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Radius Problem for Subclasses of Harmonic Univalent Functions

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Abstract: Let $f = h + \bar{g}$ be harmonic functions in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$. In this paper we find the radius of the Goodman-Ronning type starlikeness and convexity of $D_f^{\varepsilon} = zf_z - \varepsilon\bar{z}f_{\bar{z}}$ $(|\varepsilon| = 1)$, when the coefficients of h and g satisfy the harmonic Bieberbach coefficients conjecture conditions.

Keywords: Univalent functions, Harmonic functions, Goodman-Ronning type functions.

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1. Introduction

Let H denote the class of all complex-valued harmonic functions f in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$. The harmonic function f has a unique representation $f = h + \bar{g}$, where h and g are analytic and co-analytic parts of f given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n; \; g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

The Jacobian of $f = h + \bar{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. The map f is sense preserving and locally one to one in D if and only if $J_f(z) > 0$ in D (see [3]). We now recall a few basic subclasses of harmonic functions (see [3, 5]). Let H_{SP} be the subclass of H, consisting of sense preserving functions. For the functions in this class, $|b_1| < 1$. Let H^0 denote the subclass of H with $b_1 = 0$. The class $H_{SP}^0 = H_{SP} \cap H^0$ contains all sense preserving harmonic functions with $b_1 = 0$. Let S_H and S_H^0 be the subclasses of H_{SP} and H_{SP}^0 respectively containing univalent harmonic functions and S_H^*, K_H and C_H be the subclasses of S_H mapping D on to star-like, convex and close-to-convex domains, respectively. Let K_H^{*0}, S_H^{*0} denote the subclasses of S_H^0 consisting of functions mapping D on to convex and star-like domains respectively. In 1984, Clunie and Sheil-Small [3], conjectured that if $f \in S_H^0$ then the Taylor coefficients of the functions h and g satisfy the inequality

$$|a_n| \le \frac{1}{6}(2n+1)(n+1) \quad |b_n| \le \frac{1}{6}(2n-1)(n-1) \text{ for all } \; n \ge 2. \tag{2}$$

Equality holds for the harmonic Koebe function defined by

$$K(z) = H_1(z) + \overline{G_1(z)} = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \frac{\overline{\frac{1}{2}z^2 + \frac{1}{6}z^3}}{(1-z)^3}. \tag{3}$$

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In [3], if $f = h + \overline{g} \in C_H^0$, then

$$|a_n| \leq \frac{n+1}{2} \text{ and } |b_n| \leq \frac{n-1}{2}, \tag{4}$$

holds for all $n \geq 2$. Equality holds for the harmonic right half-plane mapping $L \in C_H^0$ given by

$$L(z) = H_2(z) + \overline{G_2(z)} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \overline{\frac{-\frac{1}{2}z^2}{(1-z)^2}}. \tag{5}$$

In [1, 6, 7], Thomas Rosy introduced and examined two new subclass of harmonic univalent functions of Goodman-Ronning type.

Definition 1.1 ([6]). *A function $f \in H$ is said to be in the subclass $G_H(\alpha)$ if Re $\left\{(1+e^{i\gamma})\frac{zf'(z)}{z'f(z)} - e^{i\gamma}\right\} > \alpha$, $0 \leq \alpha < 1$. Here $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z)) = \frac{\partial}{\partial\theta}f(re^{i\theta})$, $0 \leq r < 1$, $\theta \in \mathbb{R}$.*

Lemma 1.2 ([6]). *Let $f = h + \overline{g}$, where h and g are given by (1), and let*

$$\sum_{n=1}^{\infty} \left[\frac{2n-1-\alpha}{1-\alpha}|a_n| + \frac{2n+1+\alpha}{1-\alpha}|b_n|\right] \leq 2 \tag{6}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$, then f is harmonic univalent in D and $f \in G_H(\alpha)$.

Definition 1.3 ([7]). *A function $f \in H$ is said to be in the subclass $GK_H(\alpha)$ if Re $\left\{1 + (1+e^{i\gamma})\frac{zf''(z)}{f'(z)} - e^{i\gamma}\right\} > \alpha$, $0 \leq \alpha < 1$. Here $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z)) = \frac{\partial}{\partial\theta}f(re^{i\theta})$, $0 \leq r < 1$, $\theta \in \mathbb{R}$.*

Lemma 1.4 ([7]). *Let $f = h + \overline{g}$, where h and g are given by (1) and let*

$$\sum_{n=1}^{\infty} \left[\frac{n(2n-1-\alpha)}{1-\alpha}|a_n| + \frac{n(2n+1+\alpha)}{1-\alpha}|b_n|\right] \leq 2 \tag{7}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$, then f is harmonic univalent in D and $f \in GK_H(\alpha)$.

