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A General Family of Generating Relations Involving Multivariable Hypergeometric Polynomials

Research Article

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Abstract: The aim of this paper is to derive two general theorems on generating relations for a certain sequence of functions. The generating functions for multivariable generalized hypergeometric polynomials are shown here as special cases of a general class of generating relations. A number of results associated with Jacobi, Khandekar polynomials of several variables and other results of multiple Gaussian hypergeometric functions scattered in the literature of special functions follow as applications of main results.

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1. Introduction and Preliminaries

Special functions play a vital role in both classical as well as quantum physics. There are many special functions, in which one of the most convenient depends on the particular problem at hand. Classical orthogonal polynomials play a central role in optics and certain parts of quantum mechanics.

In the endeavor of attempting to unify several results in the theory of polynomials, also in hypergeometric functions of two or more variables, Khandekar [15] defined the generalized Rice polynomials of one variable in the following form:-

$$H_n^{(\alpha,\beta)}[\xi,p,\nu]=\frac{(1+\alpha)_n}{n!}\,{}_3F_2\left[\begin{array}{lr}-n,\,1+\alpha+\beta+n,\,\xi\;; & \\ & \nu\\ 1+\alpha,\,p\;; & \end{array}\right] \tag{1.1}$$

where $n=0,1,2,3,\ldots,$ and $p\neq -n-1,-n-2,-n-3,\ldots.$

If $\alpha=\beta=0$ in (1.1), we get Rice polynomials [19]

$$H_n[\xi,p,\nu]=H_n^{(0,0)}[\xi,p,\nu] \tag{1.2}$$

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If $p=\xi$ and $\nu=\frac{1}{2}(1-x)$, (1.1) becomes:-

$$H_n^{(\alpha,\beta)}\left[\xi,\xi,\frac{1}{2}(1-x)\right]=P_n^{(\alpha,\beta)}(x) \tag{1.3}$$

or

$$H_n^{(\alpha,\beta)}[\xi,\xi,x]=P_n^{(\alpha,\beta)}(1-2x) \tag{1.4}$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomials defined as [18]:-

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{array}{lc} -n,\,1+\alpha+\beta+n\;\; ; & \\ & \frac{1-x}{2} \\ 1+\alpha \qquad\qquad\quad ; & \end{array}\right] \tag{1.5}$$

and ${}_pF_q$ denotes generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters, defined by [34]:-

$${}_pF_q\left[\begin{array}{lc} a_1,\,a_2,\,\ldots,a_p\;\; ; & \\ & x \\ b_1,\,b_2,\,\ldots,b_q\;\; ; & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\,(a_2)_n\cdots(a_p)_n}{(b_1)_n\,(b_2)_n\cdots(b_q)_n}\frac{x^n}{n!} \tag{1.6}$$

where $(a)_n$ is the Pochhammer symbol, defined as:-

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases} 1 & ;\ \text{if } n=0 \\ a(a+1)(a+2)\cdots(a+n-1) & ;\ \text{if } n=1,2,3,\cdots \end{cases} \tag{1.7}$$

The denominator parameters are neither zero nor negative integers, the numerator parameters may be zero and negative integers.

The generalized Jacobi polynomials of r variables of Shrivastava [21] are defined by:-

$$P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2;\ldots;\alpha_r,\beta_r)}(x_1,x_2,\ldots,x_r)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n\ldots(1+\alpha_r)_n}{(n!)^r}\times$$

$$\times F_{0:1;\ldots;1}^{1:1;\ldots;1}\left[\begin{array}{l} -n:1+\alpha_1+\beta_1+n;\ldots;1+\alpha_r+\beta_r+n;\\ -\;\;:\qquad 1+\alpha_1\qquad ;\ldots;\qquad 1+\alpha_r\qquad ;\end{array}\frac{1-x_1}{2},\ldots,\frac{1-x_r}{2}\right] \tag{1.8}$$

where multivariable Kampé de Fériet function $F_{0:1;\ldots;1}^{1:1;\ldots;1}[\ldots]$ is defined by equation (1.10).

