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The Univarite and Multivariate Generalized Slash Student Distribution

Research Article

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Abstract: In this paper, a new family of univariate and multivariate generalized slash student distribution is presented as the scale mixture of the student and the beta distributions. We called it generalized slash student distribution. It is shown that the new family of distributions can have heavier tails than the slash student distribution and slash normal distribution. Furthermore, moments and the invariant property under linear transformations are addressed. A simulation study is performed to investigate asymptotically the bias properties of the estimators.

Keywords: Slash distribution, slash student distribution, moments, heavy-tailed distribution.

1. Introduction

Heavy-tailed distributions have been the subject of much study in the statistical literature. Heavy-tailed distributions are the distributions which have more observations in the tails and to be thinner in the midrange than a normal distribution. First heavy-tailed alternative distributions to the normal distribution are the student and the slash distributions, which have been very popular in robust statistical analysis (Rogers and Tukey [14]; Kafadar [5]; Morgenthaler [13]; Lange et al.[11]; Kafadar [6]; Jamshidian [9]; Kashid and Kulkarni [7]). Both of these distributions can be derived by mixing a normally distributed random variable with a nonnegative scale random variable. They both belong to the scale mixture of normal distribution family. It is known that the standard normal distribution has the following density function

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2}), \quad -\infty < z < \infty. \tag{1}$$

and the student distribution with r degree of freedom has the following density function

$$g(t) = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{r\pi}\Gamma(\frac{r}{2})}\left(1+\frac{t^2}{r}\right)^{-\left(\frac{r+1}{2}\right)}, -\infty < t < \infty. \tag{2}$$

and the beta distribution has the following density function

$$h(y) = \frac{1}{Beta(\alpha,\beta)}y^{\alpha-1}(1-y)^{\beta-1}, \quad 0 < y < 1. \tag{3}$$

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where α and $\beta > 0$ are shape parameters. The slash random variable is defined as the ratio of two independent random variables: Let the standard normal random variable Z be independent of the uniform random variable U on $(0,1)$, then the random variable $S = Z/U^{1/q}$ is said to have slash normal distribution with the following density:

$$\Psi(s;q) = q\int_0^1 t^q f(st)dt, \qquad -\infty < s < \infty. \tag{4}$$

where $q > 0$ is the shape parameter and $f(.)$ denotes the standard normal distribution density function defined in (1). For $q = 1$ the distribution is called the standard slash normal distribution and it has the following density:

$$\Psi(s;1) = \begin{cases} \frac{1}{\sqrt{2\pi s^2}}\left(1 - \exp(-\frac{s^2}{2})\right), & \text{if } s \neq 0. \\ \frac{1}{2\sqrt{2\pi}}, & \text{otherwise.} \end{cases} . \tag{5}$$

The standard slash normal density has heavier tails than those of the normal. Let the random variable T has a student distribution with degree of freedom r and be independent of the uniform random variable U on $(0,1)$, then the random variable $M = T/U^{1/q}$ is said to have slash student distribution with the following density:

$$\Phi(m;q) = q\int_0^1 v^q g(mv)dv, \qquad -\infty < m < \infty. \tag{6}$$

where $q > 0$ is the shape parameter and $g(.)$ denotes the student distribution density function defined in (2). For $q = 1$ the distribution is called the standard slash student distribution. The standard slash student density has heavier tails than those of the standard slash normal and those of the student distribution. EL-Bassiouny. A. H and Abdo. N. F generalized the known family of the slash distribution. Since the beta distribution with the two parameters α and β on the interval $(0,1)$ reduces to the uniform distribution when $\alpha = \beta = 1$, then they replaced the uniform random variable in the denominator of the slash random variable by the beta random variable with the parameters α and β and introduce what they called the generalized slash distribution, symbolically written $GSL(\mu,\sigma,\alpha,\beta,q)$ with probability density function pdf given by

$$j(y) = \frac{q}{\sigma\sqrt{2\pi}Beta(\alpha,\beta)}\int_0^1 v^{q\alpha}(1 - v^q)^{\beta-1} e^{-\frac{(y-\mu)^2v^2}{2\sigma^2}} dv, -\infty < t < \infty. \tag{7}$$

In literature, many authers studied multivariate and skew multivariate extensions of the slash distribution such as Wang and Genton [17], Arslan [1], Arslan and Genc [2]. The main objective of this paper is to generalize the slash student distribution, our main idea is to replace the uniform random variable in the denominator of the slash student random variable by the beta random variable with the parameters α and β.

This paper is organized as follows: In section 2, we introduce the univariate generalized slash student distribution and some special cases are discussed. The moments of the univariate generalized slash student distribution are obtained in section 3. The unimodility is discussed in section 4. In section 5, the likelihood estimation is presented. In section 6 , the multivariate generalized slash student distribution is introduced and some special cases are discussed. The first two moments of the mutlivariate generalized slash student distribution are presented in section 7. Sum properties of the generalized slash student distribution are discussed in section 8. The applications are introduced in section 9. The conclusion is presented in section 10.

2. Univariate Generalized Slash Student Distribution

In this section we define the univariate generalized slash student distribution according to the following theorem.