2. Radius of the Goodman-Ronning type Starlikeness and Convexity of D_f^ε

The sharp radius of univalence, fully starlikeness and fully convexity of the harmonic linear differential operators $D_f^\varepsilon = zf_z - \varepsilon\bar{z}f_{\bar{z}}$ ($|\varepsilon| = 1$) are obtained for $f = h + \overline{g}$, where h and g are given by (1) and $F_\lambda(z) = (1-\lambda)f + \lambda D_f^\varepsilon$ ($0 \leq \lambda \leq 1$) when the coefficients of h and g satisfy the harmonic Bieberbach coefficients conjecture conditions (see [2,4]). We consider the function from the classes $G_H(\alpha)$, $GK_H(\alpha)$ and obtain the radius of starlikeness and convexity of order α for the harmonic linear differential operator D_f^ε. We now recall certain standard sums,

(a). $\sum\limits_{n=2}^{\infty} nr^{n-1} = \frac{r(2-r)}{(1-r)^2}$.

(b). $\sum\limits_{n=2}^{\infty} n^2r^{n-1} = \frac{r(4-3r+r^2)}{(1-r)^3}$.

(c). $\sum\limits_{n=2}^{\infty} n^3r^{n-1} = \frac{r(8-5r+4r^2-r^3)}{(1-r)^4}$.

(d). $\sum\limits_{n=2}^{\infty} n^4r^{n-1} = \frac{r(16+r+11r^2-5r^3+r^4)}{(1-r)^5}$.

(e). $\sum\limits_{n=2}^{\infty} n^5r^{n-1} = \frac{r(32+51r+46r^2-14r^3+6r^4-r^5)}{(1-r)^5}$.

For $0 < r < 1$ and $f = h + \bar{g} \in H$, the operator $D_f^{\varepsilon,r}$ is defined by

$$D_f^{\varepsilon,r}(z) = \frac{D_f^{\varepsilon}(rz)}{r} = z + \sum_{n=2}^{\infty} n a_n r^{n-1} z^n - \varepsilon \overline{\sum_{n=2}^{\infty} n b_n r^{n-1} z^n}, \ |\varepsilon| = 1. \tag{8}$$

Theorem 2.1. *Let $f = h + \bar{g}$, where h and g are given by (1) and the coefficients satisfy the condition (2) for $n \geq 2$. Then,*

(1). $D_f^{\varepsilon,r} \in G_H(\alpha)$ for $r \leq r_1(\alpha)$, where $r_1(\alpha)$ is the unique real root of the equation

$$(1-\alpha) - (25 - 9\alpha)r + (5 - 21\alpha)r^2 - 21(1-\alpha)r^3 + 10(1-\alpha)r^4 - 2(1-\alpha)r^5 = 0 \tag{9}$$

in the interval (0,1).

(2). $D_f^{\varepsilon,r} \in G_H(0)$ for $r \leq r_1(0)$, where $r_1(0) \approx 0.040270$ is the unique real root of the equation

$$1 - 25r + 5r^2 - 21r^3 + 10r^4 - 2r^5 = 0 \tag{10}$$

in the interval (0,1).

Proof. Let $0 < r < 1$, it suffices to show that $D_f^{\varepsilon,r} \in G_H(\alpha)$ where $D_f^{\varepsilon,r}$ is defined as in (8). Consider,

$$S_1 = \sum_{n=2}^{\infty} \frac{2n-1-\alpha}{1-\alpha} |na_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{2n+1+\alpha}{1-\alpha} |nb_n| r^{n-1}.$$