Motivated by the works on different types of Hypergeometric polynomials of one and more variables, we define a generalized Rice's polynomials of r variables in the following form:-

$$H_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2;\ldots;\alpha_r,\beta_r)}\left[(b_{B_1}^{(1)});(d_{D_1}^{(1)}):\ldots:(b_{B_r}^{(r)});(d_{D_r}^{(r)}):x_1,\ldots,x_r\right]=\frac{(1+\alpha_1)_n(1+\alpha_2)_n\ldots(1+\alpha_r)_n}{(n!)^r}\times$$

$$\times F^{1:1+B_1;\ldots;1+B_r}_{0:1+D_1;\ldots;1+D_r}\left[\begin{array}{l} -n:1+\alpha_1+\beta_1+n,(b^{(1)}_{B_1});\ldots;1+\alpha_r+\beta_r+n,(b^{(r)}_{B_r});\\ -\;:\qquad 1+\alpha_1,(d^{(1)}_{D_1})\qquad;\ldots;\qquad 1+\alpha_r,(d^{(r)}_{D_r})\qquad; \end{array} x_1,\ldots,x_r\right] \tag{1.9}$$

where multivariable Kampé de Fériet function $F^{1:1+B_1;\ldots;1+B_r}_{0:1+D_1;\ldots;1+D_r}[\ldots]$ is defined by equation (1.10) and $(b^{(r)}_{B_r})$ denotes B_r parameters given by $b^{(r)}_1, b^{(r)}_2, \ldots, b^{(r)}_{B_r}$ with similar interpretation for others.

On specializing the parameters and arguments, the polynomials (1.9) reduce to generalized Sister Celine's polynomials of Shah [20], Fasenmyer Sister Celine's polynomials [7, 8], multivariable Sister Celine's polynomials of Shrivastava [22, 27–29], Bateman's polynomials $Z_n(x)$ and $F_n(z)$ [1–3], Pasternack's polynomials $F^{(m)}_n(z)$ [17], generalized Rice's polynomials of Khandekar $H^{(\alpha,\beta)}_n[\xi,p,\nu]$ [6, 15, 16, 30, 31], Rice's polynomials $H_n[\xi,p,\nu]$ [19], Classical Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$ [18], Classical Legendre polynomials $P_n(x)$ [18] and classical Ultraspherical (Gegenbauer) polynomials $C^{\nu}_n(x)$ [18], Classical Chebyshev polynomials of I kind $T_n(x)$ and II kind $U_n(x)$ [18], Cohen polynomials [4, 5], Extended Jacobi polynomials of Fujiwara $F^{(\alpha,\beta)}_n(x;a,b,c)$ [9], Khan polynomials [11–13], Khan-Shukla polynomials [14], multivariable Jacobi polynomials of Shrivastava's [21, 23–26] and others.

The multivariable extension of Kampé de Fériet function is given in the form [32, 34]

$$F^{p:q_1;q_2;\ldots;q_n}_{l:m_1;m_2;\ldots;m_n}\left[\begin{array}{l} (a_p):(b^{(1)}_{q_1});\ldots;(b^{(n)}_{q_n});\\ \\ (\alpha_l):(\beta^{(1)}_{m_1});\ldots;(\beta^{(n)}_{m_n}); \end{array} x_1,\ldots,x_n\right]=\sum_{s_1,\ldots,s_n=0}^{\infty}\Lambda(s_1,\ldots,s_n)\frac{{x_1}^{s_1}}{s_1!}\cdots\frac{{x_n}^{s_n}}{s_n!} \tag{1.10}$$

$$\Lambda(s_1,\ldots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\ldots+s_n}\prod_{j=1}^{q_1}(b^{(1)}_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b^{(n)}_j)_{s_n}}{\prod_{j=1}^{l}(\alpha_j)_{s_1+\ldots+s_n}\prod_{j=1}^{m_1}(\beta^{(1)}_j)_{s_1}\cdots\prod_{j=1}^{m_n}(\beta^{(n)}_j)_{s_n}}$$

