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Δ^m-lacunary Statistical Convergence of Order α Defined by a Modulus Function

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Abstract: In this paper, we introduce the concepts of lacunary Δ^m-statistically convergent sequence of order α and strongly $N_\theta^\alpha(\Delta^m, f, p)$-summable sequences of order α are introduced. Some inclusion relations between the set of lacunary Δ^m-statistically convergent sequences of order α and strongly $N_\theta^\alpha(\Delta^m, f, p)$-summable sequences of order α and the relations between two arbitrary lacunary sequences are discussed.

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1. Introduction

The notion of statistical convergence which is closely related to the concept of natural density or asymptotic density of a subset of the set of natural numbers $\mathbb{N}$ was first introduced by Fast [11] and Schoenberg [26] independently. From the point of view of sequence spaces, the concept of statistical convergence was generalized and developed by *Šalát* [24], Fridy [13], Connor [4] and many others. However, the order of statistical convergence for sequences of positive linear operators was introduced by Gadjiev and Connor [16] and it was generalized for the method of A-statistical convergence by Duman [6]. Later on, the concept of order of statistical convergence for sequences of numbers was generalized and developed Çolak [3], Bhunia [2], Et [7] and many others. Freedman [12] were introduced a related concept of convergence with the help of a lacunary sequence $\theta = (k_r)$ and called it as a lacunary statistical convergence. Later on, this concept was generalized by Fridy [13], Li [18] and many others.

The order of statistical convergence for sequences of positive linear operators was introduced by Gadjiev and Connor [16] and it was generalized for the method of A-statistical convergence by Duman et al. [6] and for sequences of numbers by Çolak [3]. Later on, for sequences of numbers, this concept was generalized and developed by Altınok [1], Bhunia [2], Et [7] and many others.

Kızmaz [17] was introduced the difference operator on the basic sequence spaces ℓ_∞, c and c_0. Later on, Et and Çolak [8] were introduced the m-th order difference operator on the basic sequence spaces. The concept of modulus function was introduced by Nakano [22]. Later on, Ruckle [23], Maddox [20, 21] and several authors have constructed various types of sequence spaces by using modulus function.

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The purpose of this paper is to introduce and study the space of lacunary strongly Δ^m-summable sequence of order α defined by a modulus function, lacunary Δ^m-statistically convergent sequences of order α and discuss some inclusion relations between the set of lacunary Δ^m-statistical convergence of order α and lacunary strongly Δ^m-summable sequences of order α.

2. Definitions and Preliminaries

Definition 2.1. *Let $f : [0,\infty) \to [0,\infty)$. Then f is called a modulus if*

(i). $f(x) = 0$ if and only if $x = 0$,

(ii). $f(x + y) \leq f(x) + f(y)$, for all $x > 0, y > 0$,

(iii). f is increasing,

(iv). f is continuous from the right at 0.

It is immediate from (ii) and (iv) that f is continuous everywhere on $[0,\infty)$.

Definition 2.2 ([13]). *A sequence $x = (x_k)$ is said to be statistically convergent to the number L, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences will be denoted by S.

Definition 2.3. *A sequence $x = (x_k)$ is said to be Δ^m-statistically convergent to the number L, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}| = 0.$$

The set of all Δ^m-statistically convergent sequences will be denoted by $S(\Delta^m)$.

Definition 2.4 ([3]). *Let $0 < \alpha \leq 1$ be given. The sequence $x = (x_k)$ is said to be statistically convergent of order α to the number L, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S^\alpha - \lim x_k = L$. The set of all statistically convergent sequences of order α will be denoted by S^α.

Definition 2.5. *Let $0 < \alpha \leq 1$ be given. The sequence $x = (x_k)$ is said to be Δ^m-statistically convergent of order α to the number L, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S^\alpha(\Delta^m) - \lim x_k = L$. The set of all Δ^m-statistically convergent sequences of order α will be denoted by $S^\alpha(\Delta^m)$.

Definition 2.6 ([14]). *By a lacunary sequence, we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this article, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.*

Definition 2.7 ([14]). *A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L, if for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

The set of all lacunary statistically convergent sequences will be denoted by S_θ.

