

Formulation of Finite Difference Method Algorithm for solution of One Dimensional Systems

Pallavi P. Chopade^{1,*} and Prabha S. Rastogi¹

¹ Department of Mathematics, Shri J.J.T. University, Jhunjhunu, Rajasthan, India.

Abstract: In this paper, algorithm for effective numerical method (finite difference method) is designed for the solution of one dimensional systems (Laplace equation). Finite Difference method with Dirichlet boundary conditions is used to obtain the solutions of differential equations. We have solved one dimensional Laplace equation with analytical and numerical approach. These approaches are developed in MATLAB and their solutions are compared and verified for different step size.

Keywords: Differential equation, Dirichlet boundary condition, Finite Difference method, Laplace equation, Poisson equation.

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1. Introduction

Differential equations are used to model different physical quantities, widely used in many aspects of engineering disciplines such as electric fields, pressure, stress, displacement, etc. They become the heart and soul of every numerical in applied mathematics as numerous problems' solutions is calculated by it. Additionally, the usage of PDEs has been exponentially increased fields of electronics engineering, computer science, chemistry, economics, etc. The formulation of problems having functions of more than one independent variables are done by Partial differential equations (PDEs). The order of the highest derivative defines the order of the equation. The three-dimensional Laplace equation for a function $V(x, y, z)$, describing steady-state temperature distribution in the absence of heat sources and sinks in heat and mass transfer theory or describing the electrostatic potential when unpaired electric charge is absent is given as [1, 3],

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \text{ or } \nabla^2 V = 0 \quad (1)$$

where x, y and z are the independent variables, V is the unknown function, and ∇^2 is the Laplacian operator. The solution of equation to be found is supplemented by initial and/or boundary conditions. The Poisson equation is the non-homogeneous form of the Laplace equation given by,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = F(x, y, z) = -\frac{\rho}{\epsilon_0} \text{ or } \nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (2)$$

where ϵ_0 is the vacuum permittivity, ρ is the charge density [6]. The analytical solution of partial differential equation is complex and difficult. For problem involving complex shapes, boundary conditions, material properties, etc. we adhere to

* E-mail: pallavichopade@gmail.com (Research Scholar)

the numerical methods that provide approximate but acceptable values of unknown quantities only at discrete number of points in the structure. Instead of solving the problem for entire structure in one operation, the solutions are formulated on each constituent unit and thereby combined to obtain the solution for the original structure. In this paper, focus on solution of one dimensional differential equation to understand the concept of finite difference method in engineering field is done.

2. Finite Difference Solution to One Dimensional Differential Equation

Consider the variation of potential in one of the three independent variables. The Laplace's equation becomes an ordinary derivative and reduces to one dimensional as,

$$\frac{d^2V}{dx^2} = 0, \text{ or } \frac{d^2V}{dy^2} = 0, \text{ or } \frac{d^2V}{dz^2} = 0 \quad (3)$$

with initial and final points as boundary conditions [4]. The solution for one dimensional differential equations can be obtained using analytical methods, but it becomes more cumbersome as the dimensions increases with different boundary conditions and complex shapes. So, to ease the process of solving complicated numerical, different numerical approaches are used. In this paper, the basic concept of FDM is studied so that we can apply it to the higher dimensions with different boundary conditions. This basic concept is applied to one dimensional differential equations along with the algorithm developed in MATLAB [2, 7]. The finite difference method is one of the powerful numerical method for solving partial differential equations. While applying this method, the problem is defined by partial differential equation, a solution region or boundary conditions. In FDM, the solution is divided into finite discrete points and the PDEs are replaced with a set of difference equations. Thus, the solutions obtained are approximate and not the exact solution. However, the error in the solution can be reduced to a tolerable level if very fine discretization is made. In FDM, the derivatives in differential equation are replaced with difference expression modifying the differential equation into finite difference equations [9, 10]. Using the central difference method, the representation for one dimensional system is given as [5, 8],

$$\frac{d^2V}{dx^2} = \frac{V_1 - 2V_0 + V_{-1}}{h^2} \quad (4)$$

The flowchart for FDM is shown in Figure 1.

