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$\tilde{g}$-closed Sets in Ideal Topological Spaces

Research Article

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Abstract: The notion of $\tilde{g}$-closed sets is introduced in ideal topological spaces. Characterizations and properties of $\mathcal{I}_{\tilde{g}}$-closed sets and $\mathcal{I}_{\tilde{g}}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\tilde{g}}$-open sets. Also, it is established that an $\mathcal{I}_{\tilde{g}}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

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1. Introduction and Preliminaries

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

1. $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$ and
2. $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$ [17].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^\star$: $\wp$(X)$\rightarrow\wp$(X), called a local function [17] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A⊆X, $A^\star(\mathcal{I},\tau)$={x∈X | U∩A∉$\mathcal{I}$ for every U∈τ(x)} where τ(x)={U∈τ | x∈U}.

We will make use of the basic facts about the local functions [14] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(A)=A\cup A^\star(\mathcal{I},\tau)$ [31]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I},\tau)$.

If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed [14] (resp. $\star$-dense in itself [10]) if $A^\star\subseteq A$ (resp. $A\subseteq A^\star$). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [2] if $A^\star\subseteq$U whenever A⊆U and U is open.

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By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A$\subseteq$X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and int*(A) will denote the interior of A in (X, $\tau^\star$).

A subset A of a space (X, τ) is an α-open [26] (resp. semi-open [18], preopen [21], regular open [30]) set if A$\subseteq$int(cl(int(A))) (resp. A$\subseteq$cl(int(A)), A$\subseteq$int(cl(A)), A = int(cl(A))).

The complement of semi-open set is called semi-closed. The semi closure of a subset A of (X, τ), scl(A), is the intersection of all semi-closed sets of X containing A.

The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by cl_α(A).

Definition 1.1. *A subset A of a space (X, τ) is said to be*

1. *g-closed [19] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is open.*
2. *g-open [19] if its complement is g-closed.*
3. *$\hat{g}$-closed [32] or ω-closed [29] or $s^\star g$-closed [16, 22, 27] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is semi-open.*
4. *$\hat{g}$-open [32] if its complement is $\hat{g}$-closed.*
5. *$^\star g$-closed [33] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is $\hat{g}$-open.*
6. *$^\star g$-open [33] if its complement is $^\star g$-closed.*
7. *$^\# gs$-closed [34] if scl(A)$\subseteq$U whenever A$\subseteq$U and U is $^\star g$-open.*
8. *$^\# gs$-open [34] if its complement is $^\# gs$-closed.*
9. *$\tilde{g}$-closed [12] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is $^\# gs$-open.*
10. *$\tilde{g}$-open [12] if its complement is $\tilde{g}$-closed.*

Definition 1.2. *An ideal $\mathcal{I}$ is said to be*

1. *codense [3] or τ-boundary [25] if $\tau \cap \mathcal{I}$={ϕ},*
2. *completely codense [3] if PO(X) $\cap$ $\mathcal{I}$ = {ϕ}, where PO(X) is the family of all preopen sets in (X, τ).*

Lemma 1.3. *Every completely codense ideal is codense but not conversely [3].*

Lemma 1.4 ([14])**.** *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

1. *$A\subseteq B \Rightarrow A^\star \subseteq B^\star$,*
2. *$A^\star = cl(A^\star) \subseteq cl(A)$,*
3. *$(A^\star)^\star \subseteq A^\star$,*
4. *$(A\cup B)^\star = A^\star \cup B^\star$,*
5. *$(A\cap B)^\star \subseteq A^\star \cap B^\star$.*

Lemma 1.5. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $A \subseteq X$. *If* $A \subseteq A^\star$, *then* $A^\star = cl(A^\star) = cl(A) = cl^\star(A)$ *[28].*

Lemma 1.6. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. Then* $\mathcal{I}$ *is codense if and only if* $G \subseteq G^\star$ *for every semi-open set* G *in* X *[28].*

Lemma 1.7. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* $\mathcal{I}$ *is completely codense, then* $\tau^\star \subseteq \tau^\alpha$ *[28].*

Result 1.8. *For a subset of a topological space, the following properties hold:*

1. *Every closed set is* $\tilde{g}$*-closed but not conversely [12].*
2. *Every* $\tilde{g}$*-closed set is* $\hat{g}$*-closed but not conversely [12].*
3. *Every* $\hat{g}$*-closed set is* g*-closed but not conversely [32].*

Definition 1.9. *An ideal topological space* $(X, \tau, \mathcal{I})$ *is said to be a* $T_{\mathcal{I}}$*-space [2] if every* $\mathcal{I}_g$*-closed subset of* X *is a* $\star$*-closed set.*

