \neq \frac{2}{3}; \\ \left(\dfrac{K^{a-1}}{16^\beta}\sigma\left(2 + \dfrac{1}{3^{\beta\alpha}}\right)\dfrac{16}{|16 - 4^{3\beta\alpha}|}||x||^{3\alpha}\right)^p, & \alpha \neq \frac{2}{3} \end{cases} \tag{14}$$

4. Odd Case Stability Analysis

In this section, we analyze the generalized Ulam - Hyers stability of the functional equation (1) when g is odd.

Theorem 4.1. *Suppose that $\Phi : \mathcal{X}^3 \to [0, \infty)$ is a function satisfies the condition*

$$\lim_{a\to\infty}\frac{\Phi\left(3^{aj}x, 3^{aj}y, 3^{aj}z\right)}{3^{aj}} < \infty \tag{15}$$

for all $x, y, z \in \mathcal{X}$. Assume $g_o : \mathcal{X} \to \mathcal{Y}$ be an odd function satisfies the inequality

$$\|D\ g_o(x,y,z)\|_{\mathcal{Y}} \leq \Phi\,(x,y,z) \tag{16}$$

for all $x, y, z \in \mathcal{X}$. Then there exists one and only additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g_o(x) - A(x)\|_{\mathcal{Y}}^p \leq \left(\frac{K^a}{3^\beta} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{1}{3^{ij}} \Psi_O\left(3^{ij}x\right) \right)^p \tag{17}$$

where

$$\Psi_O(3^{ij}x) = \Phi\left(\frac{3^{ij}x}{2}, \frac{3^{ij}x}{2}, \frac{3^{ij}x}{2}\right) + \Phi\left(\frac{3^{ij}x}{2}, \frac{3^{ij}x}{2}, \frac{3^{ij}x}{6}\right) \tag{18}$$

for all $x \in \mathcal{X}$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{a\to\infty} \frac{g_o(3^{aj}x)}{3^{aj}} \tag{19}$$

for all $x \in \mathcal{X}$, where $j \in \{-1, 1\}$.

Proof. Assume $j = 1$. Letting (x, y, z) by $\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$ in (16), and using oddness of g_o, we obtain

$$\left\|g_o(3x) - g_o(x) + g_o(2x) - 8g_o\left(\frac{x}{2}\right)\right\|_{\mathcal{Y}} \leq \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \tag{20}$$

for all $x \in \mathcal{X}$. Replacing (x, y, z) by $\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{6}\right)$ in (16) and using oddness of g_o, we get

$$\left\|g_o(2x) + g_o(x) + g_o(x) - 8g_o\left(\frac{x}{2}\right)\right\|_{\mathcal{Y}} \leq \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{6}\right) \tag{21}$$

for all $x \in \mathcal{X}$. Now, from (20) and (21), we have

$$\begin{aligned}
\|g_o(3x) - 3g_o(x)\|_{\mathcal{Y}} &= \left\|g_o(3x) + g_o(2x) - g_o(x) - 8g_o\left(\frac{x}{2}\right) - g_o(2x) - g_o(x) - g_o(x) + 8g_o\left(\frac{x}{2}\right)\right\|_{\mathcal{Y}} \\
&\leq K\left\{\left\|g_o(3x) + g_o(2x) - g_o(x) - 8g_o\left(\frac{x}{2}\right)\right\|_{\mathcal{Y}} + \left\|g_o(2x) + g_o(x) + g_o(x) - 8g_o\left(\frac{x}{2}\right)\right\|_{\mathcal{Y}}\right\} \\
&\leq K\left\{\Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{6}\right)\right\} = K\Psi_O(x)
\end{aligned} \tag{22}$$

for all $x \in \mathcal{X}$. From (22), we reach

$$\left\|\frac{g_o(3x)}{3} - g_o(x)\right\|_{\mathcal{Y}} \leq \frac{K}{3^\beta}\Psi_O(x) \tag{23}$$

for all $x \in \mathcal{X}$. Changing x by $3x$ in (23) and multiply by $\frac{1}{3}$, we arrive

$$\left\|\frac{g_o(3^2x)}{3^2} - \frac{g_o(3x)}{3}\right\|_{\mathcal{Y}} \leq \frac{K}{3^{\beta+1}}\Psi_O\,(3x) \tag{24}$$

for all $x \in \mathcal{X}$. Combining (23) and (24), we find that

$$\left\|\frac{g_o(3^2x)}{3^2} - g_o(x)\right\|_{\mathcal{Y}} \leq \frac{K^2}{3^\beta}\left[\Psi_O\,(x) + \frac{\Psi_O\,(3x)}{3}\right] \tag{25}$$

for all $x_1 \in \mathcal{X}$. Using induction on a positive integer a , we obtain that

$$\left\|\frac{g_o(3^ax)}{3^a} - g_o(x)\right\|_{\mathcal{Y}} \leq \frac{K^a}{3^\beta}\sum_{i=0}^{a-1}\frac{1}{3^i}\Psi_O\left(3^ix\right) \tag{26}$$

for all $x \in \mathcal{X}$. The rerst of the proof is similar ideas to that of Theorem 3.1. □

The following corollary is a immediate consequence of Theorem 4.1 concerning the stabilities of Ulam - Hyers [17], Ulam - TRassias [26], Ulam - GRassias [23], Ulam - JRassias [30] of (1).

