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Some Generating Relations Involving Laguerre Polynomials of Several Variables

Research Article

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Abstract: The present paper deals with m-variable polynomial sets generated by the functions in the form $e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt)$.Two generating relations are obtained as applications of general theorems associated with multiple series identities.

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1. Introduction and Preliminaries

Throughout in present paper, we use the following standard notations:

$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers. The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases}$$

it is being understood *conventionally* that $(0)_0 := 1$ and assumed tacitly that the Gamma quotient exists. In 1997-98, the Laguerre polynomials of m-variable are defined by Khan and Shukla [1, 2] in the following form

$$L_n^{(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_m)}(x_1, x_2, x_3, \ldots, x_m) = \frac{\prod_{j=1}^{m}(1+\alpha_j)_n}{(n!)^m}\Psi_2^{(m)}\left[-n; 1+\alpha_1, 1+\alpha_2, 1+\alpha_3, \ldots, 1+\alpha_m; x_1, x_2, x_3, \ldots, x_m\right] \tag{1}$$

In 1920-21 P. Humbert defined multi-variable hypergeometric function $\psi_2^{(m)}$ in the form [3]

$$\psi_2^{(m)}\left[a; c_1, c_2, c_3, \ldots, c_m; x_1, x_2, x_3, \ldots, x_m\right]$$

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$$= \sum_{k_1,k_2,k_3,\ldots,k_m=0}^{\infty} \frac{(a)_{k_1+k_2+k_3+\ldots+k_m}}{(c_1)_{k_1}(c_2)_{k_2}(c_3)_{k_3}\ldots(c_m)_{k_m}} \frac{x_1^{k_1}x_2^{k_2}x_3^{k_3}\ldots x_m^{k_m}}{(k_1)!(k_2)!(k_3)!\ldots(k_m)!} \tag{2}$$

where $|x_1| < \infty, \quad |x_2| < \infty, \quad |x_3| < \infty, \quad \ldots, |x_m| < \infty.$

Binomial theorem:

$$(1-t)^{-\alpha} = {}_1F_0\left[\begin{matrix}\alpha; \\ -;\end{matrix}\; t\right] = \sum_{n=0}^{\infty}(\alpha)_n \frac{t^n}{n!} \quad ; \qquad |t| < 1 \tag{3}$$

Multiple series identities [3]:

$$\sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}\phi(n;k_1,k_2,k_3,\ldots,k_m)\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k_1,k_2,k_3,\ldots,k_m=0}^{k_1+k_2+k_3+\ldots+k_m\leq n}\phi(n-k_1-k_2-k_3-\ldots-k_m;k_1,k_2,k_3,\ldots,k_m)\right) \tag{4}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k_1,k_2,k_3,\ldots,k_m=0}^{k_1+k_2+k_3+\ldots+k_m\leq n}\phi(n;k_1,k_2,k_3,\ldots,k_m)\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}\phi(n+k_1+k_2+k_3+\ldots+k_m;k_1,k_2,k_3,\ldots,k_m)\right) \tag{5}$$

If m-variable polynomial sets $\sigma_n(x_1,x_2,x_3,\ldots,x_m)$ has a generating function of the form $e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt)$ given by equation (24) then Theorem 4.1 yields another generating function in the form $(1-t)^{-c}G\left(\frac{x_1t}{1-t},\frac{x_2t}{1-t},\frac{x_3t}{1-t},\ldots,\frac{x_mt}{1-t}\right)$ given by equation (26) for same polynomial set $\sigma_n(x_1,x_2,x_3,\ldots,x_m)$.