Lemma 1.10. *If* $(X, \tau, \mathcal{I})$ *is a* $T_{\mathcal{I}}$*-space and* A *is an* $\mathcal{I}_g$*-closed set, then* A *is a* $\star$*-closed set [23].*

Lemma 1.11. *Every* g*-closed set is* $\mathcal{I}_g$*-closed but not conversely [2].*

Definition 1.12. *A subset* A *of an ideal topological space* $(X, \tau, \mathcal{I})$ *is said to be*

1. $\mathcal{I}_{rg}$*-closed [24] if* $A^\star \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* $(X, \tau, \mathcal{I})$.
2. $pre^\star_{\mathcal{I}}$*-open [4] if* $A \subseteq int^\star(cl(A))$.
3. $pre^\star_{\mathcal{I}}$*-closed [4] if* $X \backslash A$ *is* $pre^\star_{\mathcal{I}}$*-open.*
4. $\mathcal{I}$*-R closed [1] if* $A = cl^\star(int(A))$.

Remark 1.13 ([5])**.** *In any ideal topological space, every* $\mathcal{I}$*-R closed set is* $\star$*-closed but not conversely.*

Definition 1.14 ([5])**.** *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. A subset* A *of* X *is said to be a weakly* $\mathcal{I}_{rg}$*-closed set if* $(int(A))^\star \subseteq U$ *whenever* $A \subseteq U$ *and* U *is a regular open set in* X.

Remark 1.15 ([5])**.** *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. The following diagram holds for a subset* $A \subseteq X$:

$$\begin{array}{ccccc} \mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} & \longrightarrow & \text{weakly } \mathcal{I}_{rg}\text{-closed} \\ \uparrow & & & & \uparrow \\ \star\text{-closed} & \longleftarrow & \mathcal{I}\text{-R-closed} & & pre^\star_{\mathcal{I}}\text{-closed} \end{array}$$

These implications are not reversible.

Definition 1.16 ([6, 7])**.** *A subset* A *of an ideal topological space* $(X, \tau, \mathcal{I})$ *is said to be*

1. *semi***-*$\mathcal{I}$*-open if* $A \subseteq cl(int^\star(A))$,
2. *semi***-*$\mathcal{I}$*-closed if its complement is semi***-*$\mathcal{I}$*-open.*

Definition 1.17 ([6])**.** *The semi***-*$\mathcal{I}$*-closure of a subset* A *of an ideal topological space* $(X, \tau, \mathcal{I})$*, denoted by* $s^\star_{\mathcal{I}}cl(A)$*, is defined by the intersection of all semi***-*$\mathcal{I}$*-closed sets of* X *containing* A.

Theorem 1.18 ([6])**.** *For a subset* A *of an ideal topological space* $(X, \tau, \mathcal{I})$, $s^\star_{\mathcal{I}}cl(A) = A \cup int(cl^\star(A))$.

Definition 1.19 ([8]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. A is called*

1. *generalized semi*-$\mathcal{I}$-closed ($gs^\star_\mathcal{I}$-closed) in $(X, \tau, \mathcal{I})$ if $s^\star_\mathcal{I}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in $(X, \tau, \mathcal{I})$.*

2. *generalized semi*-$\mathcal{I}$-open ($gs^\star_\mathcal{I}$-open) in $(X, \tau, \mathcal{I})$ if $X \backslash A$ is a $gs^\star_\mathcal{I}$-closed set in $(X, \tau, \mathcal{I})$.*

2. $\mathcal{I}_{\tilde{g}}$-closed Sets

Definition 2.1. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

1. *$\mathcal{I}_{\tilde{g}}$-closed if $A^\star \subseteq U$ whenever $A \subseteq U$ and U is $^\#gs$-open.*

2. *$\mathcal{I}_{\tilde{g}}$-open if its complement is $\mathcal{I}_{\tilde{g}}$-closed.*

Theorem 2.2. *If $(X, \tau, \mathcal{I})$ is any ideal topological space, then every $\mathcal{I}_{\tilde{g}}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Proof. It follows from the fact that every open set is $^\#gs$-open. □

Example 2.3. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{c\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{a\}\}$. It is clear that $\{b\}$ is $\mathcal{I}_g$-closed but not $\mathcal{I}_{\tilde{g}}$-closed.*

Theorem 2.4. *If $(X, \tau, \mathcal{I})$ is any ideal topological space and $A \subseteq X$, then the following are equivalent.*

1. *A is $\mathcal{I}_{\tilde{g}}$-closed.*

2. *$cl^\star(A) \subseteq U$ whenever $A \subseteq U$ and U is $^\#gs$-open in X.*

Proof.