Definition 2.8. *A sequence $x = (x_k)$ is said to be lacunary Δ^m-statistically convergent to the number L, if for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| = 0.$$

The set of all lacunary Δ^m-statistically convergent sequences will be denoted by $S_\theta(\Delta^m)$.

Definition 2.9 ([25]). *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k)$ is said to be lacunary statistically convergent of order α to the number L, if for every $\varepsilon > 0$*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denote the αth power $(h_r)^\alpha$ of h_r. That is, $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \ldots, h_r^\alpha, \ldots)$. In this case, we write $S_\theta^\alpha - \lim x_k = L$. The set of all lacunary statistically convergent sequences of order α will be denoted by S_θ^α.

Definition 2.10. *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k)$ is said to be lacunary Δ^m-statistically convergent of order α to the number L, if for every $\varepsilon > 0$*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S_\theta^\alpha(\Delta^m) - \lim x_k = L$. The set of all lacunary Δ^m-statistically convergent sequences of order α will be denoted by $S_\theta^\alpha(\Delta^m)$.

Definition 2.11 ([12]). *The lacunary sequence $\beta = (l_r)$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subseteq (l_r)$.*

Lemma 2.12 ([21]). *Let a_k, b_k for all k be sequences of complex numbers and (p_k) be a bounded sequence of positive real numbers, then*

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k})$$

and

$$|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$$

where $C = \max(1, 2^{H-1}), H = \sup p_k$ and λ is any complex number.

Lemma 2.13 ([21]). *Let $a_k \geq 0,\ b_k \geq 0$ for all k be sequences of complex numbers and $1 \leq p_k \leq \sup p_k < \infty$, then*

$$\Big(\sum_k |a_k + b_k|^{p_k}\Big)^{\frac{1}{M}} \leq \Big(\sum_k |a_k|^{p_k}\Big)^{\frac{1}{M}} + \Big(\sum_k |b_k|^{p_k}\Big)^{\frac{1}{M}}$$

where $M = \max(1, H),\ H = \sup p_k$.

Definition 2.14. *Let $\theta = (k_r)$ be a lacunary sequence, f be any modulus function, $\alpha \in (0, 1]$ be any real number and let $p = (p_k)$ be a sequence of strictly positive real numbers. A sequence $x = (x_k)$ is said to be lacunary strongly $N_\theta^\alpha(\Delta^m, f, p)$-summable of order α, if there is a real number L such that*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} (f(|\Delta^m x_k - L|))^{p_k} = 0.$$

In this case, we write $N_\theta^\alpha(\Delta^m, f, p) - \lim x_k = L$. The set of all lacunary strongly $N_\theta^\alpha(\Delta^m, f, p)$-summable sequences of order α will be denoted by $N_\theta^\alpha(\Delta^m, f, p)$.

3. Main Results

Theorem 3.1. *Let (p_k) be a bounded sequence of positive real numbers. Then the class $N_\theta^\alpha(\Delta^m, f, p)$ is a linear space over $\mathbb{C}$.*

Proof. Using Lemma 2.12, the subadditivity property of a modulus function f and the result $f(\lambda x) \leq (1 + [|\lambda|])f(x)$, it is easy to show that $N_\theta^\alpha(\Delta^m, f, p)$ is a linear space over $\mathbb{C}$. □

Theorem 3.2. *Let (p_k) be a bounded sequence of positive real numbers such that $\inf p_k > 0$ and let $\alpha_0 \in (0, 1]$ be fixed. Then the sequence space $N_\theta^{\alpha_0}(\Delta^m, f, p)$ is a complete metric space with respect to the metric*

$$h(X, Y) = \sum_{i=1}^{m} f(|x(i) - y(i)|) + \sup_r \left(\frac{1}{h_r^{\alpha_0}} \sum_{k \in I_r} (f(|\Delta^m x_k - \Delta^m y_k|))^{p_k} \right)^{\frac{1}{M}},$$

where $M = \max\{1, \sup p_k\}$.