Using the coefficient bounds (2),

$$\begin{aligned} S_1 &\leq \sum_{n=2}^{\infty} \frac{(2n-1-\alpha)}{1-\alpha} \left(\frac{n(2n+1)(n+1)}{6} \right) r^{n-1} + \sum_{n=2}^{\infty} \frac{(2n+1+\alpha)}{1-\alpha} \left(\frac{n(2n-1)(n-1)}{6} \right) r^{n-1} \\ &= \frac{1}{6(1-\alpha)} \left[\sum_{n=2}^{\infty} n(2n-1-\alpha)(2n+1)(n+1) r^{n-1} + \sum_{n=2}^{\infty} n(2n+1+\alpha)(2n-1)(n-1) r^{n-1} \right] \\ &= \frac{1}{3(1-\alpha)} \sum_{n=2}^{\infty} n(4n^3 - n - 3\alpha n) r^{n-1} \\ &= \frac{1}{3(1-\alpha)} \left[\sum_{n=2}^{\infty} 4n^4 r^{n-1} - (1+3\alpha) \sum_{n=2}^{\infty} n^2 r^{n-1} \right]. \end{aligned}$$

Now, using the standard sums,

$$S_1 = \frac{1}{3(1-\alpha)} \left[\frac{4r(16 + r + 11r^2 - 5r^3 + r^4)}{(1-r)^5} - (1+3\alpha) \frac{r(4 - 3r + r^2)}{(1-r)^3} \right].$$

By Lemma 1.1, we need to show that $S_1 \leq 1$, this is equivalent to

$$4r(16 + r + 11r^2 - 5r^3 + r^4) - (1+3\alpha) r(4 - 3r + r^2)(1-r)^2 \leq 3(1-\alpha)(1-r)^5.$$

Let $p_\alpha(r) = (1-\alpha) - (25 - 9\alpha)r + (5 - 21\alpha)r^2 - 21(1-\alpha)r^3 + 10(1-\alpha)r^4 - 2(1-\alpha)r^5 \geq 0$. Since $p_\alpha(0) = 1 - \alpha > 0$ as $0 \leq \alpha < 1$ and $p_\alpha(1) = -33$, $p_\alpha(r)$ has at least one zero in the interval (0, 1).

$$\begin{aligned} p'_\alpha(r) &= 9\alpha - 25 + 2(5 - 21\alpha)r - 63(1-\alpha)r^2 + 40(1-\alpha)r^3 - 10(1-\alpha)r^4 \\ &= 9\alpha - 25 + 2(5 - 21\alpha)r - (1-\alpha)r^2[23 + 10(r-2)^2] \end{aligned}$$

$$< 9\alpha - 25 + 2(5 - 21\alpha)r - 33(1-\alpha)r^2 = q_\alpha(r).$$

Now, it is enough to show that the polynomial $q_\alpha(r)$ is negative in the interval $r \in (0,1)$ for every $\alpha \in [0,1)$ since $q'_\alpha(r) = 2(5-21\alpha) - 66(1-\alpha)r$ and the critical point for the equation is $r_0 = \frac{5-21\alpha}{33(1-\alpha)}$, $q'_\alpha(r) > 0$ for $0 \le r < r_0$ and $q'_\alpha(r) < 0$ for $r_0 < r < 1$. We have two cases: $0 \le \alpha \le 5/21$ and $5/21 < \alpha < 1$.

Case 1: When $0 \le \alpha \le 5/21$, $q_\alpha(r_0) = \frac{16(9\alpha^2+57\alpha-50)}{33(1-\alpha)} < 0$ in the interval $r \in (0,1)$.

Case 2: When $5/21 < \alpha < 1$, $q_\alpha(0) = 9\alpha - 25 < 0$ since $0 \le \alpha < 1$ and $q_\alpha(1) = -48$, $q_\alpha(r) < 0$ in the interval $r \in (0,1)$ for all $5/21 < \alpha < 1$.

Combining the two cases, we conclude that $q_\alpha(r) < 0$ in the interval $r \in (0,1)$ and for all $\alpha \in [0,1)$. This proves that $p'_\alpha(r) < q_\alpha(r) < 0$ in the interval $r \in (0,1)$ and for all $\alpha \in [0,1)$. Thus $D_f^{\varepsilon,r} \in G_H(\alpha)$ for $r \le r_1(\alpha)$ where $r_1(\alpha)$ is the unique real root of the Equation (9). Also the roots of the Equation (9) are decreasing as a function of $\alpha \in [0,1)$. When $\alpha = 0$, Equation (9) reduces to $1 - 25r + 5r^2 - 21r^3 + 10r^4 - 2r^5 = 0$, consequently $r_1(\alpha) \le r_1(0)$, where $r_1(0) \approx 0.040270$ is the unique real root of the Equation (10) in the interval (0, 1). $\square$