and, for convergence of the multiple hypergeometric series in (1.10),

$$1+l+m_k-p-q_k \geqq 0, \qquad k=1,\ldots,n;$$

the equality holds when, in addition, either

$$p>l \qquad \text{and} \qquad |x_1|^{\frac{1}{p-l}}+\ldots+|x_n|^{\frac{1}{p-l}}<1$$

or

$$p\leqq l \qquad \text{and} \qquad max\{|x_1|,\ldots,|x_n|\}<1$$

A further generalization of the Kampé de Fériet function is due to Srivastava and Daoust who indeed, defined an extension of Wright's ${}_p\psi_q$ function in two variables. More generally recalling here the following extension of the Wright's function ${}_p\psi_q$ in *several* variables, which is referred in the literature as the generalized Lauricella function of several variables, it is also due to Srivastava and Daoust [32].

$$F^{A\,:\,B^{(1)}\,;\ldots;\,B^{(n)}}_{C\,:\,D^{(1)}\,;\ldots;\,D^{(n)}}\left(\begin{array}{l} [(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)}):\phi^{(1)}];\ldots;[(b^{(n)}):\phi^{(n)}];\\ \\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]:[(d^{(1)}):\delta^{(1)}];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{array} z_1,\ldots,z_n\right)$$

$$= \sum_{m_1,\ldots,\ m_n=0}^{\infty} \Omega(m_1,\ldots,\ m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!} \tag{1.11}$$

where, for convenience,

$$\Omega(m_1,\ldots,\ m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \tag{1.12}$$

the coefficients

$$\begin{cases} \theta_j^{(k)}, j=1,\ldots,A; \phi_j^{(k)}, j=1,\ldots,B^{(k)}; \psi_j^{(k)}, j=1,\ldots,C; \\ \qquad \delta_j^{(k)}, j=1,\ldots,D^{(k)}; \forall\ \ k\ \in\{1,\ldots,n\} \end{cases} \tag{1.13}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}, j=1,\ldots,B^{(k)}; \forall\ \ k\ \in\{1,\ldots,n\}$$

with similar interpretations for (c) and $(d^{(k)}), k=1,\ldots,n;$*et cetera*

For the precise conditions under with the multiple hypergeometric series in (1.11) converges absolutely [10, 33].

It has been observed that the generating functions play a remarkable role in the study of the polynomial sets. Therefore, an attempt has been made to establish some interesting generating relations for generalized Hypergeometric polynomials by making use of series iteration techniques in the present study.

2. General Multiple Series Identities

Theorem 2.1. *Let $\{S_q(n,k_1,k_2,\ldots,k_m)\}$, $q=1,2,3$ be bounded multiple sequences of arbitrary real or complex numbers, for every n, $k_r\in\{0,1,2,\ldots\}$; $r=1,2,3,\ldots,m$. Also let $(\lambda)_n$ type notations denote the Pochhammer symbol defined by (1.7) and $z_1,z_2,\ldots,z_m$ are several complex variables; $I_1,I_2,\ldots,I_m$ are arbitrary positive integers, then*

$$\sum_{n=0}^{\infty} S_1(n)\sum_{k_1,k_2,\ldots,k_m=0}^{L\le n}(-n)_L\,S_2(k_1,k_2,\ldots,k_m)\prod_{j=1}^{m}\left\{(\theta_j+n)_{k_j}\frac{{z_j}^{k_j}}{(k_j)!}\right\}\frac{t^n}{n!}$$

$$= \sum_{n,k_1,k_2,\ldots,k_m=0}^{\infty} S_1(n+L)\,S_2(k_1,k_2,\ldots,k_m)\prod_{j=1}^{m}\left\{\frac{(\theta_j)_{n+L+k_j}(-t)^{I_jk_j}\,{z_j}^{k_j}}{(\theta_j)_{n+L}\,(k_j)!}\right\}\frac{t^n}{n!} \tag{2.1}$$

provided that each of the multiple series involved in (2.1) converges absolutely and $L=I_1k_1+I_2k_2+\cdots+I_mk_m$.