Corollary 4.2. *Let σ and α be nonnegative real numbers. If a odd function $g_o : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|Dg_o(x,y,z)\|_{\mathcal{Y}} \leq \begin{cases} \sigma, \\ \sigma\left(||x||^{\alpha} + ||y||^{\alpha} + ||z||^{\alpha}\right), \\ \sigma||x||^{\alpha}||y||^{\alpha}||z||^{\alpha}, \\ \sigma\left\{||x||^{\alpha}||y||^{\alpha}||z||^{\alpha} + \left(||x||^{3\alpha}||y||^{3\alpha}||z||^{3\alpha}\right)\right\}, \end{cases} \tag{27}$$

for all x,y,z in $\mathcal{X}$. Then there exists a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g_o(x) - A(x)\|_{\mathcal{Y}}^p \leq \begin{cases} \left(\frac{K^a}{3^\beta} \cdot \frac{6\sigma}{|2|}\right)^p, & \\ \left(\frac{K^a \cdot \sigma}{3^\beta}\left(\frac{5}{2^{\beta\alpha}} + \frac{1}{6^{\beta\alpha}}\right)\frac{||x||^{\alpha}}{|3-3^{\beta\alpha}|}\right)^p, & \alpha \neq 1; \\ \left(\frac{K^a \cdot \sigma}{3^\beta}\left(\frac{1}{2^{3\beta\alpha}} + \frac{1}{2^{2\beta\alpha}\cdot 6^{\beta\alpha}}\right)\frac{||x||^{3\alpha}}{|3-3^{3\alpha}|}\right)^p, & \alpha \neq \frac{1}{3}; \\ \left(\frac{K^a \cdot \sigma}{3^\beta}\left(\frac{6}{2^{3\beta\alpha}} + \frac{1}{6^{3\beta\alpha}}\left(1 + \frac{1}{2^{2\beta\alpha}}\right)\right)\frac{||x||^{3\alpha}}{|3-3^{3\alpha}|}\right)^p, & \alpha \neq \frac{1}{3} \end{cases} \tag{28}$$

for all $x \in \mathcal{X}$.

5. Even - Odd Case Stability Analysis

In this section, we analyze the generalized Ulam - Hyers stability of the functional equation (1) when g is even-odd.

Theorem 5.1. *Suppose that $\Phi : \mathcal{X}^3 \to [0,\infty)$ is a function satisfies the conditions (2) and (15) for all $x,y,z \in \mathcal{X}$. Assume $g : \mathcal{X} \to \mathcal{Y}$ be a function satisfies the inequality*

$$\|D\ g(x,y,z)\|_{\mathcal{Y}} \leq \Phi\left(x,y,z\right) \tag{29}$$

for all $x,y,z \in \mathcal{X}$. Then there exists a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ and a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g(x) - Q(x) - A(x)\|_{\mathcal{Y}} \leq \tfrac{K^{p+1}}{2^{p\beta}}\left\{\left(\tfrac{K^{a-1}}{16^\beta}\sum_{i=\frac{1-j}{2}}^{\infty}\tfrac{1}{16^{ij}}\left[\Psi_E\left(4^{ij}x\right) + \Psi_E\left(-4^{ij}x\right)\right]\right)^p + \left(\tfrac{K^a}{3^\beta}\sum_{i=\frac{1-j}{2}}^{\infty}\tfrac{1}{3^{ij}}\left[\Psi_O\left(3^{ij}x\right) + \Psi_O\left(-3^{ij}x\right)\right]\right)^p\right\} \tag{30}$$

for all $x \in \mathcal{X}$. The mapping $A(x)$ and $Q(x)$ respectively defined in (6) and (19) for all $x \in \mathcal{X}$, where $j \in \{-1,1\}$.

Proof. Define a function $g_e(x)$ by

$$g_e(x) = \frac{1}{2}\left\{g\left(x\right) + g\left(-x\right)\right\}$$

for all $x \in \mathcal{X}$. Then $g_e\left(0\right) = 0, g_e\left(x\right) = g_e\left(-x\right)$. Thus

$$\begin{aligned} \|Dg_e\left(x,y,z\right)\|_{\mathcal{Y}} &= \left\|\frac{1}{2}\left\{Dg\left(x,y,z\right) + Dg\left(-x,-y,-z\right)\right\}\right\|_{\mathcal{Y}} \\ &\leq \frac{K}{2^\beta}\left\{\|Dg\left(x,y,z\right)\|_{\mathcal{Y}} + \|Dg\left(-x,-y,-z\right)\|_{\mathcal{Y}}\right\} \\ &\leq \frac{K}{2^\beta}\left\{\Phi\left(x,y,z\right) + \Phi\left(-x,-y,-z\right)\right\} \end{aligned}$$

for all $x \in \mathcal{X}$. It follows from Theorem 3.1 that there exits a unique quadratic function $Q : \mathcal{X} \to Y$ such that