Theorem 2.2. *Let $f = h + \bar{g}$, where h and g are given by (1), and the coefficients satisfy the condition (2) for $n \ge 2$. Then,*

(1). $D_f^{\varepsilon,r} \in GK_H(\alpha)$ for $r \le r_2(\alpha)$ where $r_2(\alpha)$ is the unique real root of the equation

$$3(1-\alpha) + (42\alpha - 138)r - 180(1+\alpha)r^2 + (126\alpha - 222)r^3 + 87(1-\alpha)r^4 - 36(1-\alpha)r^5 + 6(1-\alpha)r^6 = 0 \tag{11}$$

in the interval (0,1).

(2). $D_f^{\varepsilon,r} \in GK_H(\alpha)$ for $r \le r_2(0)$ where $r_2 \approx 0.021141082$ is the unique real root of the equation

$$3 - 138r - 180r^2 - 222r^3 + 87r^4 - 36r^5 + 6r^6 = 0 \tag{12}$$

in the interval (0,1).

Proof. Consider,

$$S_2 = \sum_{n=2}^{\infty} \frac{n(2n-1-\alpha)}{1-\alpha} |na_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{n(2n+1+\alpha)}{1-\alpha} |nb_n| r^{n-1}.$$

Using the coefficient bounds (2),

$$\begin{aligned}
S_2 &\le \sum_{n=2}^{\infty} \frac{n(2n-1-\alpha)}{1-\alpha} \left(\frac{n(2n+1)(n+1)}{6}\right) r^{n-1} + \sum_{n=2}^{\infty} \frac{n(2n+1+\alpha)}{1-\alpha} \left(\frac{n(2n-1)(n-1)}{6}\right) r^{n-1} \\
&= \frac{1}{6(1-\alpha)} \left[\sum_{n=2}^{\infty} n^2(2n-1-\alpha)(2n+1)(n+1)r^{n-1} + \sum_{n=2}^{\infty} n^2(2n+1+\alpha)(2n-1)(n-1)r^{n-1}\right] \\
&= \frac{1}{3(1-\alpha)} \sum_{n=2}^{\infty} 4n^5 - (3\alpha+1)n^3)r^{n-1} \\
&= \frac{1}{3(1-\alpha)} \left[\sum_{n=2}^{\infty} 4n^5 r^{n-1} - (1+3\alpha) \sum_{n=2}^{\infty} n^3 r^{n-1}\right].
\end{aligned}$$

Using standard sums,

$$= \frac{1}{3(1-\alpha)} \left[\frac{4r(32 + 51r + 46r^2 - 14r^3 + 6r^4 - r^5)}{(1-r)^6} - (1+3\alpha)\frac{r(8 - 5r + 4r^2 - r^3)}{(1-r)^4}\right].$$

By Lemma 1.4 we need to show that $S_2 \leq 1$ or r has to satisfy the inequality

$$4r(32+51r+46r^2-14r^3+6r^4-r^5)-(1+3\alpha)r(8-5r+4r^2-r^3)(1-r)^2 \leq 3(1-\alpha)(1-r)^6$$

Let $t_\alpha(r) = 3(1-\alpha)+(42\alpha-138)r-180(1+\alpha)r^2+(126\alpha-222)r^3+87(1-\alpha)r^4-36(1-\alpha)r^5+6(1-\alpha)r^6 \geq 0$. That is $t_\alpha(r) \geq 0$. Here $t_\alpha(0) \geq (1-\alpha) > 0$ and $t_\alpha(1) = -160 < 0$ so $t_\alpha(r)$ has at least one root in the interval (0, 1). By similar arguments in Theorem 2.3 we conclude that $D_f^{\varepsilon,r} \in GK_H(\alpha)$ for $r \leq r_2(\alpha)$, where $r_2(\alpha)$ is the unique real root of the Equation (11). Also the roots of the Equation (11), are decreasing as a function of $\alpha \in [0,1)$. When $\alpha = 0$, (11) reduces to the equation $3-138r-180r^2-222r^3+87r^4-36r^5+6r^6 = 0$. Consequently, $r_2(\alpha) \leq r_2(0)$, where $r_2(0) \approx 0.021141$ is the unique real root of the Equation (12) in the interval (0, 1). $\square$

