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A Stochastic Inventory System with Two Types of Services and a Finite Populations

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Abstract : In this article, we consider a continuous review (s, S) inventory system with a service facility, wherein the demand of a customer is satisfied only after performing some service on the item which is assumed to be of random duration. We also assume that the demands are generated by a finite homogeneous population. We consider that the server provides two types of service, type 1 with probability p_1 and type 2 with probability p_2. Each arriving customer may choose either type of service. The joint probability distribution of the number of customers in the system and the inventory level is obtained in the steady state case. The measures of system performance in the steady state are derived and the total expected cost rate is also derived.

Keywords : Inventory with service time, Continuous review, Markov process, Finite populations.

AMS Subject Classification: 90B05, 60J27.

1 Introduction

Several researchers have studied the inventory systems in which demanded items are instantaneously distributed from stock (if available) to the customers. During stock out period, the demands of a customer are either not satisfied (lost sales case) or satisfied only after getting the receipt of the ordered items (backlog case). In the backlog case, either all demands (full backlog case) or only a limited number of demands (partial backlog case) are satisfied during stock out period. To know the review of these works see Çakanyildirim et al., [6], Durán et al. [7], Elango and Arivarignan[8], Goyal and Giri [9], Kalpakam and Arivarignan([10], [11]), Liu and Yang[13], Nahmias[14], Raafat[15] and Yadavalli et al.[16] and the references therein.

However, in the case of inventories maintained at service facilities, after some service is performed on the demanded items they are distributed to the customers. In such situations, the items are issued not on demanding rather it is done after a random time of service. It causes the formation of queues

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in front of service centers. As a result there is a need for study of both the inventory level and the queue length in the long run. Berman and Kim [1] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [4] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. The optimal value of the maximum allowable inventory which minimizes the long - run expected cost rate has been obtained.

Berman and Sapna [5] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [2] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [3] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. Krishnamoorthy et al., [12] introduced an additional control policy (N-policy) into (s, S) inventory system with positive service time.

In all the above models, the authors assumed that at the time of service completion the outgoing customer was taken the serviced item and hence the inventory level was decreased by one. But in a real life problems, an arriving customer may demand an item with service or he/she may ask only service not the item. For example, in a car battery sales and service shop, a customer may buy a new battery from the shop or he/she brings a new battery and ask the server to fit the battery in car only.

Finite source (s, S) inventory model with two types of services is motivated by the service facility system with restricted customers for example military canteen providing service to soldiers or a company canteen serving the members of the specific working community in the company. Machine service problems within an industry is also a problem which motivated me to create the present stochastic model. The problem we consider is more relevant to the real life situation.

We model this situation in this paper. In this paper, we consider an inventory system with service facility and finite source. The system has two types of service, type 1 with probability p_1 and type 2 with probability p_2 with the service time following exponential distribution. An (s, S) ordering policy with positive random lead time is adopted. The joint distribution of the number of customers in the system and the inventory level is obtained in the steady state case. Various system performance measures are derived and the total expected cost rate is derived under a suitable cost structure.

The remainder of this paper is organized as follows. In Section 2, we present the mathematical model and the notations used in this paper are defined. Analysis of the model and the steady state solution of the model are given in Section 3. In Section 4, we derive various measures of system performance in steady state. The total expected cost rate is derived in Section 5. Section 6 has concluding remarks.

2 Formulation of the Model

In this paper finite-source inventory systems with the following assumptions are investigated.

Consider a continuous review inventory system with service facility and maximum inventory of S units. The customers are generated by a finite number of homogeneous sources N, $(1 < N < \infty)$ and the demand time points form a quasi-random distribution with parameter $\lambda(>0)$. That is, the probability that any particular source generates a demand in any interval $(t, t+dt)$ is $\lambda dt + o(dt)$ as $dt \to 0$ if the source is idle at time t, and zero if the source is in the service facility at time t, independently of the behavior of any other sources. The demand is for single item per customer. The demanded item is delivered to the customer after a random time of service. The server provides two types of service, type 1 with probability p_1 and type 2 with probability p_2 with the service time following exponential distribution with rate $p_i\mu(>0), i = 1, 2$. Each arriving customer may choose either type of service. A completion of type 1 service causes one customer to leave the system and type 2 service causes one customer to leave the system and reduces the inventory by one item.

