Characterization of the Generalized Weibull-Gompertz Distribution Based on the Upper Record Values

Research Article

A.H.El-Bassiouny[1], Medhat EL-Damcese[2], Abdelfattah Mustafa[1] and M.S.Eliwa[1]*

1 Mathematics Department, Faculty of Science, Mansoura University, Egypt.

2 Mathematics Department, Faculty of Science, Tanta University, Egypt.

Abstract: This paper introduces characterization of the generalized Weibull-Gompertz distribution based on the upper record values. Several properties are studied in this paper such as reversed (hazard) function, moments, maximum likelihood estimation, mean residual (past) lifetime. A real data set is analyzed.

Keywords: Upper record values, reversed (hazard) function, generalized Weibull-Gompertz distribution, mean residual (past) lifetime.
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1. Introduction

Record values and the associated statistics are of importance in many life fields such that weather, economic, sports data "Olympic records or world records in sport"and life testing. Also, one of the most important applications of records in reliability studies, the structure $k - out - of - n : G(F)$ where the structure functions (fails) when or more out of the components are functioning (failing).

This structure includes the cases of series and parallel systems when $k = n$ and $k = 1$, respectively. Suppose we are interested in predicting the lifetime of the system when r components have failed, for some $1 \leq r < k \leq n$. The lifetime of the system corresponds to the $(n - k - r + 1)^{th}$ future record value. On the other hand, in industry many products fail under stress, an electronic component ceases to function in an environment of too high temperature and a battery dies under the stress of time.

But the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only the record values are observed. Thus, the number of measurements made is considerably smaller than the complete sample size. This "measurement saving"can be important when the measurements of these experiments are costly if the entire sample was destroyed. So; many authors have studied record values and the associated statistics; see, for example, Balakrishnan and Chan [4], Pawlas and Szynal [8], Raqab [9], Al Zaid and Ahsanullah [2], Ahsanullah [1], Soliman et al. [10] and Grine Azedine [5].

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* *E-mail: msabereliwa@mans.edu.eg*

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with CDF $F(x)$ and PDF $f(x)$.

Set $Y_n = \max(X_1, X_2, \ldots, X_n)$, $n \geq 1$, X_j is said to be an upper record and is denoted by $X_{U(j)}$ if $Y_j > Y_{j-1}$, $j > 1$. We can transform from upper records to lower records by replacing the original sequence of $\{X_j\}$ by $\{-X_i, i \geq 1\}$.

2.   Generalized Weibull-Gompertz Distribution(GWGD)

The random variable X is said to be has GWGD if it has the following CDF for $a, b, c, d > 0$ as follows:

$$F_X(x; a, b, c, d, \theta) = 1 - e^{-ax^b\left(e^{cx^d}-1\right)}, x > 0 \tag{1}$$

where b and d are shape parameters, a is scale parameter and c is an acceleration parameter. The probability density function $f_X(x; a, b, c, d)$ is

$$f_X(x; a, b, c, d) = abx^{b-1}e^{-ax^b\left(e^{cx^d}-1\right)+cx^d}\left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right). \tag{2}$$

The survival function can be obtained as follows

$$R(x; a, b, c, d) = e^{-ax^b\left(e^{cx^d}-1\right)}, x > 0 \tag{3}$$

The hazard function $h(x)$ is

$$h(x; a, b, c, d, \theta) = abx^{b-1}e^{cx^d}\left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right). \tag{4}$$

The reversed hazard function $r(x)$ is

$$r(x; a, b, c, d) = \frac{abx^{b-1}e^{-ax^b\left(e^{cx^d}-1\right)+cx^d}\left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right)}{1 - e^{-ax^b\left(e^{cx^d}-1\right)}}. \tag{5}$$