Theorem 2.1. *Let T has the student distribution with r degree of freedom, symbolically we write $T \sim t(t;r)$ and $Y \sim Beta(\alpha,\beta)$ over the interval $(0,1)$. Assume that T and Y are independent random variables. Define a new random variable $X = \mu + \sigma Y^{-1/q}T$, where $q,\ \sigma > 0$ and $-\infty < \mu < \infty$. The random variable X has the univariate generalized slash student distribution, symbolically we writte $GSLT(\mu,\sigma,\alpha,\beta,q,r)$. The pdf of X is given by*

$$f(x) = \frac{q\Gamma(\frac{r+1}{2})}{\sigma\sqrt{r\pi}\Gamma(\frac{r}{2})Beta(\alpha,\beta)} \int_0^1 v^{q\alpha}\left(1-v^q\right)^{\beta-1}\left(1+\frac{(x-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} dv. \tag{8}$$

where $\alpha,\ \beta > 0, -\infty < x < \infty$.

Proof. Since T and Y are independent, then the joint probability density function of (T,Y) will be

$$g(t,y) = \frac{\Gamma(\frac{r+1}{2})y^{\alpha-1}(1-y)^{\beta-1}}{\sqrt{r\pi}\Gamma(\frac{r}{2})Beta(\alpha,\beta)}\left(1+\frac{t^2}{r}\right)^{-\left(\frac{r+1}{2}\right)}, -\infty < t < \infty, 0 < y < 1.$$

From the transformation $t = \left(\frac{x-\mu}{\sigma}\right)y^{1/q}$, the jpdf of (X,Y) is given by

$$h(x,y) = \frac{\Gamma(\frac{r+1}{2})y^{\alpha-1}(1-y)^{\beta-1}y^{1/q}}{\sigma\sqrt{r\pi}\Gamma(\frac{r}{2})Beta(\alpha,\beta)}\left(1+\frac{(x-\mu)^2 y^{2/q}}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)}, -\infty < x < \infty.$$

where $\frac{y^{1/q}}{\sigma}$ is the value of the jacobian, then the marginal pdf of X is given by

$$f(x) = \frac{\Gamma(\frac{r+1}{2})}{\sigma\sqrt{r\pi}\Gamma(\frac{r}{2})Beta(\alpha,\beta)}\int_0^1 y^{\alpha+1/q-1}(1-y)^{\beta-1}dy, -\infty < x < \infty. \tag{9}$$

Using the transformation $v = y^{1/q}$ in (9), then the pdf of X will be found as claimed. If we putting $\mu = 0$ and $\sigma = 1$ in (8), then we get the standard form of a univariate generalized slash student distribution $GSLT(0,1,\alpha,\beta,q,r)$.

Special cases:

1. If we putting $\alpha = \beta = 1$ in (8), then the pdf in (8) tends to the pdf of the univariate slash student distribution given in (6).

2. If $q \to \infty$, then the pdf given in (8) tends to the pdf of the student distribution given in (2).

3. If $r \to \infty$, then the pdf given in (8) tends to the pdf of the generalized slash distribution given in (7). From Fig (1), one can easily see that, when r increases the curve of $GSLT$ distribution approache to the curve of GSL distribution.

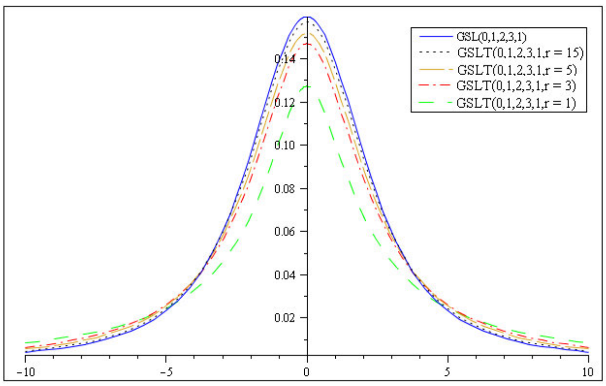

Figure 1.

4. If we putting $\alpha = \beta = 1$ and $r \to \infty$ in (8), then the pdf in (8) tends to the pdf of the univariate slash normal distribution given in (4).

□

Remark 2.1.

1. *The GSLT distribution is much more flexible with its shape parameters than the ordinary slash distributions. Heavy tails and less peak of the distribution are associated with smaller q. The width and the amplitude of GSLT have been controlled by α and β, i.e. the amplitude increases and width decreases as α increases but the amplitude decreases and the width increases as β increases , see Fig (2) and Fig (3).*

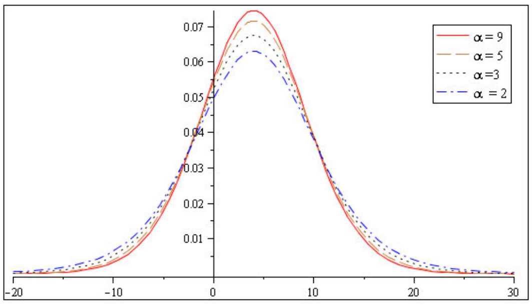

Figure 2. **Plot of the pdf of GSLT** $(4, 5, \alpha, 1, 2, 20)$ **for different values of** α.

