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Certain Formulas for Jacobi Polynomials of Several Variables

Research Article

Ahmed Ali Atash[1]* and Salem Saleh Barahmah[1]

1 Department of Mathematics, Aden University, Aden, Yemen.

Abstract: The aim of this research paper is to derive some generating functions of Jacobi polynomials of several variables [4]. Many applications of our main results lead us to obtain some hypergeometric formulas of Jacobi polynomials of two, three and several variables.Some generating functions (known or new) of Jacobi polynomials have been obtained as special cases of our main results.

MSC: 33C15, 33C20, 33C45, 33C65.

Keywords: Generating functions, Jacobi polynomials, Generalized Lauricella function of several variables, Hypergeometric functions.

1. Introduction

The Jacobi polynomials of several variables are defined by Shrivastava [4] in the following form :

$$P_n^{(\alpha_1,\beta_1\,;\cdots;\,\alpha_r\,,\beta_r)}(x_1,\cdots,x_r) = \frac{(1+\alpha_1)_n(1+\alpha_2)_n\cdots(1+\alpha_r)_n}{(n\,!)^r} \times F^{1:\;1;\;\cdots\;;\;1}_{0:\;1;\;\cdots\;;\;1}$$

$$\left(\begin{array}{ccccccccc} (-n:1,\cdots,1) & : & (1+\alpha_1+\beta_1+n:1) & ; & \cdots & ; & (1+\alpha_r+\beta_r+n:1) & ; & \frac{1-x_1}{2},\cdots,\frac{1-x_r}{2} \\ - & : & (1+\alpha_1\,:\,1) & ; & \cdots & ; & (1+\alpha_r:1) & ; & \end{array}\right), \tag{1}$$

where F- function on the right-hand side of (1) denotes a generalized Lauricella function of several variables defined by [5, 6], as follows

$$F^{A\;:\;B^{'}\;;\;\cdots\;;\;B^{(n)}}_{C\;:\;D^{'}\;;\;\cdots\;;\;D^{(n)}}\left(\begin{array}{l} [(a)\;:\;\theta^{'}\;,\;\cdots\;,\;\theta^{(n)}]\;:\;[(b')\;:\;\phi']\;;\;\cdots\;;\;\left[(b^{(n)})\;:\;\phi^{(n)}\right]\;; \\ [(c)\;:\;\psi^{'}\;,\;\cdots\;,\;\psi^{(n)}]\;:\;[(d')\;:\;\delta']\;;\;\cdots\;;\;\left[(d^{(n)})\;:\;\delta^{(n)}\right]\;; \end{array} x_1\,,\cdots,\,x_n\right)$$

$$= \sum_{m_1,\cdots,m_n=0}^{\infty} \Omega\,(m_1,\cdots,m_n)\,\frac{x_1^{m_1}}{m_1\,!}\cdots\frac{x_n^{m_n}}{m_n\,!} \tag{2}$$

where, for convenience,

$$\Omega\,(m_1,\cdots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\,\theta'_j+\cdots+m_n\,\theta_j^{(n)}}\;\prod_{j=1}^{B'}(b'_j)_{m_1\,\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\,\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\,\psi'_j+\cdots+m_n\,\psi_j^{(n)}}\;\prod_{j=1}^{D'}(d'_j)_{m_1\,\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\,\delta_j^{(n)}}} \tag{3}$$

* *E-mail: ah-a-atash@hotmail.com*

From (1), we have

$$P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(x,1) = \frac{(1+\alpha_2)_n}{n!} P_n^{(\alpha_1,\beta_1)}(x) \tag{4}$$

$$P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(1,y) = \frac{(1+\alpha_1)_n}{n!} P_n^{(\alpha_2,\beta_2)}(y) \tag{5}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomials [3]

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[\begin{array}{ll} -n, 1+\alpha+\beta+n & ; \frac{1-x}{2} \\ 1+\alpha & ; \end{array}\right] \tag{6}$$