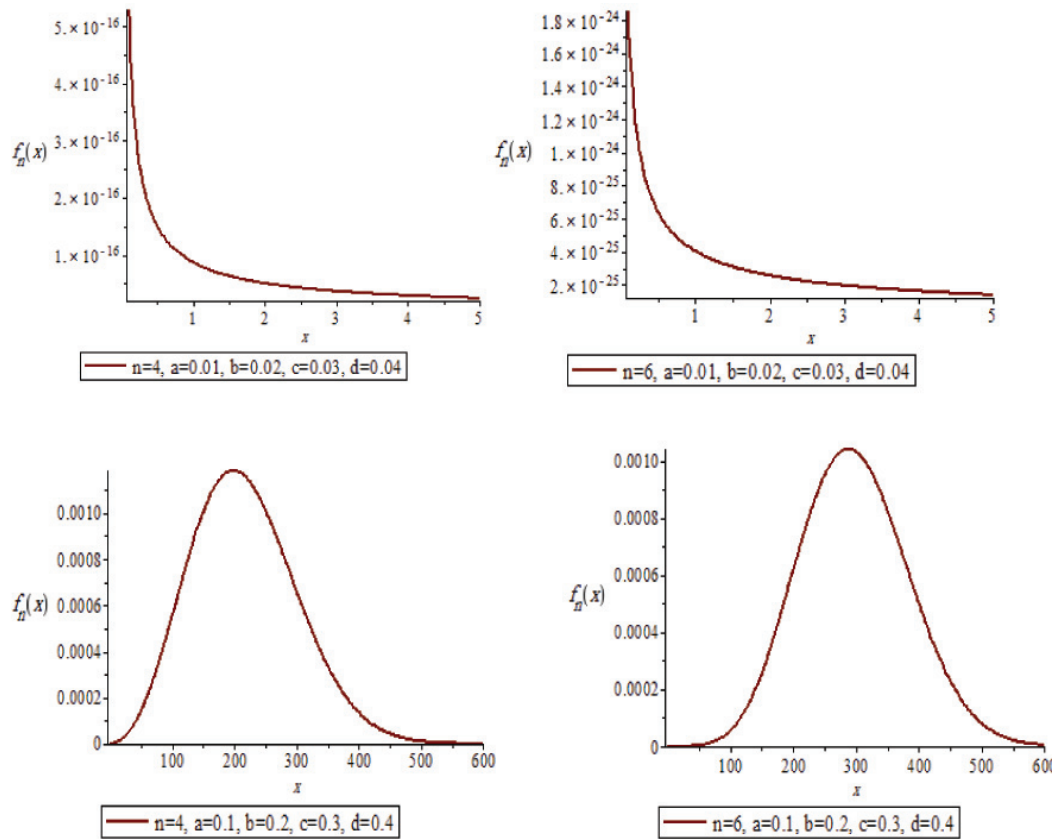### 2.1.   The Cumulative Distribution Function of GWGD Based on the Upper Record Values

Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the first upper record values of size n then the probability density function $f_n(x)$ of upper record values $X_{U(n)}$ is

$$f_n(x) = \frac{1}{\Gamma(n)}f(x)(-\ln R(x))^{n-1}, -\infty < x < \infty \tag{6}$$

substituting from Equations (2) and (3) into Equation (6), we get the probability density function of the nth upper record $X_{U(n)}$ is given by

$$f_n(x) = \frac{a^n b}{\Gamma(n)}x^{nb-1}\left(1 - e^{-cx^d}\right)^{n-1}\left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right)e^{-ax^b\left(e^{cx^d}-1\right)+ncx^d}, x > 0. \tag{7}$$

Figure 1 provides the PDFs of GWGD(a, b, c, d) based on the upper record values for different parameter values, it is immediate that the PDFs can be decreasing and unimodal.

Figure 1. The PDF plots for upper record values from GWGD

Lemma 2.1. *Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the first upper record values of size n which have GWGD then, the cumulative distribution function $F_n(x)$ of upper record values $X_{U(n)}$ is*

$$F_n(x) = \frac{a^n b}{\Gamma(n)} \sum_{i=o}^{\infty} \sum_{j=o}^{n+i-1} \sum_{l=o}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i!l!} \binom{n+i-1}{j} x^{bi+ld+nb} \times$$

$$\left(\frac{(n+i-j)^l - (n+i-j-1)^l}{bi+ld+nb} + \frac{cd(n+i-j)^l x^d}{b(bi+ld+d+nb)} \right). \tag{8}$$

Proof.