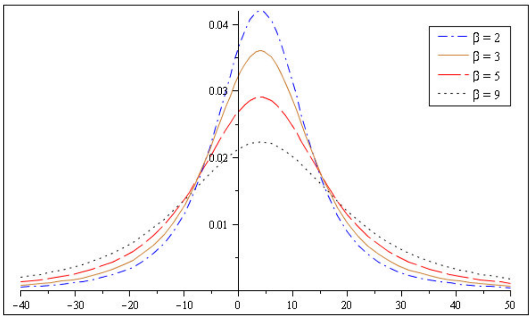

Figure 3. **Plot of the pdf of GSLT** $(4, 5, 1, \beta, 2, 20)$ **for different values of** β.

2. *From Fig (4), one can easily see that, the generalized slash student distribution is heavier in tails than the slash student and the student distributions.*

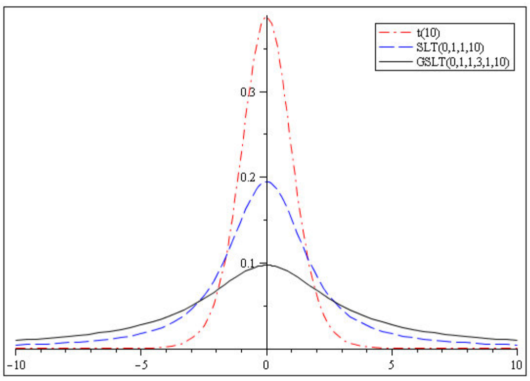

Figure 4.

In the following section, the moments of $GSLT(\mu,\sigma,\alpha,\beta,q,r)$ distribution is calculated.

3. Moments

The moments of the random variable $X=\mu+\sigma Y^{-1/q}T$ is given in the following proposition.

Proposition 3.1. *The kth moment of the random variable $X\sim GSLT\ (\mu,\sigma,\alpha,\beta,q,r)$ is given by*

$$E(X^k)=\sum_{c=0}^{k}\binom{k}{c}\sigma^c\mu^{k-c}E(Y^{-c/q})E(T^c),\ k=1,2,...... \tag{10}$$

where $E(Y^{-c/q})$ and $E(T^c)$ are the cth moment of a beta random variable $Y\sim Beta(\alpha,\beta)$ and a student random variable $T\sim t(t;r)$, they are given by

$$E(Y^{-c/q})=\frac{\Gamma(\alpha-\frac{c}{q})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta-\frac{c}{q})},\ \alpha>\frac{c}{q},\ c=1,2,...... \tag{11}$$

$$E(T^c)=\begin{cases}0 & \textit{if } c \textit{ is odd },\ 0<c<r.\\ \frac{\Gamma(\frac{c+1}{2})\Gamma(\frac{r-c}{2})r^{c/2}}{\sqrt{\pi}\Gamma(\frac{r}{2})} & \textit{if } c \textit{ is even },0<c<r.\end{cases} \tag{12}$$

where $q,\ \alpha,\ \beta>0$.

Proof. From the definition of the random variable X, one can easily get

$$\begin{aligned}E(X^k) &= E\left(\left(\mu+\sigma Y^{-1/q}T\right)^k\right).\\ &= E\left(\sum_{c=0}^{k}\binom{k}{c}\left(\sigma Y^{-1/q}T\right)^c\mu^{k-c}\right)=\sum_{c=0}^{k}\binom{k}{c}\mu^{k-c}\sigma^cE\left(Y^{-c/q}T^c\right).\end{aligned}$$

Since T and Y are independent, then (10) is follow immediately. The cth moment of a beta random variables given by

$$\begin{aligned}E(Y^{-c/q}) &= \frac{1}{Beta(\alpha,\beta)}\int_0^1 Y^{\alpha-c/q-1}(1-y)^{\beta-1}dy.\\ &= \frac{Beta(\alpha-\frac{c}{q},\beta)}{Beta(\alpha,\beta)}=\frac{\Gamma(\alpha-\frac{c}{q})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta-\frac{c}{q})},\ \alpha>\frac{c}{q},\ c=1,2,....\end{aligned}$$

□

3.1. The First Four Moments of $GSLT(\mu,\sigma,\alpha,\beta,q,r)$

The first four moments about the origin of the random variable X are given by

$$\mu_1^{'}=E(X)=\mu, \tag{13}$$

$$\mu_2^{'}=E(X^2)=\mu^2+\frac{r\,\sigma^2\Gamma\left(\alpha-\frac{2}{q}\right)\Gamma(\alpha+\beta)}{(r-2)\Gamma\left(\alpha+\beta-\frac{2}{q}\right)\Gamma(\alpha)},\alpha>\frac{2}{q}, \tag{14}$$

$$\mu_3^{'}=E(X^3)=\mu^3+\frac{3\mu\sigma^2r\ \Gamma\left(\alpha-\frac{2}{q}\right)\Gamma(\alpha+\beta)}{(r-2)\Gamma\left(\alpha+\beta-\frac{2}{q}\right)\Gamma(\alpha)},\alpha>\frac{2}{q}, \tag{15}$$

and

$$\mu_4^{'}=E(X^4)=\mu^4+\frac{6\mu^2\sigma^2r\Gamma\left(\alpha-\frac{2}{q}\right)\Gamma(\alpha+\beta)}{(r-2)\Gamma\left(\alpha+\beta-\frac{2}{q}\right)\Gamma(\alpha)}+\frac{3\sigma^4r^2\Gamma\left(\alpha-\frac{4}{q}\right)\Gamma(\alpha+\beta)}{(r-2)(r-4)\Gamma\left(\alpha+\beta-\frac{4}{q}\right)\Gamma(\alpha)}, \tag{16}$$

where $\alpha > \frac{4}{q}$.

