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Application of Double Sumudu Transform in Partial Integro-Differential Equations

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Abstract: In this paper,we apply the double Sumudu transform to solve the linear partial integro-differential equations.This method transform the given partial integro-differential equations into an algebraic equation,which we can solve by using the inverse Sumudu transform.

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Keywords: Partial integro-differential equations, Sumudu Transform.

1. Introduction

In the field of applied mathematics, when we try to find the solution to the real world phenomenon then it leads to convert the said phenomenon into integral equation and partial integro differential equations. The partial integro-differential equations occurs in various branches of science,engineering and applied mathematics [2–4]. Literature survey revels that many authors have contributed to have the solution of partial integro-differential equations [17, 18]. Integral transform method was the most favorable method that has been used to solve ordinary and partial differential equations. In this paper ,we wish to apply double Sumudu transform to solve partial integro differential equations. Tchuenche and Nyimvua [6] express the double Sumudu transform as polynomials or convergent infinite series and studied its applications. Adem Kiliçman [7] use the double Sumudu transform to solve wave equation in one dimension having singularity at initial conditions. Hassan Eltayeb and Adem Kiliçman [8] applied two techniques double Laplace transform and double Sumudu transform to solve the new wave equation with non-constant coefficient and establish a relationship between double Laplace transform and double Sumudu transform. Mohammed S. Mechee [10] studied the double Sumudu transform for solving differential equations with some applications. Recently Shams A. Ahmed [11, 12] have applied double transform named the Laplace–Sumudu transform (DLST) to unravel integral differential equations. Some more applications of double Sumudu transform can be seen in [9, 13–16]

In this method, we apply the double Sumudu transform to partial integro-differential equation which transform partial integro-differential equation into an algebric equation instead of transforming to ordinary differential equation. Solving this algebric equation and using the inverse Sumudu transform we can have the exact solution to given partial integro-differential equation. This method is more illustrated by solving some partial integro-differential equations.

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1.1. Sumudu Transform

For the function $f(t)$ the Sumudu transform is defined by Watugala [1]

$$\mathbb{S}[f(t)] = G(u) = \int_0^\infty e^{-t} f(ut)dt \qquad u \in (-\tau_1, \tau_2) \tag{1}$$

provided the integral on the right hand side exists.The Sumudu transform of functions $f(t)$ $(t \geq 0)$ are come to exists which are piecewise continuous and of exponential order defined over the set $\mathfrak{A}$=[$f(t)/\exists$ M,τ_1, $\tau_2 > 0$,|f(t)| $<$ M $e^{\frac{|t|}{\tau_j}}$,if $t \in (-1)^j \times [0, \infty)$]. The above transform can be reduced to following form with suitable change in the variable

$$\mathbb{S}[f(t)] = G(u) = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f(t)dt \tag{2}$$

The inverse Sumudu transform of function $G(u)$ is denoted by symbol $\mathbb{S}^{-1}[G(u)] = f(t)$ and is defined with Bromwich contour integral [5]

$$\mathbb{S}^{-1}[G(u)] = f(t) = \lim_{T\to\infty} \frac{1}{2\Pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}G(u)du \tag{3}$$

1.2. Double Sumudu Transform

Let$f(t,x); t, x \in \mathbb{R}_+$ be a function which can be expressed as a convergent infinite series, then its double Sumudu transform is given by [6]

$$F(u,v) = S_2[f(t,x);(u,v)] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})} f(t,x)dtdx \tag{4}$$

In this paper, we wish to solve some partial integro differential equations so that we require the double Sumudu transform of the partial derivatives with respect to x and t, which are as follows

$$\begin{aligned}
S_2\left[\frac{\partial f(t,x)}{\partial x};(u,v)\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}\frac{\partial f(t,x)}{\partial x}(t,x)dtdx \\
&= \frac{1}{u}F(u,v) - \frac{1}{u}F(0,v) \\
S_2\left[\frac{\partial^2 f(t,x)}{\partial x^2};(u,v)\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}\frac{\partial^2 f(t,x)}{\partial x^2}(t,x)dtdx \\
&= \frac{1}{u^2}F(u,v) - \frac{1}{u^2}F(0,v) - \frac{1}{u}\frac{\partial F(0,v)}{\partial x}
\end{aligned}$$

Double Sumudu transform of first and second order partial derivative with respect to t is of form

$$\begin{aligned}
S_2\left[\frac{\partial f(t,x)}{\partial t};(u,v)\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}\frac{\partial f(t,x)}{\partial t}(t,x)dtdx \\
&= \frac{1}{v}F(u,v) - \frac{1}{v}F(u,0) \\
S_2\left[\frac{\partial^2 f(t,x)}{\partial t^2};(u,v)\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}\frac{\partial^2 f(t,x)}{\partial t^2}(t,x)dtdx \\
&= \frac{1}{v^2}F(u,v) - \frac{1}{v^2}F(u,0) - \frac{1}{v}\frac{\partial F(u,0)}{\partial t}
\end{aligned}$$