$$\|g_e(x) - Q(x)\|_{\mathcal{Y}}^p \leq \frac{K^p}{2^{p\beta}} \left(\frac{K^{a-1}}{16^\beta} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{1}{16^{ij}} \left[\Psi_E \left(4^{ij} x\right) + \Psi_E \left(-4^{ij} x\right) \right] \right)^p \tag{31}$$

for all $x \in \mathcal{X}$. Again define a function $g_o(x)$ by $g_o(x) = \frac{1}{2} \{g(x) - g(-x)\}$ for all $x \in \mathcal{X}$. Then $g_o(0) = 0, g_o(x) = -g_o(-x)$. Thus

$$\begin{aligned} \|Dg_o(x, y, z)\|_{\mathcal{Y}} &= \left\| \frac{1}{2} \{Dg(x, y, z) + Dg(-x, -y, -z)\} \right\|_{\mathcal{Y}} \\ &\leq \frac{K}{2^\beta} \{\|Dg(x, y, z)\|_{\mathcal{Y}} + \|Dg(-x, -y, -z)\|_{\mathcal{Y}}\} \\ &\leq \frac{K}{2^\beta} \{\Phi(x, y, z) + \Phi(-x, -y, -z)\} \end{aligned}$$

for all $x \in \mathcal{X}$. It follows from Theorem 4.1 that there exits a unique additive function $A : \mathcal{X} \to Y$ such that

$$\|g_o(x) - A(x)\|_{\mathcal{Y}}^p \leq \frac{K^p}{2^{p\beta}} \left(\frac{K^a}{3^\beta} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{1}{3^{ij}} \left[\Psi_O \left(3^{ij} x\right) + \Psi_O \left(-3^{ij} x\right) \right] \right)^p \tag{32}$$

for all $x \in \mathcal{X}$. Now, define a function $g(x)$ by

$$g(x) = g_e(x) + g_o(x), \tag{33}$$

for all $x \in \mathcal{X}$. Then it follows from (31), (32) and (33), we arrive our desired result. □

Corollary 5.2. *Let σ and α be nonnegative real numbers. If a function $g : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|Dg(x, y, z)\|_{\mathcal{Y}} \leq \begin{cases} \sigma, & \\ \sigma \left(||x||^\alpha + ||y||^\alpha + ||z||^\alpha\right), & s \neq 1, 2; \\ \sigma ||x||^\alpha ||y||^\alpha ||z||^\alpha, & s \neq \frac{1}{3}, \frac{2}{3}; \\ \sigma \left\{||x||^\alpha + ||y||^\alpha + ||z||^\alpha + \left(||x||^{3s} ||y||^{3s} ||z||^{3s}\right)\right\}, & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{34}$$

for all x, y, z in $\mathcal{X}$. Then there exists a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ and a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g(x) - A(x) - Q(x)\|_{\mathcal{Y}} \leq \begin{cases} \frac{K^{p+1}}{2^{p\beta}} \left\{ \left(\frac{32\sigma\ K^{a-1}}{16^\beta |15|} \right)^p + \left(\frac{12\sigma\ K^a}{3^\beta |2|} \right)^p \right\} \\ \frac{K^{p+1}}{2^{p\beta}} \left\{ \left(\frac{32\sigma\ K^{a-1}}{16^\beta} \left(2 + \frac{1}{3^{\beta\alpha}}\right) \frac{||x||^\alpha}{|16 - 4^{\beta\alpha}|} \right)^p + \left(\frac{2\sigma\ K^a}{3^\beta} \left(\frac{5}{2^{\beta\alpha}} + \frac{1}{6^{\beta\alpha}} \right) \frac{||x||^\alpha}{|3 - 3^{\beta\alpha}|} \right)^p \right\} \\ \frac{K^{p+1}}{2^{p\beta}} \left\{ \left(\frac{32\sigma\ K^{a-1}}{16^\beta} \left(\frac{1}{3^{\beta\alpha}} \right) \frac{||x||^{3\alpha}}{|16 - 4^{3\beta\alpha}|} \right)^p + \left(\frac{2\sigma\ K^a}{3^\beta} \left(\frac{1}{2^{3\beta\alpha}} + \frac{1}{2^{2\beta\alpha} \cdot 6^{\beta\alpha}} \right) \frac{||x||^{3\alpha}}{|3 - 3^{3\alpha}|} \right)^p \right\} \\ \frac{K^{p+1}}{2^{p\beta}} \left\{ \left(\frac{32\sigma\ K^{a-1}}{16^\beta} \left(1 + \frac{1}{3^{3\beta\alpha}}\right) \frac{||x||^{3\alpha}}{|16 - 4^{3\beta\alpha}|} \right)^p + \left(\frac{2\sigma\ K^a}{3^\beta} \left(\frac{6}{2^{3\beta\alpha}} + \frac{1}{6^{3\beta\alpha}} \left(1 + \frac{1}{2^{2\beta\alpha}}\right) \right) \frac{||x||^{3\alpha}}{|3 - 3^{3\alpha}|} \right)^p \right\} \end{cases} \tag{35}$$

for all $x \in \mathcal{X}$.

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