Then the first four moments about the mean of the random variable X are given by

$$\mu_1 = \mu_1^{'} = \mu, \tag{17}$$

$$\begin{aligned} Var(X) &= \mu_2 = \mu_2^{'} - (\mu_1^{'})^2. \\ &= \frac{r\,\sigma^2 \Gamma\left(\alpha - \frac{2}{q}\right)\Gamma(\alpha+\beta)}{(r-2)\Gamma\left(\alpha+\beta-\frac{2}{q}\right)\Gamma(\alpha)}, \alpha > \frac{2}{q}, \end{aligned} \tag{18}$$

$$\begin{aligned} \mu_3 &= \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\left(\mu_1^{'}\right)^3. \\ &= 0, \end{aligned} \tag{19}$$

and

$$\begin{aligned} \mu_4 &= \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}(\mu_1^{'})^2 - 3(\mu_1^{'})^4. \\ &= \frac{3r^2\sigma^4\Gamma\left(\alpha-\frac{4}{q}\right)\Gamma(\alpha+\beta)}{(r-2)(r-4)\Gamma\left(\alpha+\beta-\frac{4}{q}\right)\Gamma(\alpha)}, \alpha > \frac{4}{q}. \end{aligned} \tag{20}$$

Thus the skewness γ_1 and kurtosis γ_2 are given by

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = 0. \tag{21}$$

and

$$\begin{aligned} \gamma_2 &= \frac{\mu_4}{(\mu_2)^2}. \\ &= \frac{3(r-2)\Gamma\left(\alpha-\frac{4}{q}\right)\left(\Gamma\left(\alpha+\beta-\frac{2}{q}\right)\right)^2\Gamma(\alpha)}{(r-4)\,\Gamma\left(\alpha+\beta-\frac{4}{q}\right)\left(\Gamma(\alpha-\frac{2}{q})\right)^2\Gamma(\alpha+\beta)}, \ \alpha > \frac{4}{q}. \end{aligned} \tag{22}$$

4. Unimodility

The pdf given in (8) has a unimode. One can show this by verify this inequality

$$\mu < x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2),$$

and this is as, since

$$\begin{aligned} x_1 &\le x_2 \Rightarrow x_1 - \mu \le x_2 - \mu. \\ &\Rightarrow \frac{(x_1-\mu)v}{\sigma} \le \frac{(x_2-\mu)v}{\sigma}. \\ &\Rightarrow \frac{(x_1-\mu)^2 v^2}{r\sigma^2} \le \frac{(x_2-\mu)^2 v^2}{r\sigma^2}. \\ &\Rightarrow 1 + \frac{(x_1-\mu)^2 v^2}{r\sigma^2} \le 1 + \frac{(x_2-\mu)^2 v^2}{r\sigma^2}. \\ &\Rightarrow \left(1 + \frac{(x_1-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} \ge \left(1 + \frac{(x_2-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)}. \\ &\Rightarrow v^{q\alpha}(1-v^q)^{\beta-1}\left(1 + \frac{(x_1-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} \ge \\ &\qquad v^{q\alpha}(1-v^q)^{\beta-1}\left(1 + \frac{(x_2-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)}. \end{aligned}$$

$$\Rightarrow \frac{q\Gamma(\frac{r+1}{2})}{\sigma\sqrt{r\pi}\Gamma(\frac{r}{2})Beta(\alpha,\beta)} \int_0^1 v^{q\alpha}\left(1-v^q\right)^{\beta-1}\left(1+\frac{(x_1-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} dv \geq$$
$$\frac{q\Gamma(\frac{r+1}{2})}{\sigma\sqrt{r\pi}\Gamma(\frac{r}{2})Beta(\alpha,\beta)} \int_0^1 v^{q\alpha}\left(1-v^q\right)^{\beta-1}\left(1+\frac{(x_2-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} dv.$$
$$\Rightarrow f(x_1) \geq f(x_2).$$

since $f(x) \geq 0$. Thus from the inequality and the symmetry of the distribution, the pdf in (8) is a unimode.