Proof. Using Lemma 2.12, Lemma 2.13, the subadditivity property of a modulus function f and the result $f(\lambda x) \leq (1 + [|\lambda|])f(x)$, it is easy to show that $N_\theta^{\alpha_0}(\Delta^m, f, p)$ is a complete metric space with respect to the above metric. □

Theorem 3.3. *If $0 < \alpha < \beta \leq 1$, then $S_\theta^\alpha(\Delta^m) \subset S_\theta^\beta(\Delta^m)$ and the inclusion is strict.*

Proof. $\frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \leq \frac{1}{h_r^\alpha} |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}|$. To show the inclusion is strict, let θ be given and the sequence $x = (x_k)$ be defined such that

$$\Delta^m x_k = \begin{cases} [\sqrt{h_r}] & \text{if } k = 1, 2, \ldots, [\sqrt{h_r}], \\ 0 & \text{otherwise} \end{cases}$$

Now $\lim_{r\to\infty} \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^m x_k - 0| \geq \varepsilon\}| \leq \lim_{r\to\infty} \frac{1}{h_r^\beta}[\sqrt{h_r}] \leq \frac{\sqrt{h_r}}{h_r^\beta} = \lim_{r\to\infty} \frac{1}{h_r^{\beta - \frac{1}{2}}}$. Then $x \in S_\theta^\beta(\Delta^m)$ for $\frac{1}{2} < \beta \leq 1$ but $x \neq S_\theta^\alpha(\Delta^m)$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 3.4. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. For $0 < p < \infty$, $N_\theta^\alpha(\Delta^m, f, p) \subset S_\theta^\beta(\Delta^m, p)$ and the inclusion is strict for some α and β.*

Proof. Let $X \in N_\theta^\alpha(\Delta^m, f, p)$. So

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} (f(|\Delta^m x_k - L|))^{p_k} = 0.$$

Let $\varepsilon > 0$ be given. Consider

$$\begin{aligned} \sum_{k \in I_r} (f(|\Delta^m x_k - L|))^{p_k} &= \sum_{\substack{k \in I_r, \\ |\Delta^m x_k - L| \geq \varepsilon}} (f(|\Delta^m x_k - L|))^{p_k} + \sum_{\substack{k \in I_r, \\ |\Delta^m x_k - L| < \varepsilon}} (f(|\Delta^m x_k - L|))^{p_k} \\ &\geq \sum_{\substack{k \in I_r, \\ |\Delta^m x_k - L| \geq \varepsilon}} (f(|\Delta^m x_k - L|))^{p_k} \\ &\geq |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \max\{f(\varepsilon^h), f(\varepsilon^H)\} \\ \Rightarrow \frac{1}{h_r^\alpha} \sum_{k \in I_r} (f(|\Delta^m x_k - L|))^{p_k} &\geq \frac{1}{h_r^\alpha} |\{k \in I_r : |\Delta^m x_k - L|^p \geq \varepsilon\}| \max\{f(\varepsilon^h), f(\varepsilon^H)\} \\ &\geq \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^m x_k - L|^p \geq \varepsilon\}| \max\{f(\varepsilon^h), f(\varepsilon^H)\} \end{aligned}$$

which implies $\frac{1}{h_r^\beta}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \to 0$ as $r \to \infty$. It follows that if $x = (x_k)$ is strongly $N_\theta^\alpha(\Delta^m, f, p)$-summable to L, then it is $S_\theta^\beta(\Delta^m)$-statistically convergent to L.