$$F_n(x) = P(X \leq x) = \int_0^x f_n(x) dx$$

$$= \frac{a^n b}{\Gamma(n)} \int_0^x x^{nb-1} \left(1 - e^{-cx^d}\right)^{n-1} \left(1 + \frac{cd}{b} x^d - e^{-cx^d}\right) e^{-ax^b \left(e^{cx^d} - 1\right) + ncx^d} dx$$

$$= \frac{a^n b}{\Gamma(n)} \mathbf{I_1} + \frac{a^n cd}{\Gamma(n)} \mathbf{I_2} - \frac{a^n b}{\Gamma(n)} \mathbf{I_3} \tag{9}$$

where

$$\mathbf{I_1} = \int_0^x x^{nb-1} \left(1 - e^{-cx^d}\right)^{n-1} e^{-ax^b \left(e^{cx^d} - 1\right) + ncx^d} dx$$

$$\mathbf{I_2} = \int_0^x x^{nb+d-1} \left(1 - e^{-cx^d}\right)^{n-1} e^{-ax^b \left(e^{cx^d} - 1\right) + ncx^d} dx$$

$$\mathbf{I_3} = \int_0^x x^{nb-1} \left(1 - e^{-cx^d}\right)^{n-1} e^{-ax^b \left(e^{cx^d} - 1\right) + (n-1)cx^d} dx$$

Now, we want to get the value of $\mathbf{I_1}$, since

$$e^{-ax^b\left(e^{cx^d}-1\right)} = \sum_{i=0}^{\infty} \frac{(-1)^i a^i x^{bi}}{i!}\left(e^{cx^d}-1\right)^i = \sum_{i=0}^{\infty} \frac{(-1)^i a^i x^{bi}}{i!} e^{icx^d}\left(1-e^{-cx^d}\right)^i$$

then

$$\begin{aligned}
\mathbf{I_1} &= \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \int_0^x x^{nb+bi-1}\left(1-e^{-cx^d}\right)^{n+i-1} e^{(n+i)cx^d} dx \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \frac{(-1)^{i+j} a^i}{i!}\begin{pmatrix} n+i-1 \\ j \end{pmatrix} \int_0^x x^{nb+bi-1} e^{(n+i-j)cx^d} dx \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j}(n+i-j)^l c^l a^i}{i!l!(nb+bi+ld+)}\begin{pmatrix} n+i-1 \\ j \end{pmatrix} x^{nb+bi+ld}.
\end{aligned} \tag{10}$$

Similarly;

$$\mathbf{I_2} = \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j}(n+i-j)^l c^l a^i}{i!l!(nb+bi+ld+d)}\begin{pmatrix} n+i-1 \\ j \end{pmatrix} x^{nb+bi+ld+d}. \tag{11}$$

$$\mathbf{I_3} = \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j}(n+i-j-1)^l c^l a^i}{i!l!(nb+bi+ld)}\begin{pmatrix} n+i-1 \\ j \end{pmatrix} x^{nb+bi+ld}. \tag{12}$$

Substituting from Equations (10), (11) and (12) into Equation (9), we get Equation (8).

On the other hand, the reliability function $R_n(x)$ of upper record values $X_{U(n)}$ is

$$\begin{aligned}
R_n(x) = 1 - \frac{a^n b}{\Gamma(n)} \sum_{i=o}^{\infty} \sum_{j=o}^{n+i-1} \sum_{l=o}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i!l!}\begin{pmatrix} n+i-1 \\ j \end{pmatrix} x^{bi+ld+nb} \times \\
\left(\frac{(n+i-j)^l - (n+i-j-1)^l}{bi+ld+nb} + \frac{cd(n+i-j)^l x^d}{b(bi+ld+d+nb)}\right)
\end{aligned} \tag{13}$$

and the hazard rate function $h_n(x)$ is

$$h_n(x) = \frac{\frac{a^n b}{\Gamma(n)} x^{nb-1}\left(1-e^{-cx^d}\right)^{n-1}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right) e^{-ax^b\left(e^{cx^d}-1\right)+ncx^d}}{1 - \frac{a^n b}{\Gamma(n)} \sum_{i=o}^{\infty} \sum_{j=o}^{n+i-1} \sum_{l=o}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i!l!}\begin{pmatrix} n+i-1 \\ j \end{pmatrix} x^{bi+ld+nb} \times G} \tag{14}$$