5. Likelihood Estimation

Proposition 5.1. *Let $x_1, ..., x_n$ be a data set modeled by $GSLT(\mu,\sigma,\alpha,\beta,q,r)$ distribution in the location scale form, then the estimation of μ and σ^2 are given by*

$$\hat{\mu} = \frac{\sum_{i=1}^n \omega_i(s) x_i}{\sum_{i=1}^n \omega_i(s)}. \tag{23}$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \omega_i(s)\left(x_i - \hat{\mu}\right)^2 , \tag{24}$$

where

$$\omega_i(s) = \frac{\int_0^1 v^{q\alpha+2}(1-v^q)^{\beta-1}\frac{r+1}{r}\left(1+\frac{s^2v^2}{r}\right)^{-\left(\frac{r+3}{2}\right)} dv}{\int_0^1 v^{q\alpha}(1-v^q)^{\beta-1}\left(1+\frac{s^2v^2}{r}\right)^{-\left(\frac{r+1}{2}\right)} dv}, s = \left|x_i - \hat{\mu}\right|/\hat{\sigma}. \tag{25}$$

Proof. The log-likelihood function is given by

$$\begin{aligned} L(\mu,\sigma,\alpha,\beta,q,r) &= \log \Pi_{i=1}^n f(x_i;\mu,\sigma,\alpha,\beta,q,r). \\ &= n\log\left[\frac{q\Gamma(\frac{r+1}{2})}{\sqrt{r\pi}\Gamma(\frac{r}{2})Beta(\alpha,\beta)}\right] - n\log\sigma \\ &+ \sum_{i=1}^n \log\int_0^1 v^{q\alpha}(1-v^q)^{\beta-1}\left(1+\frac{(x_i-\mu)^2 v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} dv. \end{aligned}$$

Taking partial derivatives of the log-likelihood function with respect to μ and σ, assuming the shape parameters are fixed, and equating the derivatives to 0, we get

$$\frac{\partial L(\mu,\sigma,\alpha,\beta,q,r)}{\partial\mu} = 0,$$

$$\sum_{i=1}^n \frac{\int_0^1 v^{q\alpha+2}(1-v^q)^{\beta-1}\frac{(x_i-\mu)}{\sigma^2}\frac{r+1}{r}\left(1+\frac{(x_i-\mu)^2v^2}{r\sigma^2}\right)^{-\left(\frac{r+3}{2}\right)} dv}{\int_0^1 v^{q\alpha}(1-v^q)^{\beta-1}\left(1+\frac{(x_i-\mu)^2v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} dv} = 0,$$

$$\sum_{i=1}^n \frac{\frac{(x_i-\mu)}{\sigma^2}\int_0^1 v^{q\alpha+2}(1-v^q)^{\beta-1}\frac{r+1}{r}\left(1+\frac{(x_i-\mu)^2v^2}{r\sigma^2}\right)^{-\left(\frac{r+3}{2}\right)} dv}{\int_0^1 v^{q\alpha}(1-v^q)^{\beta-1}\left(1+\frac{(x_i-\mu)^2v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)} dv} = 0.$$

Using (25), we get

$$\begin{aligned} \frac{1}{\sigma^2}\sum_{i=1}^n (x_i-\mu)\,\omega_i(s) &= 0, \\ \sum_{i=1}^n x_i\omega_i(s) &= \mu\sum_{i=1}^n \omega_i(s), \end{aligned}$$

Thus (23) is obtained.

$$\frac{\partial L(\mu,\sigma,\alpha,\beta,q,r)}{\partial \sigma}=0,$$

$$-\frac{n}{\sigma}+\sum_{i=1}^{n}\frac{\frac{(x_i-\mu)^2}{\sigma^3}\int_0^1 v^{q\alpha+2}(1-v^q)^{\beta-1}\frac{r+1}{r}\left(1+\frac{(x_i-\mu)^2v^2}{r\sigma^2}\right)^{-\left(\frac{r+3}{2}\right)}dv}{\int_0^1 v^{q\alpha}(1-v^q)^{\beta-1}\left(1+\frac{(x_i-\mu)^2v^2}{r\sigma^2}\right)^{-\left(\frac{r+1}{2}\right)}dv}=0.$$

Using (25), we obtain

$$\frac{n}{\sigma}=\frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2\,\omega_i(s),$$

Thus (24) is obtained. □

In the following section, we introduce the multivariate generalized slash student distribution.

6. Multivariate Generalized Slash Student Distribution

Tan, F., Peng, [16] introduced the multivariate slash student distribution defined as the resulting distribution of the ratio of multivariate student distribution to an independent standard uniform variable. Here we replace the standard uniform distribution with beta distribution over the intervial (0,1) and introduce what we called a multivariate generalized slash student distribution. The invariant property of this distribution under linear transformations is done. Furthermore, the moments and marginal distributions are discussed.

A continuous k-dimensional random vector T has a student distribution with degrees of freedom r, mean vector μ, and correlation matrix Σ, written $T\sim t_k(t;\mu,\Sigma,r)$, if it has the density

$$t_k(t;\mu,\Sigma,r)=\frac{\Gamma(\frac{r+k}{2})}{(r\pi)^{\frac{k}{2}}\Gamma(\frac{r}{2})\,|\Sigma|^{1/2}}\left[1+\frac{(t-\mu)^{\intercal}\Sigma^{-1}(t-\mu)}{r}\right]^{-\frac{r+k}{2}},t\in\mathbb{R}^k. \tag{26}$$

There are many variants of the definitions of student distribution and we use the above one. For more details, see, e.g., Kotz and Nadarajah [8].