The reorder level is fixed as s and an order is placed when the inventory level reaches the reorder level s. The ordering quantity for the commodity is $Q(= S - s > s + 1)$ items. The requirement $S - s > s + 1$ ensures that after a replenishment the inventory level will be always above the reorder level s. Otherwise it may not be possible to place reorder which leads to perpetual shortage. The lead time is assumed to be distributed as negative exponential with parameter $\beta(>0)$. All stochastic processes involved in the system are independent of each other.

Notations

The following notations are used in the paper.

$\mathbf{0}$: zero matrix

$\mathbf{e}^T$: $(1, 1, \ldots, 1)$.

I_N : identity matrix of order N

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

$\overline{\delta_{ij}}$ = 1-δ_{ij}

$$\sum_{j=0}^{i} a^j = \begin{cases} a^0 + a^1 + \cdots + a^i, & \text{if } i \text{ is non negative integer} \\ 0, & \text{otherwise} \end{cases}$$

$[A]_{ij}$: $(i, j)-$th element of the matrix A

3 Analysis

Let $L(t)$, $X(t)$, denote, respectively, the inventory level and number of customers (waiting and being served) in the system at time t. From the assumptions made on the input and output processes, it may be shown that

$$(L, X) = \{(L(t), X(t),) \,; t \geq 0\}$$

on the state space $E = \{(i, k) \mid i = 0, 1, 2, \ldots, S,\ k = 0, 1, 2, \ldots, N,\}$ is a Markov process. The infinitesimal generator of this process,

$$\tilde{A} = ((\ a((i, k); (j, l)),$$

may be obtained by using the following arguments:

- An arrival of a customer causes a transition from (i,k) to $(i,k+1)$, $i=0,1,\ldots,S$; $k=0,1,\ldots,N-1$;. The rate for this transition is $(N-k)\lambda$.
- A completion of type-1 service causes one customer to leave the system. Thus, a transition takes place from (i,k) to $(i,k-1)$, $i=0,1,2,\ldots,S$, $k=1,2,\ldots,N$. The rate for this transition is $p_1\mu$.
- A completion of type 2 service causes one customer to leave the system and reduces the inventory by one item. Thus, a transition takes place from (i,k) to $(i-1,k-1)$, $i=1,2,\ldots,S$, $k=1,2,\ldots,N$,. The rate for this transition is $p_2\mu$.
- A transition from (i,k) to $(i+Q,k)$ for $i=0,1,\ldots,s$; $k=0,1,\ldots,N$; with intensity β when a replenishment occurs.
- For other transition from (i,k) to (j,l) except $(i,k)\neq(j,l)$, the rate is zero.
- Finally, note that

$$a((i,k);(i,k)) = -\sum_{j}\sum_{\substack{l \\ (j,l)\neq(i,k)}} a((i,k);(j,l)).$$

Hence, $a((i,k);(j,l))$ can be writtten as

$$\begin{cases}
(N-k)\lambda, & j=i, & l=j+1, \\
 & i=0,1,2,\cdots,S, & k=0,1,\cdots,N-1 \\
p_1\mu, & j=i, & l=j-1, \\
 & i=1,\cdots,S, & k=1,2,\cdots,N \\
p_2\mu, & j=i-1, & l=j-1, \\
 & i=1,\cdots,S, & k=1,2,\cdots,N \\
\beta, & j=i+Q, & l=j, \\
 & i=s,s-1,\cdots,0, & k=0,1,\cdots,N \\
-[\ \overline{\delta_{Nj}}\ (N-k)\lambda+H(s-i)\beta+ & & \\
\overline{\delta_{0j}}\ (p_1\ \mu+\overline{\delta_{0i}}\ p_2\ \mu)\ ], & j=i, & l=j, \\
 & i=0,1,\cdots,S & k=0,1,\cdots,N \\
0, & \text{otherwise.} &
\end{cases}$$

By ordering the set of states of E as lexicographically, the infinitesimal generator

$$\tilde{A} = (a((i,k),(j,l))), \quad (i,k),(j,l)\in E,$$

can be conveniently expressed in a block partitioned matrix with entries

$$\tilde{A} = \begin{cases} A_0, & j = i, \quad i = s+1, s+2, \ldots, S \\ A_1, & j = i, \quad i = s, s-1, \ldots, 2, 1 \\ A_2, & j = i, \quad i = 0 \\ B, & j = i-1, \quad i = 1, 2, \ldots, S \\ C, & j = i+Q, \quad i = 0, 1, \ldots, s \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