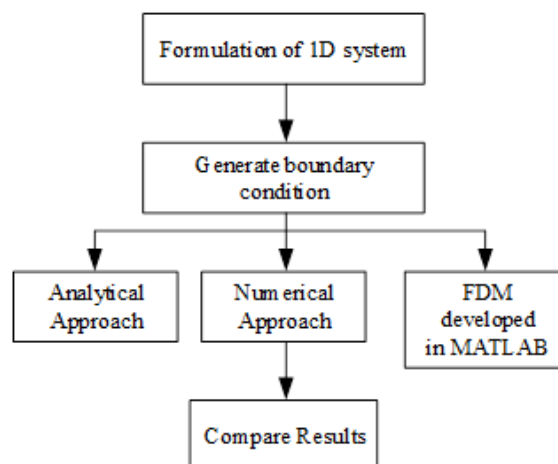


Figure 1. Flowchart of FDM

3. Finite Difference Solution To Electrostatic Problem Having Boundary Conditions

Here, two problem statements with different boundary conditions are evaluated using analytical approach, numerical approach (FDM) and FDM algorithm implemented in MATLAB.

Problem Statement 3.1. Find the potential distribution between two surfaces if $V(x = 0) = 0$ and $V(x = 1) = 5$. There is no charge distribution in the space $0 \leq x \leq 1$.

Solution.

(A). Using Analytical Approach: As the variation of potential is in one dimension, the Laplace's equation becomes an ordinary derivative and reduces to one dimensional as

$$\frac{d^2V}{dx^2} = 0 \quad (5)$$

On integrating Equation (5),

$$\frac{dV}{dx} = c_1 \quad (6)$$

Again, on integrating Equation (6), we get

$$V = c_1x + c_0 \quad (7)$$

where c_1 and c_0 are the constants to be determined from the given boundary conditions. On substituting the given boundary conditions, i.e., $x = 0, V = 0$ and $x = 1, V = 5$, the values of constants are,

$$c_0 = 0, c_1 = 5 \quad (8)$$

Hence, the analytical solution of the Equation (7) becomes,

$$V = 5x \quad (9)$$

The values of V can be obtained by substituting different values of x (step size).

(B). Finite Difference Method: Using the central difference method, the Laplace equation for the given boundary conditions and different step sizes $h = 0.5, 0.25, 0.125$ are solved. The differential equation for $h = 0.5$ becomes,

$$\left. \frac{d^2V}{dx^2} \right|_{x=0.5} = \frac{V_2 - 2V_1 + V_0}{(0.5)^2} = 0 \quad (10)$$

$$V_1 = 0.5(V_0 + V_2) \quad (11)$$

where, $V_0 = 0$ and $V_2 = 5$ (boundary condition). Therefore $V_1 = 2.5$. The second iteration with decreasing step size of $h = 0.25$, is applied to five points in the same interval, the boundary conditions are $V_0 = 0$ and $V_4 = 5$.

$$V_1 = 0.5(V_0 + V_2),$$

$$V_2 = 0.5(V_1 + V_3),$$

$$V_3 = 0.5(V_2 + V_4)$$

On solving these equations, the solutions are shown in Table 1.

x	0	0.25	0.5	0.75	1
v	0	1.25	2.5	3.75	5

Table 1. Solution for step size $h = 0.25$

The third iteration with step size, $h = 0.125$ gives nine points in same interval having boundary conditions $V_0 = 0$ and $V_9 = 5$.

$V_1 = 0.5(V_0 + V_2),$	$V_5 = 0.5(V_4 + V_6),$
$V_2 = 0.5(V_1 + V_3),$	$V_6 = 0.5(V_5 + V_7),$
$V_3 = 0.5(V_2 + V_4),$	$V_7 = 0.5(V_6 + V_8),$
$V_4 = 0.5(V_3 + V_5),$	$V_8 = 0.5(V_7 + V_9),$

On solving these equations, the values of V are shown in Table 2.

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
v	0	0.625	1.25	1.875	2.5	3.125	3.75	4.375	5

Table 2. Solution for step size $h = 0.125$

(C). FDM Algorithm in MATLAB: The FDM algorithm explained below is implemented in MATLAB and the results obtained are shown in Table III and graphically plotted in Figure 2.