2. Generating Functions

In this section, we shall prove the following generating functions of Jacobi polynomials of several variables :

$$\sum_{m,n=0}^{\infty} \prod_{j=1}^{r} \left\{\frac{(1+\alpha_j+\beta_j)_{m+n}}{(1+\alpha_j)_m}\right\} \frac{(\lambda_1)_{m+n}\cdots(\lambda_p)_{m+n}(m!)^{r-1}\ t^m v^n}{(\mu_1)_{m+n}\cdots(\mu_q)_{m+n}\, n!} P_m \begin{array}{c}(\alpha_1,\beta_1+n;\cdots;\alpha_r,\beta_r+n)\\ (x_1,\cdots,x_r)\end{array}$$

$$= F\begin{array}{l} p+r\ :\ 0\ ;\ 0\ ;\ 0\ ;\ \cdots\ ;\ 0 \\ q\ :\ 0\ ;\ 0\ ;\ 1\ ;\ \cdots\ ;\ 1 \end{array}\left[\begin{array}{l} (\lambda_1:1,\cdots,1),\cdots,(\lambda_p:1,\cdots,1), \\ (\mu_1:1,\cdots,1),\cdots,(\mu_q:1,\cdots,1) \end{array}\right.$$

$$\begin{array}{l} (1+\alpha_1+\beta_1:1,1,2,1,\cdots,1),\cdots,(1+\alpha_r+\beta_r:1,1,\cdots,1,2)\ :\ -\ ;\ -\ ;\ ---\ ;\ \cdots\ ;\ ---\ ; \\ :\ -\ ;\ -\ ;\ (1+\alpha_1:1)\ ;\ \cdots\ ;\ (1+\alpha_r:1)\ ; \end{array}$$

$$\left. t, v, \frac{(x_1-1)t}{2},\cdots,\frac{(x_r-1)t}{2} \right] \tag{7}$$

$$\sum_{m=0}^{\infty} \frac{(\lambda_1)_m\cdots(\lambda_p)_m(m!)^{r-1}\ t^m}{(\mu_1)_m\cdots(\mu_q)_m \prod_{j=1}^{r}(1+\alpha_j)_m} P_m \begin{array}{c}(\alpha_1,\beta_1-m;\cdots;\alpha_r,\beta_r-m)\\ (x_1,\cdots,x_r)\end{array}$$

$$= F\begin{array}{l} p\ :\ 1\ ;\ \cdots\ ;\ 1\ ;\ 0 \\ q\ :\ 1\ ;\ \cdots\ ;\ 1\ ;\ 0 \end{array}\left[\begin{array}{l} (\lambda_1:1,\cdots,1),\cdots,(\lambda_p:1,\cdots,1)\ : \\ (\mu_1:1,\cdots,1),\cdots,(\mu_q:1,\cdots,1)\ : \end{array}\right.$$

$$\left.\begin{array}{l} (1+\alpha_1+\beta_1:1)\ ;\ \cdots\ ;\ (1+\alpha_r+\beta_r:1)\ ;\ -\ ;\ \frac{(x_1-1)t}{2},\cdots,\frac{(x_r-1)t}{2}, t \\ (1+\alpha_1:1)\ ;\ \cdots\ ;\ (1+\alpha_r:1)\ ;\ -\ ; \end{array}\right] \tag{8}$$

Proof of (7): Denoting the left - hand side of (7) by T, expressing Jacobi polynomials of several variables as in (1) and using certain well – known properties of Pochhammer symbol, we get

$$T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p_1=0}^{m}\sum_{p_2=0}^{m-p_1}\cdots\sum_{p_r=0}^{m-p_1-\cdots-p_r} \frac{(-1)^{p_1+\cdots+p_r}\, m!}{(m-p_1-\cdots-p_r)!}\frac{(\lambda_1)_{m+n}\cdots(\lambda_p)_{m+n}}{(\mu_1)_{m+n}\cdots(\mu_q)_{m+n}}$$
$$\times\frac{(1+\alpha_1+\beta_1)_{m+n+p_1}\cdots(1+\alpha_r+\beta_r)_{m+n+p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\, m!\,n!\,p_1!\cdots p_r!} t^m v^n \left(\frac{1-x_1}{2}\right)^{p_1}\cdots\left(\frac{1-x_r}{2}\right)^{p_r}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p_1=0}^{\infty}\cdots\sum_{p_r=0}^{\infty}\frac{(\lambda_1)_{m+n+p_1+\cdots+p_r}\cdots(\lambda_p)_{m+n+p_1+\cdots+p_r}}{(\mu_1)_{m+n+p_1+\cdots+p_r}\cdots(\mu_1)_{m+n+p_1+\cdots+p_r}}$$
$$\times\frac{(1+\alpha_1+\beta_1)_{m+n+2p_1+p_2+\cdots+p_r}\cdots(1+\alpha_r+\beta_r)_{m+n+p_1+\cdots+p_{r-1}+2p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\, m!\,n!\,p_1!\cdots p_r!}\times t^m v^n \left(\frac{(x_1-1)t}{2}\right)^{p_1}\cdots\left(\frac{(x_r-1)t}{2}\right)^{p_r}$$