2. Main Results

Now to illustrate the method, we consider the general linear partial integro differential equation of the form

$$\sum_{i=0}^{m} a_i\frac{\partial^i u}{\partial t^i} + \sum_{i=0}^{n} b_i\frac{\partial^i u}{\partial x^i} + cu + \sum_{i=0}^{r} d_i\int_0^\infty K_i(t-s)\frac{\partial^i u(x,s)}{\partial x^i}ds + f(x,t) = 0 \tag{5}$$

where $f(x,t)$ and $K_i(t-s)$ are known functions and $a_i's, b_i's, d_i's$ and c are constants (with prescribed conditions). Apply double Sumudu transform to given partial integro differential equation (5), we get

$$\sum_{i=0}^{m} a_i S_2\left[\frac{\partial^i u}{\partial t^i}\right] + \sum_{i=0}^{n} b_i S_2\left[\frac{\partial^i u}{\partial x^i}\right] + cS_2[u] + \sum_{i=0}^{r} d_i S_2\left[\int_0^{\infty} K_i(t-s)\frac{\partial^i u(x,s)}{\partial x^i}ds\right] + S_2[f(x,t)] = 0$$

Using double Sumudu transform of partial derivatives and convolution theorem, we get

$$\sum_{i=0}^{m} a_i\left[\frac{1}{v^i}T(u,v) - \sum_{k=0}^{i-1}\frac{1}{v^{i-k}}S_x\left[\frac{\partial^k T(u,0)}{\partial t^k}\right]\right] + \sum_{i=0}^{n} b_i\left[\frac{1}{u^i}T(u,v) - \sum_{k=0}^{i-1}\frac{1}{u^{i-k}}S_t\left[\frac{\partial^k T(0,v)}{\partial x^k}\right]\right]$$
$$+ cT(u,v) + \sum_{i=0}^{r} d_i K_i(v)\left[\frac{1}{u^i}T(u,v) - \sum_{k=0}^{i-1}\frac{1}{u^{i-k}}S_t\left[\frac{\partial^k T(0,v)}{\partial x^k}\right]\right] + F(u,v) = 0 \quad (6)$$

where $T(u,v) = S_2[u(x,t)]$, $F(u,v) = S_2[f(x,t)]$, $K_i(v) = S_2[K_i(t)]$. Equation (6) is an algebric equation in $T(u,v)$, solving this equation and applying inverse double Sumudu transform to $T(u,v)$, we get an exact solution $u(x,t)$ of given partial integro differential equation (5)

3. Illustrative Examples

Example 3.1. *Consider the PIDE*

$$u_t - u_{xx} + u + \int_0^t e^{t-y}u(x,y)dy = (x^2+1)e^t - 2 \quad (7)$$

with initial condition $u(x,0) = x^2, u_t(x,0) = 1$ *and boundary condition* $u(0,t) = t, u_x(0,t) = 0$.

Solution: Apply double Sumudu transform to given PIDE (7)

$$S_2[u_t] - S_2[u_{xx}] + S_2[u] + S_2\left[\int_0^t e^{t-y}u(x,y)dy\right] = S_2[(x^2+1)e^t] - S_2[2] \quad (8)$$

$$\left[\frac{1}{v}T(u,v) - \frac{1}{v}T(u,0)\right] - \left[\frac{1}{u^2}T(u,v) - \frac{1}{u^2}T(0,v) - \frac{1}{u}T_x(0,v)\right] + T(u,v) + v.\frac{1}{1-v}T(u,v) = (2u^2+1)\frac{1}{1-v} - 2 \quad (9)$$

Now applying the single Sumudu transform to initial conditions and boundary condition, we get

$$S_2[u(x,0)] = S_2[x^2] = 2u^2 \qquad S_2[u_t(x,0)] = S_2[1] = 1$$
$$S_2[u(0,t)] = S_2[t] = v \qquad S_2[u_x(0,t)] = S_2[0] = 0$$

Using above conditions in equation (9), we get

$$T(u,v)\left[\frac{1}{v} - \frac{1}{u^2} + 1 + \frac{v}{1-v}\right] = \left[\frac{2u^2}{v} - \frac{v}{u^2} + \frac{2u^2+1}{1-v} - 2\right] \quad (10)$$
$$T(u,v)\left[\frac{(u^2+v^2-v)}{(vu^2)(1-v)}\right] = \left[\frac{(2u^2+v)(u^2+v^2-v)}{(vu^2)(1-v)}\right] \quad (11)$$
$$T(u,v) = 2u^2 + v \quad (12)$$

Now apply the inverse double Sumudu transform on both sides, we have

$$S_2^{-1}[T(u,v)] = S_2^{-1}[2u^2+v] = x^2 + t$$

This is the required exact solution to given PIDE.

Example 3.2. *Consider the PIDE*

$$u_{tt} = u_x + 2\int_0^t (t-y)u(x,y)dy - 2e^x \tag{13}$$

with initial condition $u(x,0) = e^x, u_t(x,0) = 0$ *and boundary condition* $u(0,t) = \cos(t)$.