Theorem 2.3. *Let $f = h+\bar{g}$ where h and g are given by (1), and the coefficients satisfy the condition (4) for $n \geq 2$. Then,*

(1). $D_f^{\varepsilon,r} \in G_H(\alpha)$ for $r \leq R_1(\alpha)$ where $R_1(\alpha)$ is the unique real root of the equation

$$s_\alpha(r) = (1-\alpha)-6(3-\alpha)r+11(1-\alpha)r^2-8(1-\alpha)r^3+2(1-\alpha)r^4 \tag{13}$$

in the interval (0,1).

(2). $D_f^{\varepsilon,r} \in G_H(\alpha)$ for $r \leq R_1(0)$ where $R_1 \approx 0.057492$ is the unique real root of the equation

$$1-18r+11r^2-8r^3+2r^4 = 0 \tag{14}$$

in the interval (0,1).

Proof. Let $0<r<1$ it suffices to show that $D_f^{\varepsilon,r} \in G_H(\alpha)$, where $D_f^{\varepsilon,r}$ is defined as in (8). Consider,

$$S_3 = \sum_{n=2}^{\infty} \frac{2n-1-\alpha}{1-\alpha}|na_n|r^{n-1} + \sum_{n=2}^{\infty} \frac{2n+1+\alpha}{1-\alpha}|nb_n|r^{n-1}.$$

Using the coefficient bounds (4),

$$\begin{aligned} S_3 &\leq \sum_{n=2}^{\infty} \frac{(2n-1-\alpha)}{1-\alpha}\left(\frac{n(n+1)}{2}\right)r^{n-1} + \sum_{n=2}^{\infty} \frac{(2n+1+\alpha)}{1-\alpha}\left(\frac{n(n-1)}{2}\right)r^{n-1} \\ &= \frac{1}{2(1-\alpha)}\left[\sum_{n=2}^{\infty} n(2n-1-\alpha)(n+1)r^{n-1} + \sum_{n=2}^{\infty} n(2n+1+\alpha)(n-1)r^{n-1}\right] \\ &= \frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty}(4n^3-2n(1+\alpha))r^{n-1} \\ &= \frac{1}{(1-\alpha)}\left[2\sum_{n=2}^{\infty} n^3r^{n-1} - (1+\alpha)\sum_{n=2}^{\infty} nr^{n-1}\right]. \end{aligned}$$

Using the standard sums we have

$$\frac{1}{(1-\alpha)}\left[\frac{2r(8-5r+4r^2-r^3)}{(1-r)^4} - (1+\alpha)\frac{r(2-r)}{(1-r)^2}\right].$$

By Lemma 1.2, it is sufficient to prove that $S_3 \leq 1$ which is equivalent to $s_\alpha(r) \geq 0$ where $s_\alpha(r)$ is given by (13) also $s_\alpha(0) = 1-\alpha > 0$ and $s_\alpha(1) = -12 < 0$ and so $s_\alpha(r)$ has at least one zero in the interval (0, 1) since $s'_\alpha(r) = -2\left\{3(3-\alpha)+(1-\alpha)r\left[4\left(r-(3/2)\right)^2+2\right]\right\} < 0$ we have $s_\alpha(r)$ is strictly decreasing in (0, 1) and hence has exactly one root in (0, 1) for all $\alpha \in [0,1)$. Thus $D_f^{\varepsilon,r} \in G_H(\alpha)$ for $r \leq R_1(\alpha)$, where $R_1(\alpha)$ is the unique real root of the Equation (13) in the interval (0, 1). Also the roots of the Equation (13) are decreasing as a function of $\alpha \in [0,1)$. Consequently, $R_1(\alpha) \leq R_1(0)$ where $R_1(0) \approx 0.057492$ is the unique real root of the equation (14) in the interval (0,1). $\square$

Theorem 2.4. *Let $f = h + \bar{g}$ where h and g are given by (1), and the coefficients satisfy the condition (4) for $n \geq 2$. Then,*

(1). $D_f^{\varepsilon,r} \in GK_H(\alpha)$ for $r \leq R_2(\alpha)$ where $R_2(\alpha)$ is the unique real root of the equation

$$u_\alpha(r) = (1-\alpha) - 3(11-3\alpha)r - 3(1+7\alpha)r^2 - 21(1-\alpha)r^3 + 10(1-\alpha)r^4 - 2(1-\alpha)r^5 \tag{15}$$

in the interval (0, 1).