More explicitly,

$$\tilde{A} = \begin{array}{c} S \\ S-1 \\ \vdots \\ s \\ s-1 \\ \vdots \\ 1 \\ 0 \end{array} \begin{pmatrix} A_0 & B & & & & & & \\ & A_0 & B & & & & & \\ & & \ddots & \ddots & & & & \\ C & & & A_1 & B & & & \\ & C & & & A_1 & B & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & C & & & A_1 & B \\ & & & & C & & & A_2 \end{pmatrix}$$

where $k = 0, 1, 2,$

$$[A_k]_{ij} = \begin{cases} (N-i)\lambda & \text{if} \quad j = i+1, \quad i = 0, 1, \cdots, N-1 \\ p_1\mu & \text{if} \quad j = i-1, \quad i = 1, 2, \cdots, N \\ -[\ \overline{\delta_{Ni}}\ (N-i)\lambda + \overline{\delta_{0k}}\beta + & \\ \overline{\delta_{0i}}\ (p_1\ \mu + \overline{\delta_{2k}}\ p_2\ \mu)\ ] & \text{if} \quad j = i, \quad i = 0, 1, \cdots, N \\ 0 & \text{otherwise.} \end{cases}$$

$$[B]_{ij} = \begin{cases} p_2\mu & \text{if} \quad j = i-1, \quad i = 1, 2, \cdots, N \\ 0 & \text{otherwise.} \end{cases}$$

$$[C]_{ij} = \begin{cases} \beta & \text{if} \quad j = i, \quad i = 0, 1, \cdots, N \\ 0 & \text{otherwise.} \end{cases}$$

It may be noted that the matrices A_0, A_1, A_2, B and C are square matrices of size $N+1$.

3.1 Steady state Results

It can be seen from the structure of the rate matrix $\tilde{A}$ that the homogeneous Markov process $\{(L(t), X(t)),$ $t \geq 0\}$ on the finite state space E is irreducible, aperiodic and persistent non-null. Hence, the limiting distribution of the Markov process

$$\phi^{(i,k)} = \lim_{t\to\infty} Pr[L(t) = i, X(t) = k | L(0), X(0)] \quad \text{exists.}$$

Let $\Phi = (\mathbf{\Phi^{(S)}}, \mathbf{\Phi^{(S-1)}}, \ldots, \mathbf{\Phi^{(1)}}, \mathbf{\Phi^{(0)}})$, and we partition the vector $\mathbf{\Phi}$, into as follows: $\mathbf{\Phi^{(i)}} = \phi^{(i,0)}, \phi^{(i,1)}, \ldots, \phi^{(i,N)}$ for $i = 0, 1, 2, \ldots, S$. Then the vector of limiting probabilities $\mathbf{\Phi}$ satisfies

$$\mathbf{\Phi} A = \mathbf{0} \text{ and } \sum_{(i,k)\in E}\sum \phi^{(i,k)} = 1 \tag{3.1}$$

The first equation of the (3.1) yields the following set of equations:

$$\begin{aligned}
\mathbf{\Phi^{(i)}} A_0 + \mathbf{\Phi^{(i-Q)}} C &= \mathbf{0}, \quad i = S \\
\mathbf{\Phi^{(i+1)}} B + \mathbf{\Phi^{(i)}} A_0 + \mathbf{\Phi^{(i-Q)}} C &= \mathbf{0}, \quad i = S-1, S-2, \ldots, Q \\
\mathbf{\Phi^{(i+1)}} B + \mathbf{\Phi^{(i)}} A_0 &= \mathbf{0}, \quad i = Q-1, Q-2, \ldots, s+1 \\
\mathbf{\Phi^{(i+1)}} B + \mathbf{\Phi^{(i)}} A_1 &= \mathbf{0}, \quad i = s, s-1, \ldots, 1 \\
\mathbf{\Phi^{(i+1)}} B + \mathbf{\Phi^{(i)}} A_2 &= \mathbf{0}, \quad i = 0
\end{aligned}$$