Proof. The L.H.S. of equation (2.1) can be written as:-

$$T=\sum_{n=0}^{\infty}S_1(n)\sum_{k_1,k_2,\ldots,k_m=0}^{I_1k_1+I_2k_2+\cdots+I_mk_m\le n}(-n)_{I_1k_1+I_2k_2+\cdots+I_mk_m}S_2(k_1,k_2,\ldots,k_m)(\theta_1+n)_{k_1}\frac{z_1^{k_1}}{(k_1)!}\times$$

$$\times\,(\theta_2+n)_{k_2}\frac{z_2^{k_2}}{(k_2)!}\cdots(\theta_m+n)_{k_m}\frac{z_m^{k_m}}{(k_m)!}\frac{t^n}{n!} \tag{2.2}$$

Replacing n by $n + I_1k_1 + I_2k_2 + \cdots + I_mk_m$ in (2.2) and using the lemma [34]

$$\sum_{n=0}^{\infty} \sum_{k_1,k_2,\ldots,k_m=0}^{I_1k_1+I_2k_2+\cdots+I_mk_m \leq n} A(n; k_1, k_2, \ldots, k_m)$$

$$= \sum_{n=0}^{\infty} \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} A(n + I_1k_1 + I_2k_2 + \cdots + I_mk_m; k_1, k_2, \ldots, k_m) \tag{2.3}$$

we get:-

$$T = \sum_{n=0}^{\infty} \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} S_1(n + I_1k_1 + I_2k_2 + \cdots + I_mk_m) S_2(k_1, k_2, \ldots, k_m) \times$$

$$\times (\theta_1 + n + I_1k_1 + I_2k_2 + \cdots + I_mk_m)_{k_1} \frac{(-t)^{I_1k_1} {z_1}^{k_1}}{(k_1)!} (\theta_2 + n + I_1k_1 + I_2k_2 + \cdots + I_mk_m)_{k_2} \frac{(-t)^{I_2k_2} {z_2}^{k_2}}{(k_2)!} \cdots \times$$

$$\times\ (\theta_m + n + I_1k_1 + I_2k_2 + \cdots + I_mk_m)_{k_m} \frac{(-t)^{I_mk_m} {z_m}^{k_m}}{(k_m)!} \frac{t^n}{n!} \tag{2.4}$$

Now using the definition of Pochhammer symbol $\Gamma(\alpha + n) = \Gamma(\alpha)(\alpha)_n$, we get the right hand side of (2.1).

This identity completes the proof of theorem (2.1) under the assumption that the series involved, are absolutely convergent.

□

Theorem 2.2. *Under the hypotheses stated of Theorem 2.1, we have:-*

$$\sum_{n=0}^{\infty} \sum_{k_1,k_2,\ldots,k_m=0}^{L \leq n} \frac{S_3(n-L)(\lambda + n - L)_{k_1+k_2+\ldots+k_m}}{(n-L)!} \Big\{ \prod_{j=1}^{m} \frac{{z_j}^{k_j}}{(k_j)!} \Big\} t^n$$

$$= \Big(1 - \sum_{j=1}^{m} z_j\ t^{I_j}\Big)^{-\lambda} \sum_{n=0}^{\infty} \frac{S_3(n)}{n!} \left(\frac{t}{1 - \sum_{j=1}^{m} z_j\ t^{I_j}} \right)^n \tag{2.5}$$

provided that each of the multiple series involved in (2.5) converges absolutely and $L = I_1k_1 + I_2k_2 + \cdots + I_mk_m$.

It is important to note that theorems proved here, are of very general nature.