2. Partial Differential Equations Associated with Polynomial Sets $\sigma_n(x_1,x_2,x_3,\ldots,x_m)$

Theorem 2.1. *If*

$$e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt) = \sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3,\ldots,x_m)t^n \tag{6}$$

then

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+\ldots+x_m\frac{\partial}{\partial x_m}\right)\sigma_0(x_1,x_2,x_3,\ldots,x_m) = 0 \tag{7}$$

and for $n \geq 1$

$$(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+\cdots+x_m\frac{\partial}{\partial x_m})\sigma_n(x_1,x_2,x_3,\ldots,x_m)-n\;\sigma_n(x_1,x_2,x_3,\ldots,x_m) = -\sigma_{n-1}(x_1,x_2,x_3,\ldots,x_m) \tag{8}$$

Proof. Let us consider the generating relation of the type

$$e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt) = \sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3\ldots,x_m)t^n \tag{9}$$

Suppose

$$F = e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt) \tag{10}$$

or

$$F = e^t\phi_1\phi_2\phi_3\ldots\phi_m$$

Then

$$\frac{\partial F}{\partial x_1} = te^t\phi_1'\phi_2\phi_3\ldots\phi_m \tag{11}$$

$$\frac{\partial F}{\partial x_2} = te^t\phi_1\phi_2'\phi_3\ldots\phi_m \tag{12}$$

$$\frac{\partial F}{\partial x_3} = te^t\phi_1\phi_2\phi_3'\ldots\phi_m \tag{13}$$

similarly

$$\frac{\partial F}{\partial x_m} = te^t\phi_1\phi_2\phi_3\ldots\phi_m' \tag{14}$$

and

$$\frac{\partial F}{\partial t} = e^t\phi_1\phi_2\phi_3\ldots\phi_m + x_1e^t\phi_1'\phi_2\phi_3\ldots\phi_m + x_2e^t\phi_1\phi_2'\phi_3\ldots\phi_m + x_3e^t\phi_1\phi_2\phi_3'\ldots\phi_m + \ldots + x_me^t\phi_1\phi_2\phi_3\ldots\phi_m' \tag{15}$$

Eliminating $\phi_1, \phi_1', \phi_2, \phi_2', \phi_3, \phi_3', \ldots\ldots, \phi_m$ and ϕ_m' from the above equations, we obtain

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + \ldots + x_m\frac{\partial}{\partial x_m}\right)F - t\frac{\partial F}{\partial t} = -tF \tag{16}$$

From equations (9) and (16), we have

$$\sum_{n=0}^{\infty}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + \ldots + x_m\frac{\partial}{\partial x_m}\right)\sigma_n(x_1, x_2, x_3, \ldots, x_m)t^n - \sum_{n=1}^{\infty} n\ \sigma_n(x_1, x_2, x_3, \ldots, x_m)t^n$$

$$= -\sum_{n=0}^{\infty}\sigma_n(x_1, x_2, x_3, \ldots, x_m)t^{n+1} = -\sum_{n=1}^{\infty}\sigma_{n-1}(x_1, x_2, x_3, \ldots, x_m)t^n$$

Now equating the coefficients of like powers of t, we get (7) and (8). □

3. Series Representation of Polynomials $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$

The series representation of $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$ is given by $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2}\cdots\sum_{k_m=0}^{n-k_1-k_2-k_3-\ldots-k_{m-1}}\frac{a_1(k_1)\ a_2(k_2)\ a_3(k_3)\ \ldots a_m(k_m){x_1}^{k_1}{x_2}^{k_2}{x_3}^{k_3}\ldots{x_m}^{k_m}}{(n-k_1-k_2-k_3-\ldots-k_m)!}$$

$$= \sum_{k_1,k_2,k_3,\ldots,k_m=0}^{k_1+k_2+k_3+\ldots+k_m\leq n}\frac{(-n)_{k_1+k_2+k_3+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ a_3(k_3)\ \ldots a_m(k_m)}{n!}(-x_1)^{k_1}(-x_2)^{k_2}(-x_3)^{k_3}\ldots(-x_m)^{k_m} \tag{17}$$

where $\{a_1(k_1)\}$, $\{a_2(k_2)\}$, $\{a_3(k_3)\}$, $\ldots, \{a_m(k_m)\}$ are bounded sequences of arbitrary real or complex numbers, $\forall\ \ k_j\ \ \epsilon\ \ \{0, 1, 2, \ldots\}; \qquad 1 \leq j \leq m.$

Proof. Suppose the functions ϕ_1, ϕ_2, ϕ_3,..........,ϕ_m in (6) have the formal power-series expansions.