Finally, expressing the series in terms of generalized Lauricella function of several variables (2), we have the right-hand side of (7). This complete the proof of (7). The result (8) can be proved by the similar manner.The following two generating functions of Jacobi polynomials of several variables can be obtained readily from (8), as follows:

$$\sum_{m=0}^{\infty} \frac{(m\,!)^{r-1}\,t^m}{\prod\limits_{j=1}^{r}(1+\alpha_j)_m} P_m \begin{matrix}(\alpha_1,\beta_1-m;\cdots;\alpha_r,\beta_r-m)\\(x_1,\cdots,x_r)\end{matrix} = e^t \prod_{j=1}^{r} {}_1F_1\left[\begin{matrix}1+\alpha_j+\beta_j & ; & \frac{1}{2}(x_j-1)t\\ 1+\alpha_j & ; & \end{matrix}\right] \tag{9}$$

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m(m\,!)^{r-1}\,t^m}{\prod\limits_{j=1}^{r}(1+\alpha_j)_m} P_m \begin{matrix}(\alpha_1,\beta_1-m;\cdots;\alpha_r,\beta_r-m)\\(x_1,\cdots,x_r)\end{matrix} = (1-t)^{-\lambda}\; F \begin{matrix}1: & 1; & \cdots & ; 1\\ 0: & 1; & \cdots & ; 1\end{matrix}$$

$$\left[\begin{matrix}(\lambda:1,\cdots,1) & : & (1+\alpha_1+\beta_1:1) & ; & \cdots & ; & (1+\alpha_r+\beta_r:1) & ; & \frac{\frac{1}{2}(x_1-1)t}{1-t},\cdots,\frac{\frac{1}{2}(x_r-1)t}{1-t}\\ - & : & (1+\alpha_1:1) & ; & \cdots & ; & (1+\alpha_r:1) & ; & \end{matrix}\right] \tag{10}$$

Note that the generating functions (9) and (10) are equivalent to a known results of Srivastava and Choe (see for example [6]).

3. Hypergeometric Formulas

In this section we apply the generating function (9) to obtain certain hypergeometric formulas of Jacobi polynomials of two, three and several variables

Now, if we consider (9) with $r=2,\ x_1=2x+1,\ x_2=1-2x,\ \alpha_1=\alpha_2=\alpha,\ \beta_1=\beta_2=\beta$ and use the result [5]

$${}_1F_1\ [\alpha;\beta\,;x]\ {}_1F_1\ [\alpha;\beta\,;-x] = {}_2F_3\left[\begin{matrix}\alpha\,,\beta-\alpha & ; & \frac{x^2}{4}\\ \beta\,,\frac{1}{2}\beta\,,\ \frac{1}{2}\beta+\frac{1}{2} & ; & \end{matrix}\right]\ , \tag{11}$$

we get

$$\sum_{m=0}^{\infty}\frac{m\,!\,t^m}{(\alpha+1)_m\,(\alpha+1)_m}P_m^{(\alpha,\,\beta-m\,;\,\alpha,\beta-m)}\,(2x+1\ ,\ 1-2x)$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]}\frac{(1+\alpha+\beta)_k(-\beta)_k(x^2/4)^k t^m}{(\alpha+1)_k\,((\alpha+1)/2)_k\ ((\alpha+2)/2)_k k\,!\,(m-2k)!} \tag{12}$$

Comparing the coefficient of t^m on both sides of (12) and using the following identities [6]:

$$(\lambda)_{2n} = 2^{2n}\left(\tfrac{1}{2}\lambda\right)_n\left(\tfrac{1}{2}\lambda+\tfrac{1}{2}\right)_n,\ n=0,1,2,\cdots \tag{13}$$