(1) $\Rightarrow$ (2) Let A $\subseteq$ U where U is $^\#gs$-open in X. Since A is $\mathcal{I}_{\tilde{g}}$-closed, $A^\star \subseteq U$ and so $cl^\star(A) = A \cup A^\star \subseteq U$.

(2) $\Rightarrow$ (1) It follows from the fact that $A^\star \subseteq cl^\star(A) \subseteq U$.

□

Theorem 2.5. *Every $\star$-closed set is $\mathcal{I}_{\tilde{g}}$-closed but not conversely.*

Proof. Let A be a $\star$-closed. To prove A is $\mathcal{I}_{\tilde{g}}$-closed, let U be any $^\#gs$-open set such that $A \subseteq U$. Since A is $\star$-closed, $A^\star \subseteq A \subseteq U$. Thus A is $\mathcal{I}_{\tilde{g}}$-closed. □

Example 2.6. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$ and $\mathcal{I} = \{\phi\}$. It is clear that $\{a, c\}$ is $\mathcal{I}_{\tilde{g}}$-closed but not $\star$-closed.*

Theorem 2.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For every $A \in \mathcal{I}$, A is $\mathcal{I}_{\tilde{g}}$-closed.*

Proof. Let $A \in \mathcal{I}$ and let $A \subseteq U$ where U is $^\#gs$-open. Since $A \in \mathcal{I}$, $A^\star = \phi \subseteq U$. Thus A is $\mathcal{I}_{\tilde{g}}$-closed. □

Theorem 2.8. *If $(X, \tau, \mathcal{I})$ is an ideal topological space, then $A^\star$ is always $\mathcal{I}_{\tilde{g}}$-closed for every subset A of X.*

Proof. Let $A^\star \subseteq U$ where U is $^\#gs$-open. Since $(A^\star)^\star \subseteq A^\star$ [14], we have $(A^\star)^\star \subseteq U$. Hence $A^\star$ is $\mathcal{I}_{\tilde{g}}$-closed. □

Theorem 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every $\mathcal{I}_{\tilde{g}}$-closed, $^\#gs$-open set is $\star$-closed.*

Proof. Let A be $\mathcal{I}_{\tilde{g}}$-closed and $^\#gs$-open. We have $A \subseteq A$ where A is $^\#gs$-open. Since A is $\mathcal{I}_{\tilde{g}}$-closed, $A^\star \subseteq A$. Thus A is $\star$-closed. □

Corollary 2.10. *If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space and A is an $\mathcal{I}_{\tilde{g}}$-closed set, then A is $\star$-closed set.*

Proof. By assumption A is $\mathcal{I}_{\tilde{g}}$-closed in (X, τ, $\mathcal{I}$) and so by Theorem 2.2, A is $\mathcal{I}_g$-closed. Since (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space, by Definition 1.9, A is $\star$-closed. □

Corollary 2.11. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be an $\mathcal{I}_{\tilde{g}}$-closed set. Consider the following statements.*

1. *A is a $\star$-closed set,*
2. *$cl^\star(A) - A$ is an $^\# gs$-closed set,*
3. *$A^\star - A$ is an $^\# gs$-closed set.*

Then (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) hold.

Proof.
(1) $\Rightarrow$ (2) By (1) A is $\star$-closed. Hence $A^\star \subseteq A$ and $cl^\star(A) - A = (A \cup A^\star) - A = \phi$ which is an $^\# gs$-closed set.
(2) $\Rightarrow$ (3) $cl^\star(A) - A = A^\star \cup A - A = (A^\star \cup A) \cap A^c = (A^\star \cap A^c) \cup (A \cap A^c) = (A^\star \cap A^c) \cup \phi = A^\star - A$ which is an $^\# gs$-closed set by (2). □

Theorem 2.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every $\tilde{g}$-closed set is an $\mathcal{I}_{\tilde{g}}$-closed set but not conversely.*

Proof. Let A be a $\tilde{g}$-closed set. Let U be any $^\# gs$-open set such that $A \subseteq U$. Since A is $\tilde{g}$-closed, $cl(A) \subseteq U$. So, by Lemma 1.4, $A^\star \subseteq cl(A) \subseteq U$ and thus A is $\mathcal{I}_{\tilde{g}}$-closed. □