FDM Algorithm

Step 1: Specify array size and boundary conditions

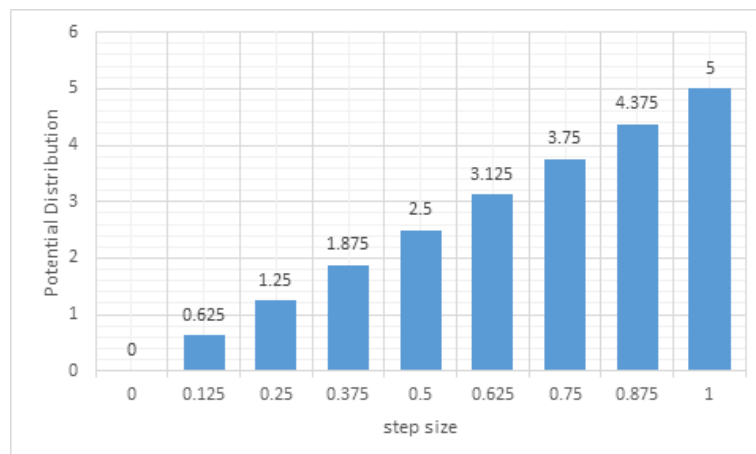
Step 2: Calculate step size and specify charge density, if needed.

Step 3: Specify boundary conditions.

Step 4: Use central difference method to calculate potential distribution and iterate it until all the values are achieved.

Step 5: Plot the results.

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
v	0	0.625	1.25	1.875	2.5	3.125	3.75	4.375	5

Table 3. Results FDM algorithm**Figure 2.** Results FDM algorithm

By decreasing the step size ($h = 0.0625$), the FDM approach results and the analytical solution of the given problem statement are compared and shown in Table IV and graphically plotted in Figure 3.

Sr. No.	Step Size	Numerical Approach	Analytical Approach
1	0	0	0
2	0.0625	0.3125	0.3125
3	0.125	0.625	0.625
4	0.1875	0.9375	0.9375
5	0.25	1.25	1.25
6	0.3125	1.5625	1.5625
7	0.375	1.875	1.875
8	0.4375	2.1875	2.1875
9	0.5	2.5	2.5
10	0.5625	2.8125	2.8125
11	0.625	3.125	3.125
12	0.6875	3.4375	3.4375
13	0.75	3.75	3.75
14	0.8125	4.0625	4.0625
15	0.875	4.375	4.375
16	0.9375	4.6875	4.6875
17	1	5	5

Table 4. Comparison of Numerical Approach and Analytical Approach

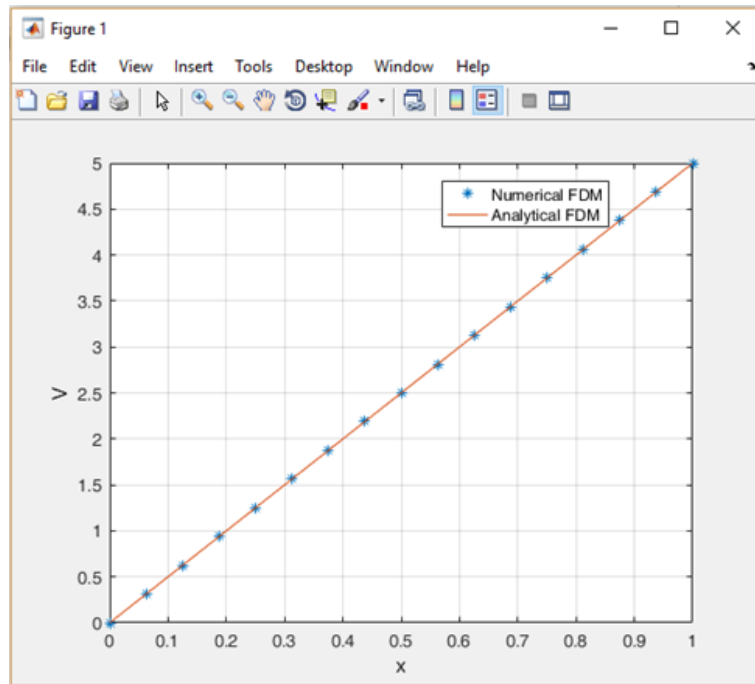


Figure 3. Comparison of Numerical Approach and Analytical Approach in MATLAB(Problem Statement 2.1)

□

Problem Statement 3.2. Find the potential distribution between two surfaces for different boundary conditions i.e., $V(x = 1) = 0$ and $V(x = 6) = 4$.

Solution.

(A). Analytical Solution: The analytical solution for this problem with boundary condition is given as,

$$V(x) = 0.8(x - 1) \quad (12)$$

(B). FDM Algorithm MATLAB: The FDM solution developed in MATLAB results in the graph shown in Figure 4.