Also; the reversed hazard rate function $r_n(x)$ is

$$h_n(x) = \frac{\frac{a^n b}{\Gamma(n)} x^{nb-1}\left(1-e^{-cx^d}\right)^{n-1}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right) e^{-ax^b\left(e^{cx^d}-1\right)+ncx^d}}{\frac{a^n b}{\Gamma(n)} \sum_{i=o}^{\infty} \sum_{j=o}^{n+i-1} \sum_{l=o}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i!l!}\begin{pmatrix} n+i-1 \\ j \end{pmatrix} x^{bi+ld+nb} \times G} \tag{15}$$

where

$$G = \left(\frac{(n+i-j)^l - (n+i-j-1)^l}{bi+ld+nb} + \frac{cd(n+i-j)^l x^d}{b(bi+ld+d+nb)}\right)$$

The following Figure 2 provides the hazard rates functions of GWGD(a, b, c, d) based on the upper record values for different parameters values, it is immediate that the hazard rates functions can be decreasing and increasing.
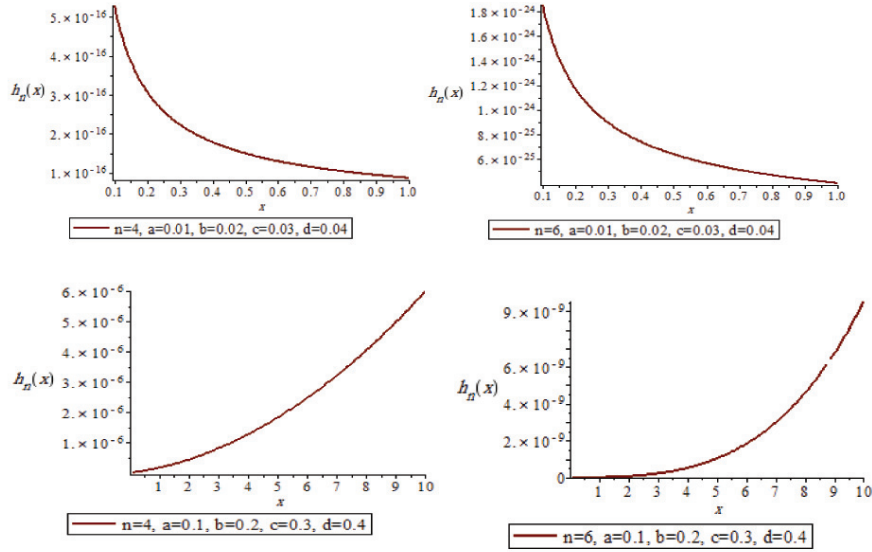
Figure 2. The hazard rate function plots for upper record values from GWGD

$\square$

3. Statistical Properties of GWGD Based on the Upper Record Values

3.1. The Median and Mode

It is observed that the mean of GWGD based on the upper record values cannot be obtained in explicit forms. It can be obtained as infinite series expansion so, in general different moments of GWGD based on the upper record values. Also, we cannot get the quartile x_q in a closed form by using the equation $F_n(x_q; a, b, c, d) - q = 0$. Thus, by using Equation (8), we find that

$$\sum_{i=o}^{\infty} \sum_{j=o}^{n+i-1} \sum_{l=o}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i!l!} \binom{n+i-1}{j} x_q^{bi+ld+nb} \times$$

$$\left(\frac{(n+i-j)^l - (n+i-j-1)^l}{bi+ld+nb} + \frac{cd(n+i-j)^l x_q^d}{b(bi+ld+d+nb)} \right) = \frac{q\Gamma(n)}{a^n b} \tag{16}$$

The median $m(X)$ of GWGD based on the upper record values can be obtained from Equation (16), when $q = 0.5$, as follows

$$\sum_{i=o}^{\infty} \sum_{j=o}^{n+i-1} \sum_{l=o}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i!l!} \binom{n+i-1}{j} x_{0.5}^{bi+ld+nb} \times$$

$$\left(\frac{(n+i-j)^l - (n+i-j-1)^l}{bi+ld+nb} + \frac{cd(n+i-j)^l x_{0.5}^d}{b(bi+ld+d+nb)} \right) = \frac{q\Gamma(n)}{2ba^n} \tag{17}$$

Moreover, the mode of GWGD based on the upper record values can be obtained as a solution of the following nonlinear equation

$$\frac{d}{dx} f_n(x : a, b, c, d) = 0$$

$$\frac{d}{dx} \left[x^{nb-1} \left(1 - e^{-cx^d} \right)^{n-1} \left(1 + \frac{cd}{b} x^d - e^{-cx^d} \right) e^{-ax^b \left(e^{cx^d} - 1 \right) + ncx^d} \right] = 0. \tag{18}$$

It is impossible to obtain the explicit solution in general case. It has to be obtained numerically.