To show the inclusion is strict, let $f(x) = x,\ p_k = 1$ for all $k \in \mathbb{N}$ and consider the following example $x = (x_k)$ be defined such that

$$\Delta^m x_k = \begin{cases} [\sqrt{h_r}] & \text{if } k = 1, 2, \ldots, [\sqrt{h_r}], \\ 0 & \text{otherwise} \end{cases}$$

Then for every $\varepsilon > 0$ and $\frac{1}{2} < \beta \leq 1$, we have

$$\frac{1}{h_r^\beta}|\{k \in I_r : |\Delta^m x_k - 0|^p \geq \varepsilon\}| = \frac{[\sqrt{h_r}]}{h_r^\beta} \to 0 \text{ as } r \to \infty.$$

That is $x_k \to 0(S_\theta^\beta(\Delta^m))$. But for $0 < \alpha < 1$,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r}|\Delta^m x_k - 0| = \frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r^\alpha} \to \infty \text{ as } r \to \infty,$$

and for $\alpha = 1$,

$$\frac{1}{h_r}\sum_{k\in I_r}|\Delta^m x_k - 0| = \frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r} \to 1 \text{ as } r \to \infty.$$

Which implies $x_k \nrightarrow 0(N_\theta^\alpha(\Delta^m, f, p))$. □

Theorem 3.5. *Let $0 < \alpha \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\liminf_r q_r > 1$, then $S^\alpha(\Delta^m) \subset S_\theta^\alpha(\Delta^m)$.*

Proof. Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \Rightarrow \left(\frac{h_r}{k_r}\right)^\alpha \geq \left(\frac{\delta}{1+\delta}\right)^\alpha \Rightarrow \frac{1}{k_r^\alpha} \geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}$$

If $x_k \to L(S^\alpha)$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned} \frac{1}{k_r^\alpha}|\{k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\}| &\geq \frac{1}{k_r^\alpha}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \\ &\geq \frac{\delta^\alpha}{(1+\delta^\alpha)}\frac{1}{h_r^\alpha}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \end{aligned}$$

This proves the sufficiency of the theorem. □

Theorem 3.6. *Let $0 < \alpha \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\limsup_r q_r < \infty$, then $S_\theta^\alpha(\Delta^m) \subset S(\Delta^m)$.*

Proof. If $\limsup_r q_r < \infty$, then there exists a constant $A > 0$ such that $q_r < A$ for all $r \in \mathbb{N}$. Suppose that $x_k \to L(S_\theta^\alpha(\Delta^m))$. So $\lim_{r\to\infty}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| = 0$. Let $\varepsilon > 0$. Suppose $N_r = |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}|$. Hence $\lim_{r\to\infty}\frac{N_r}{h_r^\alpha} = 0$. Then there exists $r_0 \in \mathbb{N}$ such that for $0 < \alpha \leq 1$,

$$\frac{N_r}{h_r^\alpha} < \varepsilon \text{ for all } r > r_0 \Rightarrow \frac{N_r}{h_r} < \varepsilon \text{ for all } r > r_0.$$

Let $B = \max\{N_r : 1 \leq r \leq r_0\}$ and choose n such that $k_{r-1} < n \leq k_r$, then we have

$$\begin{aligned} \frac{1}{n}|\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}| &\leq \frac{1}{k_{r-1}}|\{k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\}| \\ &= \frac{1}{k_{r-1}}\{N_1 + N_2 + \ldots + N_{r_0} + N_{r_0+1} + \ldots + N_r\} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{Br_0}{k_{r-1}} + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{N_{r_0+1}}{h_{r_0+1}} + \ldots + h_r\frac{N_r}{h_r}\right\} \\
&\leq \frac{Br_0}{k_{r-1}} + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\frac{N_r}{h_r}\right)\{h_{r_0+1} + \ldots + h_r\} \\
&\leq \frac{Br_0}{k_{r-1}} + \frac{1}{k_{r-1}}\varepsilon(k_r - k_0) \\
&\leq \frac{Br_0}{k_{r-1}} + \varepsilon q_r \leq \frac{Br_0}{k_{r-1}} + \varepsilon A
\end{aligned}
$$

which implies that $\frac{1}{n}\left|\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}\right| \to 0$ as $n \to \infty$ and it follows that $x_k \to L(S(\Delta^m))$. □

Theorem 3.7. *If*

$$\liminf_{r\to\infty}\frac{h_r^\alpha}{k_r} > 0, \tag{1}$$

then $S(\Delta^m) \subset S_\theta^\alpha(\Delta^m)$.