Example 2.13. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{c\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{a\}\}$. It is clear that $\{a\}$ is $\mathcal{I}_{\tilde{g}}$-closed but not $\tilde{g}$-closed.*

Theorem 2.14. *If (X, τ, $\mathcal{I}$) is an ideal topological space and A is a $\star$-dense in itself, $\mathcal{I}_{\tilde{g}}$-closed subset of X, then A is $\tilde{g}$-closed.*

Proof. Let $A \subseteq U$ where U is $^\# gs$-open. Since A is $\mathcal{I}_{\tilde{g}}$-closed, $A^\star \subseteq U$. As A is $\star$-dense in itself, by Lemma 1.5, $cl(A) = A^\star$. Thus $cl(A) \subseteq U$ and hence A is $\tilde{g}$-closed. □

Corollary 2.15. *If (X, τ, $\mathcal{I}$) is any ideal topological space where $\mathcal{I}=\{\phi\}$, then A is $\mathcal{I}_{\tilde{g}}$-closed if and only if A is $\tilde{g}$-closed.*

Proof. In (X, τ, $\mathcal{I}$), if $\mathcal{I} = \{\phi\}$ then $A^\star = cl(A)$ for the subset A. A is $\mathcal{I}_{\tilde{g}}$-closed $\Leftrightarrow$ $A^\star \subseteq U$ whenever $A \subseteq U$ and U is $^\# gs$-open $\Leftrightarrow$ $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $^\# gs$-open $\Leftrightarrow$ A is $\tilde{g}$-closed. □

Corollary 2.16. *In an ideal topological space (X, τ, $\mathcal{I}$) where $\mathcal{I}$ is codense, if A is a semi-open and $\mathcal{I}_{\tilde{g}}$-closed subset of X, then A is $\tilde{g}$-closed.*

Proof. By Lemma 1.6, A is $\star$-dense in itself. By Theorem 2.14, A is $\tilde{g}$-closed. □

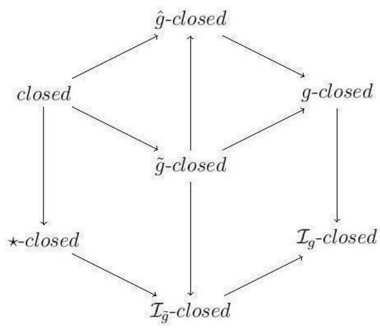
Example 2.17. *In Example 2.13, it is clear that $\{b\}$ is g-closed but not $\mathcal{I}_{\tilde{g}}$-closed.*

Example 2.18. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$ and $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. It is clear that $\{b\}$ is $\mathcal{I}_{\tilde{g}}$-closed but not g-closed.*

Example 2.19. *Let $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{c\}, \{a, b\}\}$. It is clear that $\{a\}$ is $\hat{g}$-closed but not $\tilde{g}$-closed.*

Remark 2.20. *We see that from Examples 2.17 and 2.18, g-closedness and $\mathcal{I}_{\tilde{g}}$-closedness are independent.*

Remark 2.21. *We have the following implications for the subsets stated above.*



Theorem 2.22. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If $A \subseteq B \subseteq A^\star$, then $A^\star = B^\star$ and B is $\star$-dense in itself.*

Proof. Since A⊆B, then $A^\star \subseteq B^\star$ and since $B \subseteq A^\star$, then $B^\star \subseteq (A^\star)^\star \subseteq A^\star$. Therefore $A^\star = B^\star$ and $B \subseteq A^\star \subseteq B^\star$. Hence proved. □

Theorem 2.23. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset of X is $\mathcal{I}_{\tilde{g}}$-closed if and only if every $^{\#}gs$-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{\tilde{g}}$-closed. Let U be $^{\#}gs$-open in X. Then U ⊆ U ⊆ X and U is $\mathcal{I}_{\tilde{g}}$-closed by assumption. It implies $U^\star \subseteq U$. Hence U is $\star$-closed.

Conversely, let A ⊆ X and U be $^{\#}gs$-open such that A ⊆ U. Since U is $\star$-closed by assumption, we have $A^\star \subseteq U^\star \subseteq U$. Thus A is $\mathcal{I}_{\tilde{g}}$-closed. □

Theorem 2.24. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $\mathcal{I}_{\tilde{g}}$-open if and only if $F \subseteq int^\star(A)$ whenever F is $^{\#}gs$-closed and $F \subseteq A$.*

Proof. Suppose A is $\mathcal{I}_{\tilde{g}}$-open. If F is $^{\#}gs$-closed and F⊆A, then X−A⊆X−F and so $cl^\star$(X−A)⊆X−F by Theorem 2.4(2). Therefore F⊆X−$cl^\star$(X−A)=$int^\star$(A). Hence F⊆$int^\star$(A).