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k}\ ,\ 0\le k\le n\quad , \tag{14}$$

we get

$$\frac{(m!)^2 P_m^{(\alpha,\,\beta-m\ ;\,\alpha,\,\beta-m)}\,(2x+1\,,\,1-2x)}{(\alpha+1)_m\,(\alpha+1)_m} = {}_4F_3\left[\begin{matrix}\frac{-m}{2}\,,\frac{-m}{2}+\frac{1}{2},1+\alpha+\beta,-\beta & ; & x^2\\ \alpha+1\,,\ \frac{1}{2}(\alpha+1)\,,\frac{1}{2}(\alpha+2) & ; & \end{matrix}\right] \tag{15}$$

Again applying (9) with $r=3,\ x_1=2x+1,\ x_2=1-2x,\ x_3=1-2y,\ \alpha_1=\alpha_2=\alpha,\beta_1=\beta_2=\beta,\ \alpha_3=\gamma,\ \beta_3=\delta,$ we get

$$\sum_{m=0}^{\infty}\frac{(m\,!)^2\,P_m^{(\alpha\,,\beta-m;\alpha,\beta-m;\gamma,\delta-m)}\,(2x+1\,,\,1-2x\,,\,1-2y)t^m}{(\alpha+1)_m\,(\alpha+1)_m\,(\gamma+1)_m}$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(m-k)\,!(1+\gamma+\delta)_k y^k t^m}{(\alpha+1)_{m-k}\,(\alpha+1)_{m-k}\,(\gamma+1)_k k\,!}\,P_{m-k}^{(\alpha\,,\beta-m+k\ ;\,\alpha,\,\beta-m+k)}\,(2x+1\ ,\ 1-2x) \tag{16}$$

Now, comparing the coefficient of t^m on both sides of (16) and using (14) and (15), we get

$$\frac{(m\,!)^2\,P_m^{(\alpha\,,\beta-m;\alpha,\beta-m;\gamma,\delta-m)}\,(2x+1\,,\,1-2x\,,\,1-2y)}{(\alpha+1)_m\,(\alpha+1)_m\,(\gamma+1)_m}$$

$$= \sum_{k=0}^{m} \frac{(-m)_k (1+\gamma+\delta)_k y^k}{m!\,(\gamma+1)_k k!}\, {}_4F_3 \left[\begin{matrix} \frac{1}{2}(-m+2k)\,,\, \frac{1}{2}(-m+2k+1),\ 1+\alpha+\beta,\ -\beta & ; \\ \alpha+1\,,\ \frac{1}{2}(\alpha+1)\,,\ \frac{1}{2}(\alpha+2) & ; \end{matrix} \; x^2 \right] \tag{17}$$

Expanding ${}_4F_3$ into power series and using (13), we arrive after some simplification to the following result :

$$\frac{(m!)^3 P_m^{(\alpha\,,\beta-m;\alpha,\beta-m;\gamma,\delta-m)}\,(2x+1\,,\,1-2x\,,\,1-2y)}{(\alpha+1)_m\,(\alpha+1)_m\,(\gamma+1)_m}$$

$$= X \begin{matrix} 1 : 2 ; 1 \\ 0 : 3 ; 1 \end{matrix} \left[\begin{matrix} -m & : & 1+\alpha+\beta, -\beta & ; & 1+\gamma+\delta & ; \\ - & : & \alpha+1, \frac{1}{2}(\alpha+1), \frac{1}{2}(\alpha+2) & ; & \gamma+1 & ; \end{matrix} \; \frac{x^2}{4}, y \right], \tag{18}$$

where $X_{E:G;H}^{A:B;D}$ is Exton double hypergeometric series [1].

$$X \begin{matrix} A : & B;\ B' \\ C : & D; D' \end{matrix} \left[\begin{matrix} (a) : & (b);\ (b'); \\ (c) : & (d);\ (d'); \end{matrix} \; x,\, y \right] = \sum_{m,\,n\,=\,0}^{\infty} \frac{((a))_{2m+n}\,((b))_m\,((b'))_n\ x^m\ y^n}{((c))_{2m+n}\,((d))_m\,((d'))_n\ m!\,n!}. \tag{19}$$