Proof. For a given $\varepsilon > 0$, we have

$$\{k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\} \subset \{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}$$

Therefore,

$$
\begin{aligned}
\frac{1}{k_r}|\{k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\}| &\geq \frac{1}{k_r}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \\
&= \frac{h_r^\alpha}{k_r}\frac{1}{h_r^\alpha}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}|
\end{aligned}
$$

Taking limit as $r \to \infty$ and using equation (1), we have

$$x_k \to L(S(\Delta^m)) \Rightarrow x_k \to L(S_\theta^\alpha(\Delta^m)).$$

□

Theorem 3.8. *Let* $\theta = (k_r)$ *and* $\theta' = (s_r)$ *be any two lacunary sequences such that* $I_r \subset J_r$ *for all* $r \in \mathbb{N}$ *and let* α *and* β *be such that* $0 < \alpha \leq \beta \leq 1$.

(i). If

$$\liminf_{r\to\infty}\frac{h_r^\alpha}{l_r^\beta} > 0 \tag{2}$$

then $S_{\theta'}^\beta(\Delta^m) \subseteq S_\theta^\alpha(\Delta^m)$.

(ii). If

$$\lim_{r\to\infty}\frac{l_r}{h_r^\beta} = 1 \tag{3}$$

then $S_\theta^\alpha(\Delta^m) \subseteq S_{\theta'}^\beta(\Delta^m)$.

Proof.

(i). Suppose that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let equation (2) be satisfied. For given $\varepsilon > 0$, we have

$$\{k \in J_r : |\Delta^m x_k - L| \geq \varepsilon\} \supseteq \{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}$$

and so

$$\frac{1}{l_r^\beta}|\{k \in J_r : |\Delta^m x_k - L| \geq \varepsilon\}| \geq \frac{h_r^\alpha}{l_r^\beta}\frac{1}{h_r^\alpha}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}|$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$ and $l_r = s_r - s_{r-1}$. Now taking the limit as $r \to \infty$ in the last inequality and using equation (2), we get $S_{\theta'}^\beta(\Delta^m) \subseteq S_\theta^\alpha(\Delta^m)$.

(ii). Let $x = (x_k) \in S_\theta^\alpha(\Delta^m)$ and equation (3) be satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{l_r^\beta}|\{k \in J_r : |\Delta^m x_k - L| \geq \varepsilon\}| &= \frac{1}{l_r^\beta}|\{s_{r-1} < k \leq k_{r-1} : |\Delta^m x_k - L| \geq \varepsilon\}| \\
&+ \frac{1}{l_r^\beta}|\{k_r < k \leq s_r : |\Delta^m x_k - L| \geq \varepsilon\}| + \frac{1}{l_r^\beta}|\{k_{r-1} < k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\}| \\
&\leq \frac{k_{r-1} - s_{r-1}}{l_r^\beta} + \frac{s_r - k_r}{l_r^\beta} + \frac{1}{l_r^\beta}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \\
&= \frac{l_r - h_r}{l_r^\beta} + \frac{1}{l_r^\beta}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \\
&\leq \frac{l_r - h_r^\beta}{h_r^\beta} + \frac{1}{h_r^\beta}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| \\
&\leq \left(\frac{l_r}{h_r^\beta} - 1\right) + \frac{1}{h_r^\alpha}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}|
\end{aligned}$$

for all $r \in \mathbb{N}$. Since $\lim_{r\to\infty} \frac{l_r}{h_r^\beta} = 1$ by equation (3) the first term and since $x = (x_k) \in S_\theta^\alpha(\Delta^m)$ the second term of right hand side of above inequality tend to 0 as $r \to \infty$ $\Big(\text{Note that } \Big(\frac{l_r}{h_r^\beta} - 1\Big) \geq 0\Big)$. This implies that $S_\theta^\alpha(\Delta^m) \subseteq S_{\theta'}^\beta(\Delta^m)$. □

Theorem 3.9. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. Then we have*

(i). If equation (2) holds, then $N_{\theta'}^\beta(\Delta^m, f, p) \subset N_\theta^\alpha(\Delta^m, f, p)$,

(ii). If equation (3) holds and $x \in \ell_\infty$, then $N_\theta^\alpha(\Delta^m, f, p) \subset N_{\theta'}^\beta(\Delta^m, f, p)$.