Conversely, suppose the condition holds. Let U be an $^{\#}gs$-open set such that X−A⊆U. Then X−U⊆A and so X−U⊆$int^\star$(A). Therefore $cl^\star$(X−A)⊆U. By Theorem 2.4(2), X−A is $\mathcal{I}_{\tilde{g}}$-closed. Hence A is $\mathcal{I}_{\tilde{g}}$-open. □

The following Theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{\tilde{g}}$-open sets.

Theorem 2.25. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is normal,*
2. *For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{\tilde{g}}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$,*
3. *For any closed set A and open set V containing A, there exists an $\mathcal{I}_{\tilde{g}}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*

Proof.

(1)⇒(2) The proof follows from the fact that every open set is $\mathcal{I}_{\tilde{g}}$-open.

(2)⇒(3) Suppose A is closed and V is an open set containing A. Since A and X−V are disjoint closed sets, there exist disjoint $\mathcal{I}_{\tilde{g}}$-open sets U and W such that A⊆U and X−V⊆W. Since X−V is $^{\#}gs$-closed and W is $\mathcal{I}_{\tilde{g}}$-open, X−V⊆$int^\star$(W). Then X−$int^\star$(W)⊆V. Again U∩W=ϕ which implies that U∩$int^\star$(W)=ϕ and so U⊆X−$int^\star$(W). Then $cl^\star$(U)⊆X−$int^\star$(W)⊆V and thus U is the required $\mathcal{I}_{\tilde{g}}$-open sets with A⊆U⊆$cl^\star$(U)⊆V.

(3)⇒(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and X − B an open set containing A. By hypothesis, there exists an $\mathcal{I}_{\tilde{g}}$-open set U such that A⊆U⊆$cl^\star$(U)⊆X−B. Since U is $\mathcal{I}_{\tilde{g}}$-open and A is $^{\#}gs$-closed we have, by

Theorem 2.24, A⊆int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.7, $\tau^\star \subseteq \tau^\alpha$ and so int*(U) and X−cl*(U)∈τ^α. Hence A⊆int*(U)⊆ int(cl(int(int*(U))))=G and B⊆X−cl*(U)⊆ int(cl(int(X−cl*(U)))) =H. G and H are the required disjoint open sets containing A and B respectively, which proves (1). □

Definition 2.26 ([13]). *A subset A of a topological space (X, τ) is said to be an $\tilde{g}_\alpha$-closed set if $cl_\alpha(A) \subseteq U$ whenever A⊆U and U is $^\#$gs-open. The complement of $\tilde{g}_\alpha$-closed set is said to be an $\tilde{g}_\alpha$-open set.*

If $\mathcal{I}=\mathcal{N}$, it is not difficult to see that $\mathcal{I}_{\tilde{g}}$-closed sets coincide with $\tilde{g}_\alpha$-closed sets and so we have the following Corollary.

Corollary 2.27. *Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I}=\mathcal{N}$. Then the following are equivalent.*

1. *X is normal,*
2. *For any disjoint closed sets A and B, there exist disjoint $\tilde{g}_\alpha$-open sets U and V such that A⊆U and B⊆V,*
3. *For any closed set A and open set V containing A, there exists an $\tilde{g}_\alpha$-open set U such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*

Definition 2.28. *A subset A of an ideal topological space is said to be $\mathcal{I}$-compact [9] or compact modulo $\mathcal{I}$ [25] if for every open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $A - \cup\{U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I}$. The space (X, τ, $\mathcal{I}$) is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.*

Theorem 2.29. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [[23], Theorem 2.17].*

Corollary 2.30. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an $\mathcal{I}_{\tilde{g}}$-closed subset of X, then A is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every $\mathcal{I}_{\tilde{g}}$-closed is $\mathcal{I}_g$-closed. □

Remark 2.31. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. By Remark 1.15, Definition 1.19, Definition 2.1 and Theorem 2.2, the following diagram holds for a subset G ⊆ X:*

$$\begin{array}{ccccc} gs^\star_{\mathcal{I}}\textbf{-closed} & & & & \textbf{weakly } \mathcal{I}_{rg}\textbf{-closed} \\ \uparrow & & & & \uparrow \\ \mathcal{I}_{\tilde{g}}\textbf{-closed} & \longrightarrow & \mathcal{I}_g\textbf{-closed} & \longrightarrow & \mathcal{I}_{rg}\textbf{-closed} \end{array}$$

These implications are not reversible.