Similarly, applying (9), with $r=4\,, x_1 = x\,, x_2 = 2-x\,, x_3 = y\,, x_4 = 2-y\,, \alpha_1 = \alpha_2 = \alpha, \alpha_3 = \alpha_4 = \gamma\,, \beta_1 = \beta_2 = \beta\,, \beta_3 = \beta_4 = \delta$ and using the results (11) and (15), we get

$$\frac{(m!)^4\, P_m^{(\alpha,\beta-m\,;\alpha,\beta-m\,;\gamma,\delta-m\,;\gamma,\,\delta-m\,)}\,(x, 2-x,\, y,\, 2-y)}{(\alpha+1)_m\,(\alpha+1)_m(\gamma+1)_m(\ \gamma+1)_m} = F \begin{matrix} 2 : & 2; & 2 \\ 0 : & 3; & 3 \end{matrix} \left[\begin{matrix} \frac{-m}{2}, \frac{-m}{2}+\frac{1}{2} \\ - \end{matrix} \right.$$

$$\left. \begin{matrix} : & 1+\alpha+\beta, -\beta & ; & 1+\gamma+\delta, -\delta & ; \\ : & \alpha+1\,, \frac{1}{2}(\alpha+1), \frac{1}{2}(\alpha+2) & ; & \gamma+1, \frac{1}{2}(\gamma+1), \frac{1}{2}(\gamma+2) & ; \end{matrix} \; \left(\frac{1-x}{2}\right)^2, \left(\frac{1-y}{2}\right)^2 \right] \tag{20}$$

where $F_{E:G;H}^{A:B;D}$ denotes the Kamp de Friet function of two variables [6]

$$F\,{}^{p\,:\,q\,;\,k}_{l\,:\,m\,;\,n} \left[\begin{matrix} (a_p) & : & (b_q) & ; & (c_k) & ; \\ (\alpha_l) & : & (\beta_m) & ; & (\gamma_n) & ; \end{matrix} \; x\,,\, y \right] = \sum_{r,\,s\,=\,0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}. \tag{21}$$

On the same lines of derivation the results (18) and (20), we have the following hypergeometric formulas of Jacobi polynomials of several variables:

$$P_m \begin{matrix} (\alpha_1\,, \beta_1 - m\,; \alpha_1\,, \beta_1 - m\,; \cdots; \alpha_r, \beta_r - m\,; \alpha_r\,, \beta_r - m\,; \alpha_{r+1}, \beta_{r+1} - m) \\ (1+2x_1,\, 1-2x_1, \cdots, 1+2x_r\,,\, 1-2x_r\,,\, 1-2x_{r+1}) \end{matrix} = \frac{\prod_{j=1}^{r} [(1+\alpha_j)_m(1+\alpha_j)_m]\,(1+\alpha_{r+1})_m}{(m!)^{2r+1}}$$

$$F \begin{matrix} 1 : & 2; & \cdots; & 2 & ; & 1 \\ 0 : & 3; & \cdots\,; & 3 & ; & 1 \end{matrix} \left[\begin{matrix} (-m : 2, \cdots, 2, 1\,) & : & (1+\alpha_1+\beta_1 : 1\,)\,, (-\beta_1 : 1\,) & ; & \cdots \\ - & : & (1+\alpha_1 : 1\,), ((1+\alpha_1)/2 : 1\,), ((2+\alpha_1)/2 : 1\,) & ; & \cdots \end{matrix} \right.$$

$$\left. \begin{matrix} \cdots ; & (1+\alpha_r+\beta_r : 1\,)\,,\ (-\beta_r : 1) & ; & (1+\alpha_{r+1}+\beta_{r+1} : 1\,) & ; \\ \cdots ; & (1+\alpha_r : 1), ((1+\alpha_r)/2 : 1\,), ((2+\alpha_r)/2 : 1\,) & ; & (1+\alpha_{r+1} : 1) & ; \end{matrix} \; \frac{x_1^2}{4}, \cdots, \frac{x_r^2}{4}, x_{r+1} \right] \tag{22}$$

and

$$P_m \begin{matrix} (\alpha_1\,, \beta_1 - m\,; \alpha_1\,, \beta_1 - m\,; \cdots; \alpha_r, \beta_r - m\,; \alpha_r, \beta_r - m) \\ (x_1,\, 2-x_1, \cdots, x_r\,,\ 2-x_r) \end{matrix} = \frac{\prod_{j=1}^{r} [\,(1+\alpha_j)_m(1+\alpha_j)_m]}{(m!)^{2r}}$$

$$F\begin{array}{l}2:\ 2;\ \cdots;\ 2\\ 0:\ 3;\ \cdots;\ 3\end{array}\left[\begin{array}{llll}(-\frac{1}{2}m\ :\ 1,\cdots,1),(-\frac{1}{2}m+\frac{1}{2}\ :\ 1,\cdots,1): & (1+\alpha_1+\beta_1\ :\ 1),(-\beta_1:1) & ; & \cdots\\ - & :\ (1+\alpha_1:1),(\frac{1+\alpha_1}{2}:1),(\frac{2+\alpha_1}{2}:1) & ; & \cdots\end{array}\right.$$