3.2. Moments

The following Lemma 3.1 gives the rth moment of the upper record value $X_{U(n)}$ from size biased GWGD.

Lemma 3.1. *Let* $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ *be the first upper record values of size n which have GWGD then, the* r^{th} *moment of the* n^{th} *upper record value* $X_{U(n)}$*, say* $\mu'_{r(n)}$*, is given as follows for* $a, b, c, d > 0$ *and* $x > 0$

$$\mu'_{r(n)} = \frac{a^n b}{\Gamma(n)} \sum_{i=o}^{\infty} \sum_{j=o}^{n+i-1} \sum_{l=o}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i!\, l!d(cj)^{\frac{r+(n+i)b+ld}{d}}} \begin{pmatrix} n+i-1 \\ j \end{pmatrix} \Gamma\left(\frac{r+(n+i)b+ld}{d}\right) \times$$

$$\left((n+i)^l \left\{ 1 + \frac{r+(n+i)b+ld}{bj} \right\} - (n+i-1)^l \right). \tag{19}$$

Proof.

$$\mu'_{r(n)} = \int_0^{\infty} x^r f_n(x; a, b, c, d) dx$$

$$= \frac{a^n b}{\Gamma(n)} \int_0^{\infty} x^{r+nb-1} \left(1 - e^{-cx^d}\right)^{n-1} \left(1 + \frac{cd}{b} x^d - e^{-cx^d}\right) e^{-ax^b\left(e^{cx^d}-1\right)+ncx^d} dx$$

$$\mu'_{r(n)} = \frac{a^n b}{\Gamma(n)} \mathbf{I_1} + \frac{a^n cd}{\Gamma(n)} \mathbf{I_2} - \frac{a^n b}{\Gamma(n)} \mathbf{I_3} \tag{20}$$

where

$$\mathbf{I_1} = \int_0^{\infty} x^{r+nb-1} \left(1 - e^{-cx^d}\right)^{n-1} e^{-ax^b\left(e^{cx^d}-1\right)+ncx^d} dx$$

$$\mathbf{I_2} = \int_0^{\infty} x^{r+nb+d-1} \left(1 - e^{-cx^d}\right)^{n-1} e^{-ax^b\left(e^{cx^d}-1\right)+ncx^d} dx$$

$$\mathbf{I_3} = \int_0^{\infty} x^{r+nb-1} \left(1 - e^{-cx^d}\right)^{n-1} e^{-ax^b\left(e^{cx^d}-1\right)+(n-1)cx^d} dx$$

Now, we want to get the value of $\mathbf{I_1}$, since

$$e^{-ax^b\left(e^{cx^d}-1\right)} = \sum_{i=0}^{\infty} \frac{(-1)^i a^i x^{bi}}{i!} \left(e^{cx^d}-1\right)^i = \sum_{i=0}^{\infty} \frac{(-1)^i a^i x^{bi}}{i!} e^{icx^d} \left(1 - e^{-cx^d}\right)^i$$

then

$$\mathbf{I_1} = \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \int_0^{\infty} x^{r+nb+bi-1} \left(1 - e^{-cx^d}\right)^{n+i-1} e^{(n+i)cx^d} dx$$

Since $0 < \left(1 - e^{-cx^d}\right)^{n+i-1} < 1$ for $x > 0$, then by using the binomial series expansion we have

$$\left(1 - e^{-cx^d}\right)^{n+i-1} = \sum_{j=0}^{n+i-1} (-1)^i \begin{pmatrix} n+i-1 \\ j \end{pmatrix} e^{-cjx^d}$$

then

$$\mathbf{I_1} = \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \frac{(-1)^{i+j} a^i}{i!} \begin{pmatrix} n+i-1 \\ j \end{pmatrix} \int_0^{\infty} x^{r+nb+bi-1} e^{-jcx^d} e^{(n+i)cx^d} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j}(n+i)^l c^l a^i}{i!\, l!\, d(cj)^{\frac{r+(n+i)b+ld}{d}}} \begin{pmatrix} n+i-1 \\ j \end{pmatrix} \int_0^{\infty} x^{\frac{r+(n+i)b+ld}{d}} e^{-y} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j}(n+i)^l c^l a^i}{i!\, l!\, d(cj)^{\frac{r+(n+i)b+ld}{d}}} \begin{pmatrix} n+i-1 \\ j \end{pmatrix} \Gamma\left(\frac{r+(n+i)b+ld}{d}\right). \tag{21}$$