After long simplifications, the above equations, except the first one, yield

$$\begin{aligned}
\boldsymbol{\phi}^{\boldsymbol{i}} &= \boldsymbol{\phi}^{(\mathbf{0})}(-1/B)^i A_2 A_1^{i-1}, && i = 1, 2, \cdots, s+1 \\
&= \boldsymbol{\phi}^{(\mathbf{0})}\left((-1/B)^i A_2 A_1^s A_0^{i-s-1}\right), && i = s+2, s+3, \cdots, Q \\
&= \boldsymbol{\phi}^{(\mathbf{0})}[(-1/B)^i A_2 A_1^s A_0^{i-s-1} + \\
&\quad (-1/B)^{i-Q} C A_0^{i-Q-1} + \\
&\quad (-1/B)^{i-Q} C A_2 (\sum_{j=1}^{i-Q-1} A_1^{j-1} A_0^{i-Q-j-1})] && i = Q+1, Q+2, \cdots, S
\end{aligned}$$

where $\boldsymbol{\phi}^{(\mathbf{0})}$ can be obtained by solving, $\mathbf{\Phi^{(S)}} A_0 + \mathbf{\Phi^{(s)}} C = 0$ and $\sum\limits_{i=0}^{S} \boldsymbol{\phi}^{(\boldsymbol{i})}\mathbf{e} = 1$, that is $\boldsymbol{\phi}^{(\mathbf{0})}[(-1/B)^S A_2 A_1^s A_0^Q + (-1/B)^s C A_0^s + (-1/B)^s C A_2 (\sum\limits_{j=1}^{s-1} A_1^{j-1} A_0^{s-j}) + (-1/B)^s A_2 A_1^{s-1} C] = \mathbf{0}$, and

$$\begin{aligned}
\phi^0[I + \sum_{i=1}^{s+1}((-1/B)^i A_2 A_1^{i-1}) &+ \sum_{i=s+2}^{Q}((-1/B)^i A_2 A_1^s A_0^{i-s-1}) \\
&+ \sum_{i=Q+1}^{S}[(-1/B)^i A_2 A_1^s A_0^{i-s-1} + (-1/B)^{i-Q} C A_0^{i-Q-1} + \\
&(-1/B)^{i-Q} C A_2 (\sum_{j=1}^{i-Q-1} A_1^{j-1} A_0^{i-Q-j-1})]]\mathbf{e} = 1.
\end{aligned}$$

4 System Performance Measures

In this section we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

4.1 Mean Inventory Level

Let ζ_1 denote the average inventory level in the steady state. Then we have,

$$\zeta_1 = \sum_{i=1}^{S} i \left(\sum_{k=0}^{N} \phi^{(i,k)} \right)$$

4.2 Mean Reorder Rate

Let ζ_2 denote the mean reorder rate, then we have,

$$\zeta_2 = \sum_{k=0}^{N} p_2 \mu \ \phi^{(s+1,k)}$$

4.3 Mean Waiting time

Let $\overline{W}$ denote the mean waiting time of the customers. Then, by Little's formula

$$\overline{W} = \frac{\Gamma}{\lambda_a}$$

where,

$$\Gamma = \sum_{k=1}^{N} k \left(\sum_{i=0}^{S} \phi^{(i,k)} \right).$$

The expected arrival rate is given by

$$\lambda_a = (N-k)\lambda \sum_{i=0}^{S} \sum_{k=0}^{N-1} \phi^{(i,k)}.$$

5 Cost Analysis

In order to compute the total expected cost per unit time, we introduce the following notations:

- c_h : The inventory carrying cost per unit per unit time.
- c_s : The setup cost per order.
- c_w : Waiting time cost of a customer per unit time.

Then the long-run expected cost rate is given by

$$TC(S, s, N) = c_h \zeta_1 + c_s \zeta_2 + c_w \overline{W}.$$

Substituting $\zeta' s$ and $\overline{W}$ into the above equation, we obtain

$$TC(S, s, N) = c_h \left(\sum_{i=1}^{S} i \left(\sum_{k=0}^{N} \phi^{(i,k)} \right) \right) + c_s \left(\sum_{k=0}^{N} p_2 \mu \phi^{(s+1,k)} \right)$$
$$+ c_w \left(\frac{\Gamma}{\lambda_a} \right)$$

6 Conclusion

In this paper, we discussed a continuous review stochastic (s, S) inventory system at a service facility with two types of services and a finite populations. The joint distribution of the number of customers in the system and the inventory level is obtained in the steady state case. Various system performance measures are derived and the total expected cost rate is derived under a suitable cost structure.

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