Solution: Apply double Sumudu transform to given PIDE (13)

$$S_2[u_{tt}] = S_2[u_x] + 2S_2\left[\int_0^t (t-y)u(x,y)dy\right] - S_2[e^x]$$
$$\left[\frac{1}{v^2}T(u,v) - \frac{1}{v^2}T(u,0) - \frac{1}{v}T_t(u,0)\right] = \left[\frac{1}{u}T(u,v) - \frac{1}{u}T(0,v)\right] + 2v^2.T(u,v) - 2\frac{1}{1-u} \tag{14}$$

Now applying the single Sumudu transform to initial conditions and boundary condition, we get

$$S_2[u(x,0)] = S_2[e^x] = T(u,0) = \frac{1}{1-u}$$
$$S_2[u_t(x,0)] = S_2[0] = T_t(u,0) = 0$$
$$S_2[u(0,t)] = S_2[cos(t)] = T(0,v) = \frac{1}{1+v^2}$$

Using above conditions in equation (14), we get

$$T(u,v)\left[\frac{1}{v^2} - \frac{1}{u} - 2v^2\right] = \left[\frac{1}{v^2(1-u)} - \frac{1}{u(1+v^2)} - \frac{2}{1-u}\right] \tag{15}$$
$$T(u,v)\left[\frac{(u-v^2-2uv^4)}{(vu^2)}\right] = \left[\frac{(u-v^2-2uv^4)}{(1-u)(1+v^2)(vu^2)}\right] \tag{16}$$
$$T(u,v) = \frac{1}{(1-u)(1+v^2)} \tag{17}$$

Now apply the inverse double Sumudu transform on both sides, we have

$$S_2^{-1}[T(u,v)] = S_2^{-1}\left[\frac{1}{(1-u)(1+v^2)}\right] = e^x.\cos(t)$$

This is the required exact solution to given PIDE.

Example 3.3. *Consider the PIDE*

$$u_t + u_{ttt} - 2\int_0^t sinh(t-y)u_{xxx}(x,y)dy = 0 \tag{18}$$

with initial condition $u(x,0) = 0, u_t(x,0) = x, u_{tt} = 0$ *and boundary condition* $u(0,t) = 0, u_x(0,t) = sin(t), u_{xx}(0,t) = 0$.

Solution: Apply double Sumudu transform to given PIDE (18)

$$S_2[u_t] + S_2[u_{tt}] - 2S_2\left[\int_0^t sinh(t-y)u_{xxx}(x,y)dy\right] = 0$$

$$\left[\frac{1}{v}T(u,v) - \frac{1}{v}T(u,0)\right] + \left[\frac{1}{v^3}T(u,v) - \frac{1}{v^3}T(u,0) - \frac{1}{v^2}T_t(u,0) - \frac{1}{v}T_{tt}(u,0)\right]$$
$$- \frac{v^2}{1-v^2}\left[\frac{1}{u^3}T(u,v) - \frac{1}{u^3}T(0,v) - \frac{1}{u^2}T_x(0,v) - \frac{1}{u}T_{xx}(0,v)\right] = 0 \tag{19}$$

Now applying the single Sumudu transform to initial conditions and boundary condition, we get

$$S_2[u(x,0)] = S_2[0] = T(u,0) = 0 \qquad S_2[u_t(x,0)] = S_2[x] = T_t(u,0) = u$$
$$S_2[u_{tt}(x,0)] = S_2[0] = T_{tt}(u,0) = 0 \qquad S_2[u(0,t)] = S_2[0] = T(0,v) = 0$$
$$S_2[u_x(0,t)] = S_2[sin(t)] = T_x(0,v) = \frac{v}{1+v^2} \qquad S_2[u_{xx}(0,t)] = S_2[0] = T_{xx}(0,v) = 0$$

Using above conditions in equation (19), we get

$$T(u,v)\left[\frac{1}{v} - \frac{1}{v^3} - \frac{v^2}{u^3(1-v^2)}\right] = \frac{u}{v^2} - \frac{v^3}{u^2(1+v^2)(1-v^2)} \tag{20}$$
$$T(u,v)\left[\frac{v(u^3 - u^3v^4 - v^5)}{u^3v^4(1-v^2)}\right] = \left[\frac{(u^3 - u^3v^4 - v^5)}{v^2u^2(1+v^2)(1-v^2)}\right] \tag{21}$$
$$T(u,v) = \frac{uv}{(1+v^2)} \tag{22}$$

Now apply the inverse double Sumudu transform on both sides, we have

$$S_2^{-1}[T(u,v)] = S_2^{-1}\left[\frac{uv}{(1+v^2)}\right] = x.\sin(t)$$

This is the required exact solution to given PIDE.

4. Conclusion

we have successfully applied the double Sumudu transform method to solve the linear partial integro-differential equations which converts the given partial integro-differential equations into an algebric equation, which we can solve by using the inverse Sumudu transform to have the exact solution.

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