Proof.

(i). Let $x = (x_k) \in N_{\theta'}^\beta(\Delta^m, f, p)$ and suppose that equation (2) holds. Since $I_r \subseteq J_r$ and $h_r \leq l_r$ for all $r \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{1}{l_r^\beta}\sum_{k\in J_r}(f(|\Delta^m x_k - L|))^{p_k} &\geq \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k} \\
&= \frac{h_r^\alpha}{l_r^\beta}\frac{1}{h_r^\alpha}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k}
\end{aligned}$$

Using equation (2), we get $x_k \to L(N_\theta^\alpha(\Delta^m, f, p))$.

(ii). Let $x = (x_k) \in N_\theta^\alpha(\Delta^m, f, p)$ and suppose that equation (3) holds. Since f is bounded, then there exists some $M_1 > 0$ such that $f(|\Delta^m x_k - L|) \leq M_1$ for all k. Now, since $I_r \subseteq J_r$ and $h_r \leq l_r$ for all $r \in \mathbb{N}$, we may write

$$\frac{1}{l_r^\beta}\sum_{k\in J_r}(f(|\Delta^m x_k - L|))^{p_k} = \frac{1}{l_r^\beta}\sum_{k\in J_r - I_r}(f(|\Delta^m x_k - L|))^{p_k} + \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k}$$

$$
\begin{aligned}
&\leq \left(\frac{l_r - h_r}{l_r^\beta}\right) M_1^H + \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k} \\
&\leq \left(\frac{l_r - h_r^\beta}{h_r^\beta}\right) M_1^H + \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k} \\
&\leq \left(\frac{l_r}{h_r^\beta} - 1\right) M_1^H + \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k}
\end{aligned}
$$

for every $r \in \mathbb{N}$. Therefore $x = (x_k) \in N_{\theta'}^\beta(\Delta^m, f, p)$. □

Theorem 3.10. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}, \alpha$ and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. Then*

(i). If equation (2) holds and a sequence is strongly $N_{\theta'}^\beta(\Delta^m, f, p)$-summable to L, then it is lacunary Δ^m-statistically convergent to L.

(ii). If equation (3) holds, f is a bounded modulus function and $x = (x_k)$ is lacunary Δ^m-statistically convergent to L, then it is strongly $N_{\theta'}^\beta(\Delta^m, f, p)$-summable to L.

Proof.

(i). Using the techniques of Theorem 3.7 and the condition 2, it can be easily proved.

(ii). Suppose $x_k \to L(S_\theta^\alpha(\Delta^m))$ and f is a bounded modulus function. Then there exists some $M_1 > 0$ such that $f(|\Delta^m x_k - L|) \leq M_1$ for all k, then for every $\varepsilon > 0$, we may write

$$
\begin{aligned}
\frac{1}{l_r^\beta}\sum_{k\in J_r}(f(|\Delta^m x_k - L|))^{p_k} &= \frac{1}{l_r^\beta}\sum_{k\in J_r - I_r}(f(|\Delta^m x_k - L|))^{p_k} + \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k} \\
&\leq \left(\frac{l_r - h_r}{l_r^\beta}\right) M_1^H + \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k} \\
&\leq \left(\frac{l_r - h_r^\beta}{l_r^\beta}\right) M_1^H + \frac{1}{l_r^\beta}\sum_{k\in I_r}(f(|\Delta^m x_k - L|))^{p_k} \\
&\leq \left(\frac{l_r}{h_r^\beta} - 1\right) M_1^H + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r, \\ |\Delta^m x_k - L|\geq\varepsilon}}(f(|\Delta^m x_k - L|))^{p_k} + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r, \\ |\Delta^m x_k - L|<\varepsilon}}(f(|\Delta^m x_k - L|))^{p_k} \\
&\leq \left(\frac{l_r}{h_r^\beta} - 1\right) M_1^H + \frac{M_1^p}{h_r^\beta}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| + \frac{h_r}{h_r^\beta}\max\{f(\varepsilon^h), f(\varepsilon^H)\} \\
&\leq \left(\frac{l_r}{h_r^\beta} - 1\right) M_1^H + \frac{M_1^H}{h_r^\alpha}|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| + \frac{l_r}{h_r^\beta}\max\{f(\varepsilon^h), f(\varepsilon^H)\}
\end{aligned}
$$

for all $r \in \mathbb{N}$. Using equation (3), we obtain $x_k \to L(N_{\theta'}^\alpha(\Delta^m, f, p))$. □