Similarly;

$$\mathbf{I_2} = \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j}(n+i)^l c^l a^i}{i! \, l! \, d(cj)^{\frac{r+(n+i)b+ld+d}{d}}} \begin{pmatrix} n+i-1 \\ j \end{pmatrix} \Gamma \left(\frac{r+(n+i)b+ld+d}{d} \right). \tag{22}$$

$$\mathbf{I_3} = \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j}(n+i-1)^l c^l a^i}{i! \, l! \, d(cj)^{\frac{r+(n+i)b+ld}{d}}} \begin{pmatrix} n+i-1 \\ j \end{pmatrix} \Gamma \left(\frac{r+(n+i)b+ld}{d} \right). \tag{23}$$

Substituting from Equations (21), (22) and (23) into Equation (20), we get Equation (19).

4. The Mean Past (Residual) Life Time MPL (MRL) for GWGD Based on the Upper Record Values

The mean past lifetime (MPL) corresponds to the mean time elapsed since the failure of T_i given that $T - i \le t$. In this case, the random variable of interest is $[t - T_i]T_i \le t], i = 1, \ldots, n$. This conditional random variable shows the time elapsed since the failure of T_i given that it failed at or before t. The expectation of this random variable gives the mean past lifetime (MPL) P(t).

Lemma 4.1. *Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the first upper record values of size n which have GWGD then the mean past lift time of the nth upper record value $X_{U(n)}$, say $p_n(t)$, is given as follows for $a, b, c, d > 0$ and $t > 0$*

$$p_n(t) = \frac{1}{F_n(t)} \frac{a^n b}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{j=0}^{n+i-1} \sum_{l=0}^{\infty} \frac{(-1)^{i+j} a^i c^l}{i! \, l!} \begin{pmatrix} (n+i-1) \\ j \end{pmatrix} t^{bi+ld+nd+} \times$$
$$\left(\frac{(n+i-j)^l - (n+i-j-1)^l}{(bi+ld+nb)(bi+ld+nb+1)} + \frac{cd(n+i-j)^l t^d}{b(bi+ld+d+nb)(bi+ld+d+nb+1)} \right) \tag{24}$$

Proof.   Since

$$p_n(t) = \frac{1}{F_n(t)} \int_0^t F_n(x) dx \tag{25}$$

It is easy to prove this Lemma by using Equations (8) and (25). On the other hand, in reliability theory and survival analysis to study the lifetime characteristics of a live organism there have been defined several measure such as the mean residual life time m(t). Assuming that each component of the system has survived up to time t, the survival function of $T_i - t$ given that $T_i > t$; $i = 1, \ldots, n$. This is the corresponding conditional survival function of the components at age t. $\square$

The following Lemma 4.2 obtains that we can get the MRL by MPL, $F_n(t)$ and expectation value $\mu'_{1(n)}$.

Lemma 4.2. *Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the first upper record values of size n which have GWGD then the mean residual life time of the nth upper record value $X_{U(n)}$ say $m_n(t)$, is given as follows for $a, b, c, d > 0$ and $t > 0$*

$$m_n(t) = (1 - F_n(f))^{-1} \left(\mu'_{1(n)} + p_n(t) F_n(t) - t \right) \tag{26}$$

where $F_n(t), \mu'_{1(n)}$ and $p_n(t)$ are given from Equations (8), (19) and (24) respectively.