Proof. The L.H.S. of equation (2.5) can be written as:-

$$R = \sum_{n=0}^{\infty} \sum_{k_1,k_2,\ldots,k_m=0}^{I_1k_1+I_2k_2+\cdots+I_mk_m \leq n} S_3(n - I_1k_1 - I_2k_2 - \cdots - I_mk_m) \times$$

$$\times \frac{(\lambda + n - I_1k_1 - I_2k_2 - \cdots - I_mk_m)_{k_1+k_2+\cdots+k_m}\ {z_1}^{k_1}\ {z_2}^{k_2}\ \cdots\ {z_m}^{k_m}}{(n - I_1k_1 - I_2k_2 - \cdots - I_mk_m)!\ (k_1)!(k_2)! \cdots (k_m)!} t^n \tag{2.6}$$

Replacing n by $n + I_1k_1 + I_2k_2 + \cdots + I_mk_m$ in (2.6) and using the lemma (2.3), we get

$$R = \sum_{n=0}^{\infty} S_3(n) \frac{t^n}{n!} \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} \frac{(\lambda + n)_{k_1+k_2+\cdots+k_m} (z_1t^{I_1})^{k_1} (z_2t^{I_2})^{k_2} \cdots (z_mt^{I_m})^{k_m}}{(k_1)!(k_2)! \cdots (k_m)!} \tag{2.7}$$

Now using the reduction formula [34], we get:-

$$R = \sum_{n=0}^{\infty} S_3(n) \frac{t^n}{n!} \sum_{p=0}^{\infty} \frac{(\lambda + n)_p [z_1t^{I_1} + z_2t^{I_2} + \cdots + z_mt^{I_m}]^p}{p!}$$

$$= \sum_{n=0}^{\infty} S_3(n) \frac{t^n}{n!} (1 - z_1t^{I_1} - z_2t^{I_2} - \cdots - z_mt^{I_m})^{-\lambda-n}$$

$$= (1 - z_1t^{I_1} - z_2t^{I_2} - \cdots - z_mt^{I_m})^{-\lambda} \sum_{n=0}^{\infty} \frac{S_3(n)}{n!} \left(\frac{t}{1 - z_1t^{I_1} - z_2t^{I_2} - \cdots - z_mt^{I_m}} \right)^n$$

which is the right hand side of (2.5). □

3. Special Cases

It is very interesting that formulae proved here, is leading to certain generalizations of the various well-known results due to Shrivastava, Rice et-al.

(i) Setting $m = 3, I_1 = I_2 = I_3 = 1, z_1 = x, z_2 = y, z_3 = z, \theta_1 = 1 + \alpha_1 + \beta_1, \theta_2 = 1 + \alpha_2 + \beta_2, \theta_3 = 1 + \alpha_3 + \beta_3$

$$S_1(n) = (1 + \alpha_1 + \beta_1)_n (1 + \alpha_2 + \beta_2)_n (1 + \alpha_3 + \beta_3)_n$$

and

$$S_2(k_1, k_2, k_3) = \frac{(\nu_1)_{k_1} (\nu_2)_{k_2} (\nu_3)_{k_3}}{(\sigma_1)_{k_1} (1 + \alpha_1)_{k_1} (\sigma_2)_{k_2} (1 + \alpha_2)_{k_2} (\sigma_3)_{k_3} (1 + \alpha_3)_{k_3}}$$

in Theorem 2.1, interpreting L.H.S. in terms of generalized Rice polynomials (for r = 3) defined by (1.9) and R.H.S. in terms of Srivastava-Daoust function(1.11), we get:-

$$\sum_{n=0}^{\infty} \prod_{i=1}^{3} \left\{ \frac{(1 + \alpha_i + \beta_i)_n}{(1 + \alpha_i)_n} \right\} (n!)^2 H_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3)} [\nu_1; \sigma_1 : \nu_2; \sigma_2 : \nu_3; \sigma_3 : x, y, z]\, t^n$$

$$= F^{3:0;1;1;1}_{0:0;2;2;2} \left(\begin{array}{l} [1 + \alpha_1 + \beta_1 : 1, 2, 1, 1],\ [1 + \alpha_2 + \beta_2 : 1, 1, 2, 1],\ [1 + \alpha_3 + \beta_3 : 1, 1, 1, 2] : \\ \rule{8cm}{0.4pt} : \\ \text{——} ;\quad [\nu_1 : 1] \quad ; \quad [\nu_2 : 1] \quad ; \quad [\nu_3 : 1] \quad ; \\ \text{——} ; [1 + \alpha_1 : 1], [\sigma_1 : 1] ; [1 + \alpha_2 : 1], [\sigma_2 : 1] ; [1 + \alpha_3 : 1], [\sigma_3 : 1] ; \end{array} \; t, -xt, -yt, -zt \right) \tag{3.1}$$