$$\phi_1(u_1) = \sum_{k_1=0}^{\infty} a_1(k_1)u_1^{k_1}; \qquad a_1(0) \neq 0 \tag{18}$$

$$\phi_2(u_2) = \sum_{k_2=0}^{\infty} a_2(k_2)u_2^{k_2}; \qquad a_2(0) \neq 0 \tag{19}$$

$$\phi_3(u_3) = \sum_{k_3=0}^{\infty} a_3(k_3)u_3^{k_3}; \qquad a_3(0) \neq 0 \tag{20}$$

and

$$\phi_m(u_m) = \sum_{k_m=0}^{\infty} a_m(k_m) u_m^{k_m}; \qquad a_m(0) \neq 0 \tag{21}$$

Then (6) yields

$$\sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k_1, k_2, k_3, \ldots, k_m=0}^{\infty} a_1(k_1) x_1^{k_1} a_2(k_2) x_2^{k_2} \ldots a_m(k_m) x_m^{k_m} \frac{t^{n+k_1+k_2+k_3+\ldots+k_m}}{n!} \tag{22}$$

Now using multiple series identity (4), we get

$$\sum_{n=0}^{\infty} \{\sigma_n(x_1, x_2, x_3, \ldots, x_m)\} t^n$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \sum_{k_3=0}^{n-k_1-k_2} \cdots \sum_{k_m=0}^{n-k_1-k_2-k_3-\ldots-k_{m-1}} \frac{a_1(k_1) a_2(k_2) a_3(k_3) \ldots a_m(k_m) {x_1}^{k_1} {x_2}^{k_2} {x_3}^{k_3} \ldots {x_m}^{k_m}}{(n-k_1-k_2-k_3-\ldots-k_m)!} \right\} t^n \tag{23}$$

Now comparing the coefficients of t^n in equation (23), we get the series representation (17) for $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$. □

4. General Theorem Associated with Generating Relation and Generating Function

Theorem 4.1. *If*

$$e^t \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t) \ldots \phi_m(x_m t) = \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^n \tag{24}$$

and

$$\phi_1(u_1) = \sum_{k_1=0}^{\infty} a_1(k_1) u_1^{k_1}; \qquad \phi_2(u_2) = \sum_{k_2=0}^{\infty} a_2(k_2) u_2^{k_2};$$

$$\phi_3(u_3) = \sum_{k_3=0}^{\infty} a_3(k_3) u_3^{k_3}; \qquad \ldots \phi_m(u_m) = \sum_{k_m=0}^{\infty} a_m(k_m) u_m^{k_m}; \tag{25}$$

where $a_1(0) \neq 0, \quad , a_2(0) \neq 0, \quad , a_3(0) \neq 0, \ldots, a_m(0) \neq 0$

then

$$(1-t)^{-c} G\left(\frac{x_1 t}{1-t}, \frac{x_2 t}{1-t}, \frac{x_3 t}{1-t}, \ldots, \frac{x_m t}{1-t}\right) = \sum_{n=0}^{\infty} (c)_n \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^n \tag{26}$$

where c is arbitrary and

$$G(w_1, w_2, w_3, \ldots, w_m)$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} (c)_{k_1+k_2+k_3+\ldots+k_m} a_1(k_1)\ a_2(k_2)\ a_3(k_3)\ \ldots a_m(k_m) w_1^{k_1} w_2^{k_2} w_3^{k_3} \ldots w_m^{k_m} \tag{27}$$

where $\{a_1(k_1)\}, \ \{a_2(k_2)\}, \ \{a_3(k_3)\}, \ \ldots, \{a_m(k_m)\}$ *are bounded sequences of arbitrary real or complex numbers,* $\forall \quad k_j \ \epsilon \ \{0, 1, 2, \ldots\}; \qquad 1 \leq j \leq m.$