Proof.   Since

$$m_n(t) = \frac{1}{R_n(t)} \int_t^{\infty} R_n(x) dx$$
$$= \frac{1}{1 - F_n(t)} \left\{ \mu'_{1(n)} - \int_t^{\infty} R_n(x) dx \right\} \tag{27}$$

from Equation (25), we get

$$p_n(t)F_n(t) = t - \int_t^\infty R_n(x)dx \tag{28}$$

Substituting from Equation (28) into Equation (27), we get Equation (26). □

5. Parameters Estimation

5.1. Maximum Likelihood Estimates

In this section, we derive the maximum likelihood estimates of the unknown parameters of GWGD based on upper record values where $c = 0.001$. Consider we observe n upper record values $X = \{X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}\}$ from a sequence of independent and identically distributed random variables following a GWGD with probability density function in Equations (2) and (3). Arnold et al. [3] give the likelihood function as

$$l = f(x_{U(n)}; a, b, c, d) \prod_{i=1}^{n-1} \frac{f(x_{U(i)}; a, b, c, d)}{R(x_{U(i)}; a, b, c, d)} \tag{29}$$

Substituting from Equations (2) and (3) into Equation (29), we get

$$L = -ax_n^b \left(e^{cx_n^b} - 1\right) + (b-1)\sum_{i=1}^n \ln x_i + \sum_{i=1}^n \ln\left(1 + \frac{cd}{b}x_i^d - e^{-cx_i^d}\right) + c\sum_{i=1}^n x_i^d + n(\ln a + \ln b) \tag{30}$$

Differentiate Equation (32) with respect to a,b and d, we get

$$\frac{\partial L}{\partial a} = -x_n^b \left(e^{cx_n^d} - 1\right) + \frac{n}{a}. \tag{31}$$

$$\frac{\partial L}{\partial c} = -ax_n^b x_n^d e^{cx_n d} + \sum_{i=1}^n x_i^d + \sum_{i=1}^n \frac{\frac{d}{b}x_i^d + x_i^d e^{-cx_i^d}}{\left(1 + \frac{cd}{b}x_i^d - e^{-cx_i^d}\right)}. \tag{32}$$

$$\frac{\partial L}{\partial d} = -acx_n^b x_n^d e^{cx_n d} \ln x_i + \sum_{i=1}^n \frac{\frac{c}{b}x_i^d + \frac{cd}{b}x_i^d \ln x_i + cx_i^d e^{-cx_i^d} \ln x_i}{\left(1 + \frac{cd}{b}x_i^d - e^{-cx_i^d}\right)} + c\sum_{i=1}^n x_i^d \ln x_i. \tag{33}$$

Equating Equations (??), (??) and (??) by zero and numerical solving with respect to a, b and d. The MLE of , say $\hat{a}(\hat{b}, \hat{c}, \hat{d})$ can be obtained as

$$\hat{a}(\hat{b}, \hat{c}, \hat{d}) = \frac{n}{x_n^{\hat{b}}\left(e^{\hat{c}x_n^{\hat{d}}} - 1\right)} \tag{34}$$

5.2. Asymptotic Confidence Bounds

In this section, we derive the asymptotic confidence intervals of these parameters when a, b and d, by using variance covariance matrix I_0^{-1} see Lawless [6], where I_0^{-1} is the inverse of the observed information matrix

$$I_0^{-1} = \begin{bmatrix} \frac{-\partial^2 L}{\partial a^2} & \frac{-\partial^2 L}{\partial a \partial b} & \frac{-\partial^2 L}{\partial a \partial d} \\ \frac{-\partial^2 L}{\partial b \partial a} & \frac{-\partial^2 L}{\partial b^2} & \frac{-\partial^2 L}{\partial b \partial d} \\ \frac{-\partial^2 L}{\partial d \partial a} & \frac{-\partial^2 L}{\partial d \partial b} & \frac{-\partial^2 L}{\partial d^2} \end{bmatrix} \tag{35}$$

thus,

$$
I_0^{-1} = \begin{bmatrix} var(\hat{a}) & cov(\hat{a},\hat{b}) & cov(\hat{a},\hat{d}) \\ cov(\hat{b},\hat{a}) & var(\hat{b}) & cov(\hat{b},\hat{d}) \\ cov(\hat{d},\hat{b}) & cov(\hat{b},\hat{b}) & var(\hat{d}) \end{bmatrix} \tag{36}
$$