$$\left.\begin{array}{lll}\cdots; & (1+\alpha_r+\beta_r\ :\ 1),\ (-\beta_r\ :\ 1) & ;\\ \cdots; & (1+\alpha_r:1),\ ((1+\alpha_r)/2:1),((2+\alpha_r)/2:1) & ;\end{array}\ (\tfrac{1-x_1}{2})^2,\cdots,(\tfrac{1-x_r}{2})^2\right] \tag{23}$$

4. Special Cases

In (7) putting $r=1, p=q$ and $\lambda_i=\mu_i$, $i=1,2,\cdots$,we get

$$\sum_{m,n=0}^{\infty}\frac{(1+\alpha+\beta)_{m+n}t^m v^n}{(1+\alpha)_m n!}P_m^{(\alpha,\beta+n)}(x)=(1-t-v)^{-1-\alpha-\beta}{}_2F_1\left[\begin{array}{ll}\frac{1}{2}(1+\alpha+\beta),\frac{1}{2}(2+\alpha+\beta) & ;\ \frac{2t(x-1)}{(1-t-v)^2}\\ 1+\alpha & ;\end{array}\right] \tag{24}$$

Further in (24), replacing x, t and v by $1-2x$, yt and $(1-y)t$ respectively, we get a known result of Munot and Saxena [2]

$$\sum_{m,n=0}^{\infty}\frac{(1+\alpha+\beta)_{m+n}\ y^m(1-y)^n}{(1+\alpha)_m\ n!}P_m^{(\alpha,\beta+n)}(1-2x)\,t^{m+n}$$

$$=(1-t)^{-1-\alpha-\beta}{}_2F_1\left[\begin{array}{ll}\frac{1}{2}(1+\alpha+\beta),\frac{1}{2}(2+\alpha+\beta) & ;\ \frac{-4xyt}{(1-t)^2}\\ 1+\alpha & ;\end{array}\right] \tag{25}$$

In (7) putting $r=1,\ p=1,\ q=3,\ \mu_1=\lambda_1,\ \mu_2=1+\alpha+\beta$ and $\mu_3=1+\beta$, we get

$$\sum_{m,n=0}^{\infty}\frac{t^m v^n}{(1+\alpha)_m(1+\beta)_{m+n}n!}P_m^{(\alpha,\beta+n)}(x)={}_0F_1\left[-;1+\alpha;\tfrac{1}{2}(x-1)t\right]{}_0F_1\left[-;1+\beta;\tfrac{1}{2}(x+1)t+v\right] \tag{26}$$

Further in (26), replacing x, t and v by $1-2x$, yt and $(1-y)t$ respectively, we get another known result of Munot and Saxena [2]

$$\sum_{m,n=0}^{\infty}\frac{y^m(1-y)^n}{(1+\alpha)_m(1+\beta)_{m+n}\ n!}P_m^{(\alpha,\beta+n)}(1-2x)\,t^{m+n}$$

$$={}_0F_1\left[-;1+\alpha;-xyt\right]{}_0F_1\left[-;1+\beta;(1-xy)t\right] \tag{27}$$

In (8) putting $r=2,\ p=2,\ q=1,\ \lambda_1=\lambda,\ \lambda_2=1+\alpha_1$ and $\mu_1=\mu$, we get

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m m!t^m}{(\mu)_m(1+\alpha_2)_m}P_m^{(\alpha_1,\beta_1-m;\alpha_2,\beta_2-m)}(x_1,x_2)=(1-t)^{-\lambda}$$

$$F^{(3)}\left[\begin{array}{llllllll}\lambda & ::\ 1+\alpha_1 & ;- & ;- & :\ 1+\alpha_1+\beta_1 & ;\ 1+\alpha_2+\beta_2 & ;\ \mu-\alpha_1-1 & ;\ \frac{\frac{1}{2}(x_1-1)t}{1-t},\frac{\frac{1}{2}(x_2-1)t}{1-t},\frac{t}{t-1}\\ \mu & ::\ - & ;- & ;- & :\ 1+\alpha_1 & ;\ 1+\alpha_2 & ;\ - & ;\end{array}\right] \tag{28}$$

where $F^{(3)}[x,y,z]$ is the general triple hypergeometric series [6].