Proof. Now consider the new series in the form

$$\sum_{n=0}^{\infty} (c)_n \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \sum_{k_3=0}^{n-k_1-k_2} \cdots \sum_{k_m=0}^{n-k_1-k_2-k_3-\ldots-k_{m-1}} \frac{(c)_n a_1(k_1) a_2(k_2) \ldots a_m(k_m) {x_1}^{k_1} {x_2}^{k_2} \ldots {x_m}^{k_m}}{(n-k_1-k_2-k_3-\ldots-k_m)!} \right) t^n \tag{28}$$

Now using multiple series identity (5), we get

$$\sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,\ldots,x_m)t^n$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}\frac{(c)_{n+k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}}{(n)!}\right)t^n \tag{29}$$

$$=\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}\times$$

$$\times\sum_{n=0}^{\infty}\frac{(c+k_1+k_2+\ldots+k_m)_n}{n!}t^n \tag{30}$$

$$=\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}\times$$

$$\times\ {}_1F_0\left[\begin{array}{c}c+k_1+k_2+\ldots+k_m;\\ -\quad ;\end{array}t\right] \tag{31}$$

$$=\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}\times$$

$$\times(1-t)^{-(c+k_1+k_2+\ldots+k_m)} \tag{32}$$

Therefore

$$\sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,\ldots,x_m)t^n=(1-t)^{-c}\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\times$$

$$\times\left(\frac{x_1t}{1-t}\right)^{k_1}\left(\frac{x_2t}{1-t}\right)^{k_2}\ldots\left(\frac{x_mt}{1-t}\right)^{k_m} \tag{33}$$

$$=(1-t)^{-c}G\left(\frac{x_1t}{1-t},\frac{x_2t}{1-t},\ldots,\frac{x_mt}{1-t}\right)$$

where $G\left(\frac{x_1t}{1-t},\frac{x_2t}{1-t},\ldots,\frac{x_mt}{1-t}\right)$ is defined by the equation (27). □

5. Applications of Theorems 2.1 and 4.1 in Generating Relations

If $\phi_1(u_1),\phi_2(u_2),\phi_3(u_3)\ldots\phi_m(u_m)$, are specified in hypergeometric forms, then Theorem 4.1 gives for polynomial sets $\sigma_n(x_1,x_2,x_3,\ldots,x_m)$, a class of generating relation involving m-variable hypergeometric polynomials.

Now apply Theorems 2.1 and 4.1 to Laguerre polynomials of m-variable $L_n^{(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_m)}(x_1,x_2,x_3,\ldots,x_m)$ due to Khan, M. A. and Shukla, A. K. defined explicitly by (1)

$$L_n^{(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_m)}(x_1,x_2,x_3,\ldots,x_m)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n\ldots(1+\alpha_m)_n}{(n!)^m}\times$$

$$\times\ \Psi_2^{(m)}\left[-n;1+\alpha_1,1+\alpha_2,1+\alpha_3,\ldots,1+\alpha_m;x_1,x_2,x_3,\ldots,x_m\right] \tag{34}$$

Now consider a polynomial set in the form

$$\sigma_n(x_1,x_2,x_3,\ldots,x_m)=\frac{(n!)^{m-1}L_n^{(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_m)}(x_1,x_2,x_3,\ldots,x_m)}{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n\ldots(1+\alpha_m)_n} \tag{35}$$

and

$$\phi_1(u_1) = {}_0F_1(-;1+\alpha_1;-u_1) = \sum_{k_1=0}^{\infty} \frac{(-1)^{k_1} {u_1}^{k_1}}{k_1!(1+\alpha_1)_{k_1}} \tag{36}$$

$$\phi_2(u_2) = {}_0F_1(-;1+\alpha_2;-u_2) = \sum_{k_2=0}^{\infty} \frac{(-1)^{k_2} {u_2}^{k_2}}{k_2!(1+\alpha_2)_{k_2}} \tag{37}$$

$$\phi_3(u_3) = {}_0F_1(-;1+\alpha_3;-u_3) = \sum_{k_3=0}^{\infty} \frac{(-1)^{k_3} {u_3}^{k_3}}{k_3!(1+\alpha_3)_{k_3}} \tag{38}$$

similarly

$$\phi_m(u_m) = {}_0F_1(-;1+\alpha_m;-u_m) = \sum_{k_m=0}^{\infty} \frac{(-1)^{k_m} {u_m}^{k_m}}{k_m!(1+\alpha_m)_{k_m}} \tag{39}$$