Example 2.32. *In Example 2.13, it is clear that {b} is $gs^\star_{\mathcal{I}}$-closed set but not $\mathcal{I}_{\tilde{g}}$-closed.*

Definition 2.33. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be a $s^\star C_{\mathcal{I}}$-set if A = L ∩ M, where L ∈ τ and M is a semi*-$\mathcal{I}$-closed set in X.*

Theorem 2.34. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and V ⊆ X. Then V is a $s^\star C_{\mathcal{I}}$-set in X if and only if V = G ∩ $s^\star_{\mathcal{I}}cl(V)$ for an open set G in X.*

Proof. If V is a s*C$_{\mathcal{I}}$-set, then V = G ∩ M for an open set G and a semi*-$\mathcal{I}$-closed set M. But then V ⊆ M and so V ⊆ s$^\star_{\mathcal{I}}$cl(V) ⊆ M. It follows that V = V ∩ s$^\star_{\mathcal{I}}$cl(V) = G ∩ M ∩ s$^\star_{\mathcal{I}}$cl(V) = G ∩ s$^\star_{\mathcal{I}}$cl(V). Conversely, it is enough to prove that s$^\star_{\mathcal{I}}$cl(V) is a semi*-$\mathcal{I}$-closed set. Any semi*-$\mathcal{I}$-closed set containing V contains s$^\star_{\mathcal{I}}$cl(V) also and any semi*-$\mathcal{I}$-closed set containing s$^\star_{\mathcal{I}}$cl(V) contains V. Hence s$^\star_{\mathcal{I}}$cl(V) = s$^\star_{\mathcal{I}}$cl(s$^\star_{\mathcal{I}}$cl(V))=s$^\star_{\mathcal{I}}$cl(V) ∪ int(cl*(s$^\star_{\mathcal{I}}$cl(V))) and thus int(cl*(s$^\star_{\mathcal{I}}$cl(V))) ⊆ s$^\star_{\mathcal{I}}$cl(V). Thus s$^\star_{\mathcal{I}}$cl(V) is semi*-$\mathcal{I}$-closed. □

Theorem 2.35. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A ⊆ X. The following properties are equivalent.*

1. *A is a semi***-*$\mathcal{I}$*-closed set in X.*

2. *A is a* $s^\star C_{\mathcal{I}}$*-set and a* $gs^\star_{\mathcal{I}}$*-closed set in X.*

Proof.
(1) ⇒ (2): It follows from the fact that any semi*-$\mathcal{I}$-closed set in X is a s*C$_{\mathcal{I}}$-set and a gs$^\star_{\mathcal{I}}$-closed set in X.
(2) ⇒ (1): Suppose that A is a s*C$_{\mathcal{I}}$-set and a gs$^\star_{\mathcal{I}}$-closed set in X. Since A is a s*C$_{\mathcal{I}}$-set, then by Theorem 2.34, A = G ∩ s$^\star_{\mathcal{I}}$cl(A) for an open set G in (X, τ, $\mathcal{I}$). Since A ⊆ G and A is gs$^\star_{\mathcal{I}}$-closed in X, we have s$^\star_{\mathcal{I}}$cl(A) ⊆ G. It follows that s$^\star_{\mathcal{I}}$cl(A) = A and hence A is semi*-$\mathcal{I}$-closed. □

3. $^\#gs$-$\mathcal{I}$-locally closed sets

Definition 3.1. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called an* $^\#gs$*-*$\mathcal{I}$*-locally closed set (briefly,* $^\#gs$*-*$\mathcal{I}$*-LC) if A=U∩V where U is* $^\#gs$*-open and V is* $\star$*-closed.*

Definition 3.2 ([15])**.** *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a weakly* $\mathcal{I}$*-locally closed set (briefly, weakly* $\mathcal{I}$*-LC) if A=U∩V where U is open and V is* $\star$*-closed.*

Proposition 3.3. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following hold.*

1. *If A is* $^\#gs$*-open, then A is* $^\#gs$*-*$\mathcal{I}$*-LC-set.*

2. *If A is* $\star$*-closed, then A is* $^\#gs$*-*$\mathcal{I}$*-LC-set.*

3. *If A is a weakly* $\mathcal{I}$*-LC-set, then A is an* $^\#gs$*-*$\mathcal{I}$*-LC-set.*

The converses of the above Proposition 3.3 need not be true as shown in the following examples.