(2). $D_f^{\varepsilon,r} \in GK_H(\alpha)$ for $r \leq R_2(0)$ where $R_2 \approx 0.030202$ is the unique real root of the equation

$$1 - 33r - 3r^2 - 21r^3 + 10r^4 - 2r^5 = 0 \tag{16}$$

in the interval (0,1).

Proof. Consider

$$S_4 = \sum_{n=2}^{\infty} \frac{n(2n-1-\alpha)}{1-\alpha} |na_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{n(2n+1+\alpha)}{1-\alpha} |nb_n| r^{n-1}.$$

Using the coefficient bound (4),

$$\begin{aligned} S_4 &\leq \sum_{n=2}^{\infty} \frac{n(2n-1-\alpha)}{1-\alpha} \left(\frac{n(n+1)}{2}\right) r^{n-1} + \sum_{n=2}^{\infty} \frac{n(2n+1+\alpha)}{1-\alpha} \left(\frac{n(n-1)}{2}\right) r^{n-1} \\ &= \frac{1}{2(1-\alpha)} \left[\sum_{n=2}^{\infty} n^2(2n-1-\alpha)(n+1)r^{n-1} + \sum_{n=2}^{\infty} n^2(2n+1+\alpha)(n-1)r^{n-1}\right] \\ &= \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} (4n^4 - 2(1+\alpha)n^2) r^{n-1} \\ &= \frac{1}{(1-\alpha)} \left[2\sum_{n=2}^{\infty} n^4 r^{n-1} - (1+\alpha)\sum_{n=2}^{\infty} n^2 r^{n-1}\right]. \end{aligned}$$

Using the standard sums, we have

$$= \frac{1}{(1-\alpha)} \left[\frac{2r(16 + r + 11r^2 - 5r^3 + r^4)}{(1-r)^5} - (1+\alpha)\frac{r(4-3r+r^2)}{(1-r)^3}\right].$$

By Lemma 1.4, it is sufficient to prove that $S_4 \leq 1$ which is equivalent to $u_\alpha(r) \geq 0$ where $u_\alpha(r)$ is given by (14) also $u_\alpha(0) = 1-\alpha > 0$ and $u_\alpha(1) = -48 < 0$ and so $u_\alpha(r)$ has at least one zero in the interval (0,1). By similar arguments in Theorem 2.4 we conclude that $D_f^{\varepsilon,r} \in GK_H(\alpha)$ for $r \leq R_2(\alpha)$ where $R_2(\alpha)$ is the unique real root of the Equation (15). Also the roots of the Equation (15) are decreasing as a function of $\alpha \in [0, 1)$. Consequently, $R_2(\alpha) \leq R_2(0)$ where $R_2(0) \approx 0.030202$ is the unique real root of the equation (16) in the interval (0, 1). □

References

[1] B. Adolph Stephen, Thomas Rosy, K. G. Subramanian and H. Silverman , *Subclasses of Harmonic Starlike Functions*, Journal of Analysis, 9(2001), 99-108.

[2] S. V. Bharanedhar and S. Ponnusamy, *Coefficient conditions for harmonic univalent mappings and hypergeometric mappings*, Rocky Mountain J. Math., 44(3)(2014), 753-777.

[3] J. Clinie and T. Sheil Small, *Harmonic Univalent Functions*, Ann. Acad. Aci. Ser. A.I. Math., 9(1984), 3-25.

[4] Z. Liu and S. Ponnusamy, *Radius of fully starlikeness and fully convexity of harmonic linear differential operator*, Bull. Korean Math. Soc., 55(3)(2018), 819-835.

[5] Peter Duren, *Harmonic Mappings in The Plane*, Cambridge University Press, (2004).

[6] Thomas Rosy, B. Adolph Stephen and K. G. Subramanian, *Goodman-Ronning Type Harmonic Univalent Functions*, Kyungpook Mathematical Journal, 41(2001), 45-54.

[7] Thomas Rosy, B. Adolph Stephen and K. G. Subramanian and J. M. Jahangiri, *Goodman-type Harmonic Convex Functions*, Journal of Natural Geometry, 21(2001), 39-50.