The derivatives in I_0 are given as follows:

$$
\frac{\partial^2 L}{\partial a^2} = \frac{-n}{a^2},
$$

$$
\frac{\partial^2 L}{\partial a \partial b} = -x_n^b \left(e^{cx_n^d} - 1 \right) \ln x_n,
$$

$$
\frac{\partial^2 L}{\partial a \partial d} = -cx_n^b x_n^d e^{cx_n^d} \ln x_n,
$$

$$
\frac{\partial^2 L}{\partial b^2} = -ax_n^b \left(e^{cx_n^d} - 1 \right) (\ln x_n)^2 - \frac{n}{b^2} + \sum_{i=1}^n \frac{2cdx_i^d}{\left(1 + \frac{cd}{b} x_i^d - e^{-cx_i^d} \right) b^3} + \sum_{i=1}^n \frac{(cdx_i^d)^2}{\left(1 + \frac{cd}{b} x_i^d - e^{-cx_i^d} \right)^2 b^4},
$$

$$
\frac{\partial^2 L}{\partial b \partial d} = -acx_n^b x_n^d e^{cx_n^d} (\ln x_n)^2 - \sum_{i=1}^n \frac{cx_i^d}{\left(1 + \frac{cd}{b} x_i^d - e^{-cx_i^d} \right) b^2} - \sum_{i=1}^n \frac{cdx_i^d \ln x_i}{\left(1 + \frac{cd}{b} x_i^d - e^{-cx_i^d} \right) b^2}
$$

$$
+ \sum_{i=1}^n \frac{cdx_i^d \left(\frac{cx_i^d}{b} + \frac{cdx_i^d \ln x_i}{b} + cx_i^d e^{-cx_i^d} \ln x_i \right)}{\left(1 + \frac{cd}{b} x_i^d - e^{-cx_i^d} \right)^2 b^2},
$$

$$
\frac{\partial^2 L}{\partial d^2} = -acx_n^b x_n^d e^{cx_n^d} (\ln x_n)^2 - ac^2 x_n^b x_n^{2d} e^{cx_n^d} (\ln x_n)^2 + c \sum_{i=1}^1 x_i^d (\ln x_n)^2
$$

$$
+ \sum_{i=1}^n \frac{\frac{2cx_i^d \ln x_i}{b} + \frac{cdx_i^d (\ln x_i)^2}{b} + cx_i^d e^{-cx_i^d} (\ln x_i)^2 - c^2 x_i^{2d} e^{-cx_i^d} (\ln x_i)^2}{\left(1 + \frac{cd}{b} x_i^d - e^{-cx_i^d} \right)}
$$

$$
- \sum_{i=1}^n \frac{\left(\frac{cx_i^d}{b} + \frac{cdx_i^d \ln x_i}{b} + cx_i^d e^{-cx_i^d} \ln x_i \right)^2}{\left(1 + \frac{cd}{b} x_i^d - e^{-cx_i^d} \right)^2}
$$

We can derive the $(1 - \delta)100\%$ confidence intervals of the parameters a, b, d by using variance covariance matrix as in the following forms

$$
\hat{a} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{a})}, \ \hat{b} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{b})} \ \text{ and } \ \hat{d} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{d})}
$$

where $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2} \right)$th percentile of the standard normal distribution.

6.   Data Analysis and Discussion

We apply the results of this paper to one real data set. The data is presented in Lawless [7], page 319. It consists of the survival times in days of 16 lung cancer patients:

6.96	9.30	6.96	7.24	9.30	4.90	8.42	6.05	10.18	6.82
8.58	7.77	11.94	11.25	12.94	12.94				

From this data set, we extract a sample of upper records: 6.96, 9.30, 10.18, 11.94, 12.94 . Using the method of maximum likelihood estimates, we get the values of parameters and log-likelihood (L) in the following Table 1

MLEs of the parameters of GWGD(a, b, d)			log-likelihood (L)
$\hat{a} = 0.21074$	$\hat{b} = 3.934$	$\hat{d} = 0.000779$	-6.635

Table 1.   **The MLEs of the parameters and L**

By substituting the MLE of unknown parameters in Equation (35), we get estimation of the variance covariance matrix as

$$I_0^{-1} = \begin{bmatrix} 0.94485 & -1.74507 & 0.01121 \\ -1.74507 & 0.12936 & -0.12927 \\ 0.01121 & -0.12927 & 0.12920 \end{bmatrix}$$

The approximate 95% two sided confidence interval of the parameters a, b and c are [0, 2.12], [3.22, 4.64] and [0, 0.705] respectively.

□

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