Example 3.4. 1. *In Example 2.13, it is clear that* $\{b\}$ *is an* $^\#gs$*-*$\mathcal{I}$*-LC-set but not* $\star$*-closed.*

2. *In Example 2.18, it is clear that* $\{a\}$ *is an* $^\#gs$*-*$\mathcal{I}$*-LC-set but not* $^\#gs$*-open.*

Example 3.5. *In Example 2.13, it is clear that* $\{b\}$ *is an* $^\#gs$*-*$\mathcal{I}$*-LC-set but not a weakly* $\mathcal{I}$*-LC-set.*

Theorem 3.6. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an* $^\#gs$*-*$\mathcal{I}$*-LC-set and B is a* $\star$*-closed set, then A∩B is an* $^\#gs$*-*$\mathcal{I}$*-LC-set.*

Proof. Let B be $\star$-closed, then A∩B=(U∩V)∩B=U∩(V∩B), where V∩B is $\star$-closed. Hence A∩B is an $^\#gs$-$\mathcal{I}$-LC-set. □

Theorem 3.7. *A subset of an ideal topological space (X, τ, $\mathcal{I}$) is* $\star$*-closed if and only if it is*

1. *weakly* $\mathcal{I}$*-LC and* $\mathcal{I}_g$*-closed. [11]*

2. $^\#gs$*-*$\mathcal{I}$*-LC and* $\mathcal{I}_{\tilde{g}}$*-closed.*

Proof. (2) Necessity is trivial. We prove only sufficiency. Let A be $^\#gs$-$\mathcal{I}$-LC-set and $\mathcal{I}_{\tilde{g}}$-closed set. Since A is $^\#gs$-$\mathcal{I}$-LC, A=U∩V, where U is $^\#gs$-open and V is $\star$-closed. So, we have A=U∩V⊆U. Since A is $\mathcal{I}_{\tilde{g}}$-closed, A* ⊆ U. Also since A = U∩V⊆V and V is $\star$-closed, we have A* ⊆ V. Consequently, A* ⊆U∩ V = A and hence A is $\star$-closed. □

Remark 3.8.

1. *The notions of weakly* $\mathcal{I}$*-LC-set and* $\mathcal{I}_g$*-closed set are independent [11].*

2. *The notions of $^{\#}gs$-$\mathcal{I}$-LC-set and $\mathcal{I}_{\tilde{g}}$-closed set are independent.*

Example 3.9. *In Example 2.13, it is clear that $\{b\}$ is $^{\#}gs$-$\mathcal{I}$-LC-set but not $\mathcal{I}_{\tilde{g}}$-closed.*

Example 3.10. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\phi\}$. It is clear that $\{a, c, d\}$ is $\mathcal{I}_{\tilde{g}}$-closed set but not $^{\#}gs$-$\mathcal{I}$-LC-set.*

Definition 3.11. *Let A be a subset of a topological space (X, τ). Then the $^{\#}gs$-kernel of the set A, denoted by $^{\#}gs$-ker(A), is the intersection of all $^{\#}gs$-open supersets of A.*

Definition 3.12. *A subset A of a topological space (X, τ) is called $\Lambda_{\#gs}$-set if A=$^{\#}gs$-ker(A).*

Definition 3.13. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called $\lambda_{\#gs}$-$\mathcal{I}$-closed if A=L∩F where L is a $\Lambda_{\#gs}$-set and F is $\star$-closed.*

Lemma 3.14. 1. *Every $\star$-closed set is $\lambda_{\#gs}$-$\mathcal{I}$-closed but not conversely.*

2. *Every $\Lambda_{\#gs}$-set is $\lambda_{\#gs}$-$\mathcal{I}$-closed but not conversely.*

Example 3.15. *In Example 2.13, it is clear that $\{b\}$ is $\lambda_{\#gs}$-$\mathcal{I}$-closed but not $\star$-closed.*

Example 3.16. *In Example 2.18, it is clear that $\{a\}$ is $\lambda_{\#gs}$-$\mathcal{I}$-closed but not a $\Lambda_{\#gs}$-set.*

Remark 3.17. *It is easily observed from Examples 3.15 and 3.16, that the concepts of $\Lambda_{\#gs}$-set and $\star$-closed set are independent for $\{b\}$ is a $\Lambda_{\#gs}$-set but not a $\star$-closed set whereas $\{a\}$ is $\star$-closed but not a $\Lambda_{\#gs}$-set.*

Lemma 3.18. *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *A is $\lambda_{\#gs}$-$\mathcal{I}$-closed.*
2. *A=L∩$cl^\star(A)$ where L is a $\Lambda_{\#gs}$-set.*
3. *A=$^{\#}gs$-ker(A)∩$cl^\star(A)$.*