Theorem 6.1. *Let $T\sim t_k(t;0,I_k,r)$ and $Y\sim Beta(\alpha,\beta)$ are independent. A k-dimensional continuous random vector $X=(X_1,X_2,\ldots,X_k)$ is said to have a multivariate generalized slash student distribution with location vector $\mu\in\mathbb{R}^k$, positive definite scale matrix Σ and tail parameter $q>0$, if it can be written in the form $X=\mu+\Sigma^{1/2}TY^{-1/q}$, symbolically we written $X\sim GSLT_k(\mu,\Sigma,\alpha,\beta,q,r)$. The pdf of the random vector X is*

$$\psi_k(x;\mu,\Sigma,\alpha,\beta,q,r)=\frac{q}{Beta(\alpha,\beta)}\int_0^1 v^{q\alpha+k-1}\left(1-v^q\right)^{\beta-1}t_k(xv;\mu v,\Sigma,r)dv, \tag{27}$$

where $x\in\mathbb{R}^k$.

Proof. Since T and Y are independent, then jpdf of (T,Y) will be

$$g(t,y)=\frac{1}{Beta(\alpha,\beta)}y^{\alpha-1}(1-y)^{\beta-1}t_k(t;0,I_k,r),t\in\mathbb{R}^k.$$

From the transformation $t=y^{1/q}\left(\frac{x-\mu}{\Sigma^{1/2}}\right)$, the jpdf of (X,Y) is given by

$$h(x,y)=\frac{1}{|\Sigma|^{1/2}\,Beta(\alpha,\beta)}y^{\alpha+k/q-1}(1-y)^{\beta-1}t_k(\frac{x-\mu}{\Sigma^{1/2}}y^{1/q};0,I_k,r),x\in\mathbb{R}^k,$$

where $\frac{y^{k/q}}{|\Sigma|^{1/2}}$ is the value of the jacobian. Then the marginal pdf of X is given by

$$\psi_k(x) = \frac{1}{|\Sigma|^{1/2}\, Beta(\alpha,\beta)} \int_0^1 y^{\alpha+k/q-1}(1-y)^{\beta-1} t_k(\frac{x-\mu}{\Sigma^{1/2}} y^{1/q}; 0, I_k, r) dy \tag{28}$$

Using the transformation $v = y^{1/q}$ in (28), and

$$\frac{1}{|\Sigma|^{1/2}} t_k(\frac{x-\mu}{\Sigma^{1/2}} v; 0, I_k, r) = t_k(xv; \mu v, \Sigma, r),$$

Then the pdf of X will be found as claimed.

If we putting $\mu = 0$ and $\Sigma = I_k$ in (27), then we get the standard form of a multivariate generalized slash student distribution $GSLT_k(0, I_k, \alpha, \beta, q, r)$.

Special cases:

1. If we putting $k = 1$ in (27), then the pdf in (27) tends to the pdf of the univariate generalized slash student distribution $GSLT(\mu, \sigma, \alpha, \beta, q, r)$ given in (4).

2. If we putting $k = 2$ in (27), then we obtain the bivariate generalized slash student distribution $GSLT_2(\mu, \Sigma, \alpha, \beta, q, r)$ and it is pdf will be

$$f_2(x; \mu, \Sigma, \alpha, \beta, q, r) = \frac{q}{Beta(\alpha,\beta)} \int_0^1 v^{q\alpha+1} (1-v^q)^{\beta-1} t_2(xv; \mu, \Sigma, r) dv, x \in \mathbb{R}^2. \tag{29}$$

3. If we putting $\alpha = \beta = 1$ in (27), then the pdf in (27) tends to the pdf of the multivariate slash student distribution $SLT_k(\mu, \Sigma, q, r)$,(see,Tan, F., Peng, (2005)).

4. If $q \to \infty$, then the pdf in (27) tends to the pdf of the multivariate student distribution $t_k(\mu, \Sigma, r)$ given in (26).

5. If $r \to \infty$, then the pdf in (27) tends to the pdf of the multivariate generalized slash distribution $GSL_k(\mu, \Sigma, q)$,(see, EL-Bassiouny. A. H and Abdo. N. F).

6. If $q \to \infty$ and $r \to \infty$, then the pdf in (27) tends to the pdf of the multivariate normal distribution with location vector μ and positive definite scale matrix Σ.

7. If we putting $\alpha = \beta = 1$ and $r \to \infty$ in (27), then the pdf in (27) tends to the pdf of the multivariate slash normal distribution $SL_k(\mu, \Sigma, q)$,(see, Wang and Genton [17]).

□

In the following section, we introduce the expectation, variance and the first two moments of the multivariate generalized slash student distribution.

7. Moments

The expectation, variance and the first two moments of the multivariate generalized slash student distribution are given in the following proposition.