Further in (28), putting $x=1$, replacing μ by $\alpha_1+\mu+1$ and using (4), we get a known result of Varma [7].

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m t^m}{(\alpha_1+\mu+1)_m}P_m^{(\alpha_1,\beta_1-m)}(x_1)=(1-t)^{-\lambda}F_1\left[\lambda,1+\alpha_1+\beta_1,\mu;\alpha_1+\mu+1;\frac{\frac{1}{2}(x_1-1)t}{1-t},\frac{t}{t-1}\right] \tag{29}$$

where F_1 is the Appell function of the first kind [6].

In (9) replacing α_j , β_j and x_j by $2\alpha_j - 2$, $\frac{1}{2} - \alpha_j$ and $4x_j + 1$, $j = 1, 2, \cdots, r$ respectively and using the Kummer's second formula [5].

$$ {}_0F_1\left[-\,;\,\alpha\,;\,\frac{z^2}{4}\right] = e^{-z}\ {}_1F_1\left[\alpha - \tfrac{1}{2}\,;\,2\alpha - 1\,;\,2z\right], \tag{30}$$

we get

$$\begin{aligned}&\sum_{m=0}^{\infty} \frac{(m\,!)^{r-1}\,t^m}{\prod\limits_{j=1}^{r}(2\alpha_j - 1)_m} P_m \begin{pmatrix}(2\alpha_1 - 2\,,\tfrac{1}{2} - \alpha_1 - m\,;\cdots;\ 2\alpha_r - 2\,,\tfrac{1}{2} - \alpha_r - m)\\ (4x_1 + 1\,,\cdots,4x_r + 1)\end{pmatrix}\\ &= e^{(1 + x_1 + \cdots + x_r)t}\ {}_0F_1\left[-\,;\,\alpha_1\,;\,\frac{(x_1t)^2}{4}\right] \times \cdots \times {}_0F_1\left[-\,;\,\alpha_r\,;\,\frac{(x_rt)^2}{4}\right]\end{aligned} \tag{31}$$

In (10) putting $r = 4$, $x_1 = 2x + 1$, $x_2 = 1 - 2x$, $x_3 = 2y + 1$, $x_4 = 1 - 2y$, $\alpha_1 = \alpha_2 = \alpha$, $\alpha_3 = \alpha_4 = \gamma$, $\beta_1 = \beta_2 = \beta$, $\beta_3 = \beta_4 = \delta$ and using the result [5].

$$F_2\left[\alpha\,,\,\beta,\,\beta\,;\,\gamma,\gamma\,;x,-x\right] = {}_4F_3\left[\tfrac{\alpha}{2}\,,\tfrac{\alpha}{2} + \tfrac{1}{2},\beta,\gamma - \beta\,;\,\gamma\,,\,\tfrac{\gamma}{2}\,,\tfrac{\gamma}{2} + \frac{1}{2}\,;\,x^2\right], \tag{32}$$

we get

$$\begin{aligned}&\sum_{m=0}^{\infty} \frac{(\lambda)_m (m\,!)^3\,t^m P_m^{(\alpha,\beta-m;\,\alpha,\beta-m;\gamma,\delta-m;\gamma,\delta-m)}(2x+1\,,\,1-2x,\,2y+1\,,\,,1-2y)}{(1+\alpha)_m\,(1+\alpha)_m(1+\gamma)_m(\,1+\gamma)_m} = (1-t)^{-\lambda}\\ &F^{\,2\,:\,2\,;\,2}_{\,0\,:\,3\,;\,3}\left[\begin{array}{ccccc}\tfrac{1}{2}\lambda,\tfrac{1}{2}\lambda+\tfrac{1}{2} & : & 1+\alpha+\beta\,,-\beta & ; & 1+\gamma+\delta,\ -\delta\ ;\\ - & : & 1+\alpha\,,\tfrac{1}{2}(1+\alpha),\tfrac{1}{2}(2+\alpha) & ; & 1+\gamma,\tfrac{1}{2}(1+\gamma),\tfrac{1}{2}(2+\gamma)\ ;\\ & & \left(\tfrac{xt}{1-t}\right)^2,\ \left(\tfrac{yt}{1-t}\right)^2 & & \end{array}\right]\end{aligned} \tag{33}$$

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