References

[1] H. Altionk, M. Et and M. Işik, Δ_i^m*-lacunary statistical convergence of order* α, AIP Conference Proceedings 1926, 020004(2018); 10.1063/1.5020453.

[2] S. Bhunia, P. Das and S. K. Pal, *Restricting statistical conergence*, Acta Math Hungar, 134(1-2)(2012), 153-161.

[3] R. Çolak, *Statistical convergence of order* $\tilde{\alpha}$*, Modern methods in Analysis and its Applications*, Anamaya Publ. New Delhi, India (2010), 121-129.

[4] J. S. Connor, *The statistical and strong p-Cesàro convergence of sequence*, Analysis, 8(1980), 47-63.

[5] P. Das and E. Savaş, *On I-statistical and I-lacunary statistical convergence of order α*, Bulletin of the Iranian Math. Soc., 40(2)(2014), 459-472.

[6] O. Duman, M. K. Khan and C. Orhan, *A-statistical convergence of approximating operators*, Math Inequal. Appl., 6(4)(2003), 689-699.

[7] M. Et, A. Alotaibi and S. A. Mohiuddine, *On (Δ^m, I)-statistical convergence of order $\tilde{\alpha}$*, The Scientific World Journal.

[8] M. Et and R. Çolak, *On some generalized difference sequence spaces*, Soochow J. Math., 21(4)(1995), 377-386.

[9] M. Et, *Spaces of Cesàro difference sequences of order α defined by a modulus in a locally convex space*, Taiwanese J. Math., 10(4)(2006), 865-879.

[10] M. Et and H. Şengül, *Some Cesàro type summability spaces of order α and lacunary statistical convergence of order α*, Filomat, 28(8)(2014), 1593-1602.

[11] H. Fast, *Sur la convergence statistique*, Collo. Math., 2(1951), 241-244.

[12] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesàro type summability space*, Proc. London Math. Soc., 37(3)(1978), 508-520.

[13] J. A. Fridy, *On statistical convergence*, Analysis, 5(4)(1985), 301-313.

[14] J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pacific J. Math., 160(1993), 43-51.

[15] J. A. Fridy and C. Orhan, *Lacunary statistical summability*, J. Math. Anal. Appl., 173(2)(1993), 497-504.

[16] A. D. Gadjiev and C. Orhan, *Some approximation theorem via statistical convergence*, Rockey Mountain J. Math., 32(1)(2002), 129-138.

[17] H. Kızmaz, *On certain sequence spaces*, Canad Math Bull, 24(1981), 169-176.

[18] J. Li, *Lacunary statistical convergence and inclusion properties between lacunary methods*, International J. Math. and Math. Sci., 23(3)(2000), 175-180.

[19] I. J. Maddox, *Elements of functional analysis*, Cambridge Univ. Press, (1970).

[20] I. J. Maddox, *A new type of convergence*, Math. Proc. Camb. Phil. Soc., 83(1978), 61-64.

[21] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc., 100(1986), 161-166.

[22] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, 5(1953), 29-49.

[23] W. H. Ruckle, *FK spaces in which the sequence of co-ordinate vectors is bounded*, Canad. J. Math., 25(1973), 973-978.

[24] T. Šalát, *On statistically convergent sequence of real numbers*, Math. Slovaca, 30(1980), 139-150.

[25] H. Şengül and M. Et, *Lacunary statistical convergence of order α^**, Acta Math. Scientia, 34B(2)(2014), 473-482.

[26] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer Math Monthly, 66(1959), 361-375.