Proposition 7.1. *If* $X = \mu + \Sigma^{1/2} T Y^{-1/q}$ *has* $GSLT_k(\mu, \Sigma, \alpha, \beta, q, r)$, *then its expectation, variance and the first two moments are given by*

$$\begin{aligned} E(X) &= \mu, & (30)\\ E(X^2) &= \mu^2 + \frac{\Sigma r \Gamma(\alpha - \frac{2}{q})\Gamma(\alpha+\beta)}{(r-2)\Gamma(\alpha)\Gamma(\alpha+\beta-\frac{2}{q})}, \quad \alpha > \frac{2}{q}. & (31)\\ Var(X) &= \frac{\Sigma r \Gamma(\alpha - \frac{2}{q})\Gamma(\alpha+\beta)}{(r-2)\Gamma(\alpha)\Gamma(\alpha+\beta-\frac{2}{q})}, \quad \alpha > \frac{2}{q}. & (32) \end{aligned}$$

Proof. The moments of a beta random variable $Y \sim Beta(\alpha, \beta)$ and $T \sim t(t; r)$ are given in (11) and(12) respectively. Since T and Y are independent, then the first two moments of X is

$$\begin{aligned} E(X) &= E(\mu + \Sigma^{1/2} Y^{-1/q} T) \\ &= \mu + \Sigma^{1/2} E(Y^{-1/q} T) = \mu, \end{aligned} \tag{33}$$

$$\begin{aligned} E(X^2) &= E(\left(\mu + \Sigma^{1/2} Y^{-1/q} T\right)^2) \\ &= E(\mu^2 + 2\mu\Sigma^{1/2} Y^{-1/q} T + \Sigma \left(Y^{-1/q} T\right)^2) \\ &= \mu^2 + \Sigma \frac{r \Gamma(\alpha - \frac{2}{q})\Gamma(\alpha+\beta)}{(r-2)\Gamma(\alpha)\Gamma(\alpha+\beta-\frac{2}{q})}, \quad \alpha > \frac{2}{q}. \end{aligned} \tag{34}$$

From (33) and (34), one can easily get (32). □

We note that, when $\alpha = \beta = 1$, then the expectation and variance in (30) and (32), reduce to the expectation and variance of the multivariate slash student distribution, (see, Tan, F., Peng, [16]).

8. Properties of the Distribution

8.1. Marginal Distributions

Since the marginal distributions of a multivariate student distribution are still student distributions, (see,Tan, F., Peng, [16]), the marginal distributions of a generalized slash student distribution are also generalized slash student distribution. The following proposition states this fact.

Proposition 8.1. *The marginal distributions of a generalized slash student distribution are still generalized slash student.*

Proof. It suffices to show without loss of generality that

$$\int \psi_k(x_1, .., x_k; 0, I_k, \alpha, \beta, q, r) dx_{s+1}..dx_k = \psi_s\,(x_1, .., x_s; 0, I_s, \alpha, \beta, q, r), x_1, .., x_s \in \mathbb{R}. \tag{35}$$

For every $0 \leq s \leq k$. Substitution of the formula (27) in the left hand of the above gives

$$\begin{aligned} LHS &= \int \psi_k(x_1, ..., x_k; 0, I_k, \alpha, \beta, q, r) dx_{s+1}..dx_k. \\ &= \frac{q}{Beta(\alpha, \beta)} \int_0^1 v^{q\alpha+k-1} (1-v^q)^{\beta-1} \int t_k(vx_1, .., vx_k\,; 0, I_k, r) dx_{s+1}..dx_k dv. \end{aligned}$$

With substitution $y_{s+1} = vx_{s+1}, ...y_k = vx_k$ for $v > 0$ one has

$$\int t_k(vx_1,..,vx_k;0,I_k,r)dx_{s+1}..dx_k = v^{s-k}\int t_k(vx_1,..,vx_s,y_{s+1},..,y_k;0,I_k,r)dy_{s+1}..dy_k.$$

Because the marginals of the student distribution are still student, we have

$$\int t_k(vx_1,..,vx_s\ ,y_{s+1},..,y_k;0,I_k,r)dy_{s+1}..dy_k = t_s(v\mathbf{x}_1,..,vx_s\ ;0,I_s,r).$$

The last two equalities yield the desired equality. □

8.2. Linear Combinations

Since the distribution of a linear function of a student random vector $t_k(\mu,\Sigma,r)$ is still student, (see,Tan, F., Peng, [16]), the distribution of a linear function of $GSLT_k(\mu,\Sigma,\alpha,\beta,q,r)$ random vector is also still generalized slash student distribution, i.e. the multivariate generalized slash student distribution is invariant under linear transformation. The following proposition states this fact.

Proposition 8.2. *If $X \sim GSLT_k(x;\mu,\Sigma,\alpha,\beta,q,r)$, then its linear transformation $W = b + AX \sim GSLT_k(b + A\mu, A\Sigma A^T,\alpha,\beta,q,r)$, b is a vector in $\mathbb{R}^k$, and A is a nonsingular matrix.*

Proof. From the transformation, we have $X = A^{-1}(W-b)$ therefore, the jacobian determinant of the transformation is $|A|^{-1}(J = |\frac{dX}{dW}|)$, hence the pdf of W is

$$\begin{aligned} f(w) &= |A|^{-1}\psi_k(A^{-1}(w-b);\mu,\Sigma,\alpha,\beta,q,r) \\ &= \frac{q\,|A|^{-1}}{Beta(\alpha,\beta)}\int_0^1 v^{q\alpha+k-1}(1-v^q)^{\beta-1}t_k(A^{-1}(w-b)\,v;\mu v,\Sigma,r)dv. \end{aligned}$$

We have

$$A^{-1}t_k(A^{-1}(w-b)\,v;\mu v,\Sigma,r) = t_k\,(wv;(b+A\mu)\,v,A\Sigma A^T,r). \tag{36}$$