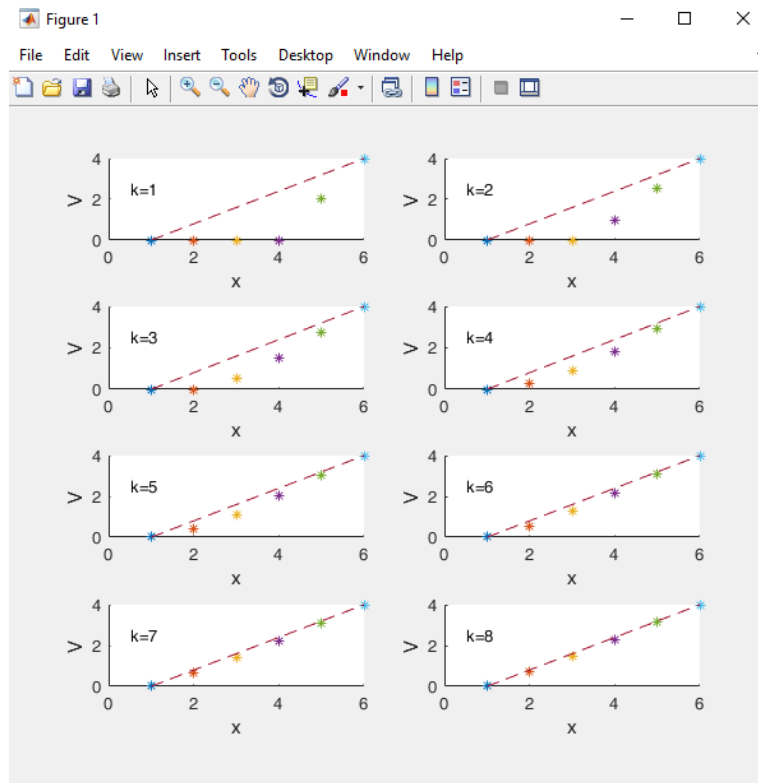


Figure 4. Comparison of Numerical Approach and Analytical Approach in MATLAB (Problem Statement 2.2)

From Figure 4, it is observed the error is more prominent for first 4 iterations and reduces as the iteration increases. The solution using both methods are approximately same at 8th iteration suggesting to halt the further iterations. Thus, the accuracy of solution is substantially good.

□

4. Conclusion

This paper presents understanding of Finite Difference Method to solve one dimensional differential equation system for electrostatic field of engineering. For simple boundary conditions, the solutions of both methods are similar, indicating infinitesimal error for smaller number of iterations, but as the boundary conditions become more complex, with increase in iterations both solutions converge to similar results and reduction in error within acceptable level is observed. For further research, the approximations of these methods can be improved and executed on software packages as applied to two dimensions and three dimensions numerical with complex boundary conditions.

References

- [1] A. K. Mitra, *Finite difference method for the solution of Laplace equation*, Department of aerospace engineering Iowa state University, (2010).
- [2] W. Y. Yang, W. Cao, T. S. Chung and J. Morris, *Applied numerical methods using MATLAB*, John Wiley & Sons, (2005).

- [3] G. Papanikos and M. C. Gousidou-Koutita, *A Computational Study with Finite Element Method and Finite Difference Method for 2D Elliptic Partial Differential Equations*, Applied Mathematics, 6(12)(2015).
- [4] W. H. Hayt and J. A. Buck, *Engineering electromagnetics*, Vol. 6, McGraw-Hill, New York, (1981).
- [5] N. Rao, *Fundamentals of Electromagnetics for Engineering*, Pearson Education India, (2008).
- [6] T. U. Chaudhari and D. M. Patel, *Finite element solution of Poisson's equation in a homogeneous medium*, International Research Journal of Engineering and Technology, 2(9)(2015), 674-679.
- [7] M. N. Sadiku, *Numerical techniques in electromagnetics with MATLAB*, CRC press, (2011).
- [8] G. Moradi and M. Marvasti, *Experimental Solution to the Laplace Equation, a Tutorial Approach*, IJARCCE, 5(9)(2016), 278-284.
- [9] S. C. Chapra and R. P. Canale, *Numerical methods for engineers*, Vol. 2, McGraw-Hill, New York, (1998).
- [10] M. K. Jain, *Numerical methods for scientific and engineering computation*, New Age International, (2003).