(ii) Setting $m = 2, I_1 = I_2 = 1, z_1 = x, z_2 = y, \theta_1 = 1 + \alpha_1 + \beta_1, \theta_2 = 1 + \alpha_2 + \beta_2,$

$$S_1(n) = (1 + \alpha_1 + \beta_1)_n$$

and

$$S_2(k_1, k_2) = \frac{(\nu_1)_{k_1} (\nu_2)_{k_2}}{(\sigma_1)_{k_1} (1 + \alpha_1)_{k_1} (\sigma_2)_{k_2} (1 + \alpha_2)_{k_2}}$$

in Theorem 2.1, interpreting L.H.S. in terms of generalized Rice polynomials (for r = 2) defined by (1.9) and using Euler's first linear transformation [18] in the R.H.S., we get:-

$$\sum_{n=0}^{\infty} \left\{ \frac{(1 + \alpha_1 + \beta_1)_n (n!)}{(1 + \alpha_1)_n (1 + \alpha_2)_n} \right\} H_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)} [\nu_1; \sigma_1 : \nu_2; \sigma_2 : x, y]\, t^n$$

$$= (1 - t)^{-1 - \alpha_1 - \beta_1} F^{3:0;1;0}_{3:0;2;0} \left(\begin{array}{l} [1 + \alpha_1 + \beta_1 : 2, 2, 1],\ [1 + \alpha_2 + \beta_2 : 2, 1, 2],\ [\nu_2 : 1, 0, 1] : \\ [1 + \alpha_2 + \beta_2 : 2, 1, 1],\ [1 + \alpha_2 : 1, 0, 1] \quad , \ [\sigma_2 : 1, 0, 1] : \end{array} \right.$$

$$\left.\begin{array}{l} \text{———};\quad [\nu_1:1]\quad;\text{———}; \\ \\ \text{———};[\sigma_1:1],[1+\alpha_1:1];\text{———}; \end{array} \frac{-yt^2}{(1-t)^2},\frac{-xt}{(1-t)^2},\frac{-yt}{(1-t)}\right) \tag{3.2}$$

$$= (1-t)^{-1-\alpha_2-\beta_2} F^{3:0;0;1}_{3:0;0;2}\left(\begin{array}{l} [1+\alpha_2+\beta_2:2,1,2],\ [1+\alpha_1+\beta_1:2,2,1],\ [\nu_1:1,1,0]: \\ \\ [1+\alpha_2+\beta_2:2,1,1],\ [1+\alpha_1:1,1,0]\quad,\ [\sigma_1:1,1,0]: \end{array}\right.$$

$$\left.\begin{array}{l} \text{———};\text{———};\quad [\nu_2:1]\quad; \\ \\ \text{———};\text{———};[\sigma_2:1],[1+\alpha_2:1]; \end{array} \frac{-xt^2}{(1-t)^2},\frac{-xt}{(1-t)},\frac{-yt}{(1-t)^2}\right) \tag{3.3}$$

(iii) In equation (3.1), replacing x by $\frac{1-x}{2}$, y by $\frac{1-y}{2}$, z by $\frac{1-z}{2}$, $\nu_1 = \sigma_1, \nu_2 = \sigma_2, \nu_3 = \sigma_3$ and using the definition of generalized Jacobi polynomials (r = 3) defined by (1.8), we get a known generating relation of Shrivastava [21]:-

$$\sum_{n=0}^{\infty}\prod_{i=1}^{3}\left\{\frac{(1+\alpha_i+\beta_i)_n}{(1+\alpha_i)_n}\right\}(n!)^2 P_n^{(\alpha_1,\beta_1\,;\,\alpha_2,\beta_2\,;\,\alpha_3,\beta_3)}(x,y,z)\,t^n$$