Then

$$a_1(k_1) = \frac{(-1)^{k_1}}{k_1!(1+\alpha_1)_{k_1}}, a_2(k_2) = \frac{(-1)^{k_2}}{k_2!(1+\alpha_2)_{k_2}}, a_3(k_3) = \frac{(-1)^{k_3}}{k_3!(1+\alpha_3)_{k_3}} \dots a_m(k_m) = \frac{(-1)^{k_m}}{k_m!(1+\alpha_m)_{k_m}} \tag{40}$$

Then generating relation is given by

$$\begin{aligned} &e^t{}_0F_1(-;1+\alpha_1;-x_1t){}_0F_1(-;1+\alpha_2;-x_2t){}_0F_1(-;1+\alpha_3;-x_3t)\dots{}_0F_1(-;1+\alpha_m;-x_mt) \\ &= \sum_{n=0}^{\infty} \frac{(n!)^{m-1} L_n^{(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_m)}(x_1,x_2,x_3,\dots,x_m)}{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n\dots(1+\alpha_m)_n} t^n \end{aligned} \tag{41}$$

We use Theorem 2.1 to conclude that

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + \dots + x_m\frac{\partial}{\partial x_m}\right) L_0^{(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_m)}(x_1,x_2,x_3,\dots,x_m) = 0$$

and for $n \geq 1$,

$$\begin{aligned} &\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + \dots + x_m\frac{\partial}{\partial x_m}\right) L_n^{(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_m)}(x_1,x_2,x_3,\dots,x_m) \\ &-n\, L_n^{(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_m)}(x_1,x_2,x_3,\dots,x_m) = -\frac{(\alpha_1+n)(\alpha_2+n)\dots(\alpha_m+n)}{n^{m-1}} L_{n-1}^{(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_m)}(x_1,x_2,x_3,\dots,x_m) \end{aligned}$$

In applying Theorem 4.1 to Laguerre polynomials of m variables $L_n^{(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_m)}(x_1,x_2,x_3,\dots,x_m)$, and

$$\begin{aligned} G(w_1,w_2,w_3,\dots,w_m) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\dots\sum_{k_m=0}^{\infty} (c)_{k_1+k_2+k_3+\dots+k_m} a_1(k_1)\ a_2(k_2)\ a_3(k_3)\dots a_m(k_m) w_1^{k_1} w_2^{k_2} w_3^{k_3}\dots w_m^{k_m} \\ &= \sum_{k_1,k_2,k_3,\dots,k_m=0}^{\infty} \frac{(c)_{k_1+k_2+k_3+\dots+k_m}(-1)^{k_1+k_2+k_3+\dots+k_m} {w_1}^{k_1}{w_2}^{k_2}{w_3}^{k_3}\dots{w_m}^{k_m}}{k_1!k_2!k_3!\dots k_m!(1+\alpha_1)_{k_1}(1+\alpha_2)_{k_2}(1+\alpha_3)_{k_3}\dots(1+\alpha_m)_{k_m}} \\ &= \psi_2^{(m)}\left[c;1+\alpha_1,1+\alpha_2,1+\alpha_3,\dots,1+\alpha_m;-w_1,-w_2,-w_3,\dots,-w_m\right] \end{aligned} \tag{42}$$

Therefore Theorem 4.1 yields

$$\begin{aligned} &(1-t)^{-c}\psi_2^{(m)}\left[c;1+\alpha_1,1+\alpha_2,1+\alpha_3,\dots,1+\alpha_m;\frac{-x_1t}{1-t},\frac{-x_2t}{1-t},\frac{-x_3t}{1-t},\dots,\frac{-x_mt}{1-t}\right] \\ &= \sum_{n=0}^{\infty} \frac{(n!)^{m-1}(c)_n L_n^{(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_m)}(x_1,x_2,x_3,\dots,x_m)}{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n\dots(1+\alpha_m)_n} t^n \end{aligned} \tag{43}$$

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