Lemma 3.19. *A subset A⊆$(X, \tau, \mathcal{I})$ is $\mathcal{I}_{\tilde{g}}$-closed if and only if $cl^\star(A)$⊆ $^{\#}gs$-ker(A).*

Proof. Suppose that A ⊆ X is an $\mathcal{I}_{\tilde{g}}$-closed set. Suppose x ∉ $^{\#}gs$-ker(A). Then there exists an $^{\#}gs$-open set U containing A such that x ∉ U. Since A is an $\mathcal{I}_{\tilde{g}}$-closed set, A ⊆ U and U is $^{\#}gs$-open implies that cl*(A) ⊆ U and so x ∉ cl*(A). Therefore cl*(A) ⊆ $^{\#}gs$-ker(A).
Conversely, suppose cl*(A) ⊆ $^{\#}gs$-ker(A). If A ⊆ U and U is $^{\#}gs$-open, then cl*(A) ⊆ $^{\#}gs$-ker(A) ⊆ U. Therefore, A is $\mathcal{I}_{\tilde{g}}$-closed. □

Theorem 3.20. *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *A is $\star$-closed.*
2. *A is $\mathcal{I}_{\tilde{g}}$-closed and $^{\#}gs$-$\mathcal{I}$-LC.*
3. *A is $\mathcal{I}_{\tilde{g}}$-closed and $\lambda_{\#gs}$-$\mathcal{I}$-closed.*

Proof.
(1)⇒(2)⇒(3) Obvious.
(3)⇒(1) Since A is $\mathcal{I}_{\tilde{g}}$-closed, by Lemma 3.19, cl*(A)⊆$^{\#}gs$-ker(A). Since A is $\lambda_{\#gs}$-$\mathcal{I}$-closed, by Lemma 3.18, A=$^{\#}gs$-ker(A)∩cl*(A)=cl*(A). Hence A is $\star$-closed. □

The following two Examples show that the concepts of $\mathcal{I}_{\tilde{g}}$-closedness and $\lambda_{\#gs}$-$\mathcal{I}$-closedness are independent.

Example 3.21. *In Example 3.10, it is clear that* $\{a, c, d\}$ *is* $\mathcal{I}_{\tilde{g}}$*-closed but not* $\lambda_{\#gs}$*-*$\mathcal{I}$*-closed.*

Example 3.22. *In Example 2.13, it is clear that* $\{b\}$ *is* $\lambda_{\#gs}$*-*$\mathcal{I}$*-closed but not* $\mathcal{I}_{\tilde{g}}$*-closed.*

4. Decompositions of ⋆-continuity

Definition 4.1. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is said to be ⋆-continuous [11] (resp.* $\mathcal{I}_g$*-continuous [11],* $^{\#}gs$*-*$\mathcal{I}$*-LC-continuous,* $\lambda_{\#gs}$*-*$\mathcal{I}$*-continuous,* $\mathcal{I}_{\tilde{g}}$*-continuous, weakly* $\mathcal{I}$*-LC-continuous [15]) if* $f^{-1}(A)$ *is ⋆-closed (resp.* $\mathcal{I}_g$*-closed,* $^{\#}gs$*-*$\mathcal{I}$*-LC-set,* $\lambda_{\#gs}$*-*$\mathcal{I}$*-closed,* $\mathcal{I}_{\tilde{g}}$*-closed, weakly* $\mathcal{I}$*-LC-set) in* $(X, \tau, \mathcal{I})$ *for every closed set* A *of* (Y, σ)*.*

Theorem 4.2. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is ⋆-continuous if and only if it is*

1. *weakly* $\mathcal{I}$*-LC-continuous and* $\mathcal{I}_g$*-continuous [11].*
2. $^{\#}gs$*-*$\mathcal{I}$*-LC-continuous and* $\mathcal{I}_{\tilde{g}}$*-continuous.*

Proof. It is an immediate consequence of Theorem 3.7. □

Theorem 4.3. *For a function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$*, the following are equivalent.*

1. *f is ⋆-continuous.*
2. *f is* $\mathcal{I}_{\tilde{g}}$*-continuous and* $^{\#}gs$*-*$\mathcal{I}$*-LC-continuous.*
3. *f is* $\mathcal{I}_{\tilde{g}}$*-continuous and* $\lambda_{\#gs}$*-*$\mathcal{I}$*-continuous.*

Proof. It is an immediate consequence of Theorem 3.20. □

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