$$= F^{3:0;0;0;0}_{0:0;1;1;1}\left(\begin{array}{l} [1+\alpha_1+\beta_1:1,2,1,1],\ [1+\alpha_2+\beta_2:1,1,2,1],\ [1+\alpha_3+\beta_3:1,1,1,2]: \\ \\ \text{—————————————————————}: \end{array}\right.$$

$$\left.\begin{array}{l} \text{———};\text{————};\text{————};\text{————}; \\ \\ \text{———};[1+\alpha_1:1];[1+\alpha_2:1];[1+\alpha_3:1]; \end{array} t,\ \frac{(x-1)t}{2},\ \frac{(y-1)t}{2},\ \frac{(z-1)t}{2}\right) \tag{3.4}$$

(iv) In (Theorem 2.1) setting $m = 1$, $I_1 = 1$, $\theta_1 = (1+\alpha+\beta)$, $S_1(n) = (1+\alpha+\beta)_n$, $S_2(k_1) = \frac{(\xi)_{k_1}}{(1+\alpha)_{k_1}(p)_{k_1}}$, $z_1 = \nu$, using the definition (1.1) of Khandekar polynomials of one variable, we get:-

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}H_n^{(\alpha,\beta)}[\xi,p,\nu]\,t^n = (1-t)^{-1-\alpha-\beta}\,{}_3F_2\left[\begin{array}{l} \Delta(2;1+\alpha+\beta),\ \xi\ ; \\ \\ 1+\alpha,\ p\ ; \end{array} \frac{-4\nu t}{(1-t)^2}\right] \tag{3.5}$$

which is well known generating function [15] for the generalized Rice polynomial defined by (1.1), where $\Delta(N;a) = \frac{a}{N}, \frac{a+1}{N}, \cdots, \frac{a+N-1}{N}$.

The equation (3.5) was obtained by Khandekar [15], using Beta integral technique.

(v) In (Theorem 2.1) setting $m = 1$, $I_1 = 1$, $\theta_1 = (1+\alpha+\beta)$, $S_1(n) = (1+\alpha+\beta)_n$, $S_2(k_1) = \frac{1}{(1+\alpha)_{k_1}}$, $z = \frac{1-x}{2}$, using the definition (1.5) of classical Jacobi polynomials, we get:-

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}P_n^{(\alpha,\beta)}(x)\,t^n = (1-t)^{-1-\alpha-\beta}\,{}_2F_1\left[\begin{array}{l} \frac{(1+\alpha+\beta)}{2},\ \frac{(2+\alpha+\beta)}{2}\ ; \\ \\ 1+\alpha\ ; \end{array} \frac{2(x-1)t}{(1-t)^2}\right] \tag{3.6}$$

which is well known generating relation [18] for the Jacobi polynomials defined by (1.5) and can also be obtained directly from (3.4).

(vi) In (Theorem 2.2) setting $m = 2$, $I_1 = I_2 = 1$, putting $S_3(n) = \frac{\prod_{i=1}^{A}(a_i)_n}{\prod_{j=1}^{B}(b_j)_n}$, using the double series identity of Srivastava [34] and replacing $(z_1 + z_2)$ by z, we get

$$\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_n}{\prod_{j=1}^{B}(b_j)_n} {}_{B+2}F_A \left[\begin{array}{lc} -n, 1-\lambda-n, 1-(b_B)-n \ ; & \\ & (-1)^{A+B}z \\ 1-(a_A)-n \ ; & \end{array} \right] \frac{t^n}{n!}$$

$$= (1-zt)^{-\lambda} {}_AF_B \left[\begin{array}{lc} (a_A) \ ; & \\ & \frac{t}{(1-zt)} \\ (b_B) \ ; & \end{array} \right] \tag{3.7}$$

which is a particular case of known generating relation of Srivastava [31, 34]].

For different values of $m, I_1, I_2, I_3, \ldots$ and bounded sequences we can derive a number of known and unknown generating relations involving generalized hypergeometric polynomials and Kampé de Fériet function of two variables. Srivastava function $F^{(3)}$ and its special cases, Exton's double hypergeometric functions and multivariable Srivastava-Daoust function.

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