Hence from (27) and (36) the pdf of W is

$$\begin{aligned} f(w) &= \frac{q}{Beta(\alpha,\beta)}\int_0^1 v^{q\alpha+k-1}(1-v^q)^{\beta-1}t_k\,(wv;(b+A\mu)\,v,A\Sigma A^T,r)dv \\ &= \psi_k(w;b+A\mu,A\Sigma A^T,\alpha,\beta,q,r). \end{aligned}$$

□

This shows that W has a multivariate generalized slash student distribution $GSLT_k(b+A\mu,A\Sigma A^T,\alpha,\beta,q,r)$. It implies that the multivariate generalized slash sudent distribution is invariant under linear transformation. For $\alpha=\beta=1$, this property is valid for the multivariate slash student distribution, (see, Tan, F., Peng, [16]).

9. Application

We perform a simulation study to investigate bias properties of the estimators asymptotically. All computations were performed using R program.and all program codes are availabe from the author on request.

9.1. Simulation Results

Because of the complexity of the log-likelihood function, one cannot derive the information matrix. It is impossible to find a theoretical asymptotic property of the maximum likelihood estimators. Therefore, we investigate the properties of the estimators numerically. We perform simulations to investigate the properties (bias and variance) of the estimators depending on the shape parameters. We first generate 500 samples of different sizes from the $GSLT$ distribution for fixed shape parameters then use the iterative forms of the estimators given in (23) and (24) to compute the estimates. The mean and variances of the estimates are given in Table (1),

n	$M(\hat{\mu})$	$V(\hat{\mu})$	$M(\hat{\sigma})$	$V(\hat{\sigma})$
		$(\alpha,\beta,q)=(1,1,2)$		
20	1.049566	0.6115208	1.939781	0.1921698
50	1.006134	0.23005	1.999149	0.0826686
250	0.9980009	0.04518617	2.043632	0.01731016
500	0.9983787	0.02015888	1.994506	0.008053589
		$(\alpha,\beta,q)=(3,1,3)$		
20	0.9978287	0.2834570	2.000467	0.1416984
50	1.025751	0.1177836	2.013447	0.04421132
250	1.003973	0.02301845	2.059674	0.00846452
500	1.003072	0.01140199	2.048103	0.004683706
		$(\alpha,\beta,q)=(3,1,2)$		
20	0.9757207	0.2790355	2.040304	0.1368656
50	0.992953	0.1144998	2.074444	0.05528438
250	1.001507	0.02409617	2.098894	0.01162367
500	1.000022	0.01209849	2.098481	0.005489125
		$(\alpha,\beta,q)=(2,1,3)$		
20	1.037093	0.3076454	1.994529	0.1344278
50	0.9746617	0.1239348	2.031742	0.05213233
250	0.9923936	0.02641631	2.083123	0.00859241
500	0.9944237	0.01193728	2.063900	0.00542433
		$(\alpha,\beta,q)=(2,2,3)$		
20	1.015662	0.4365982	1.836518	0.1319999
50	1.020890	0.1524086	1.838626	0.04815509
250	0.9919161	0.02842560	1.858968	0.008212608
500	0.9894796	0.01353102	1.881916	0.004621278
		$(\alpha,\beta,q)=(3,1,1)$		
20	0.9680027	0.4385706	2.314456	0.27541
50	1.018360	0.1916392	2.333695	0.08958936
250	1.004492	0.04158386	2.330454	0.01765299
500	0.9988717	0.01736267	2.356971	0.008150121
		$(\alpha,\beta,q)=(1,3,3)$		
20	1.09693	0.833911	1.384517	0.08370373
50	0.9657198	0.2877105	1.448615	0.03766548
250	0.9932118	0.05880109	1.436281	0.00745219
500	1.006750	0.03134134	1.427915	0.00400891
		$(\alpha,\beta,q)=(2,1,3)$		
20	1.037093	0.3076454	1.994529	0.1344278
50	0.9746617	0.1239348	2.031742	0.05213233
250	0.9960281	0.02407785	2.051052	0.01061027
500	0.9944237	0.01193728	2.063900	0.00542433

Table 1. Means and variances of the location-scale estimates of 500 samples of sizes $n = 20,\ 50,\ 250,\ 500$ from the $GSLT$, with $\mu = 1$ and $\sigma = 2$ and $(\alpha,\beta,q) = (1,1,2), (3,1,3), (3,1,2),\ (2,1\ ,3),\ (2,2,3), (3,1,1),\ (1,1,3), (2,1,3)$.

Table (1) tells us that the estimates $\hat{\mu}$ and $\hat{\sigma}$ seem asymptotically unbiased. As the sample size increase, the variance of the estimates approaches to 0, as expected.

10. Conclusion

We have introduced a new generalized family of slash student distribution for univariate and multivariate distributions. The basic idea is to replace the uniform random variable in the denominator of the slash student random variable by the beta random variable with the parameters α and β. The width and the amplitude of the symmetric univariate generalized slash student distribution of the random variable X, have been controlled by α and β. The multivariate generalized slash student distribution is invariant under linear transformation. The likelihood estimation is also studied. A simulation study is performed to investigate asymptotically the bias properties of the estimators.

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