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On Some Properties of p-k-Generalized Mittag-Leffler Type Function

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Abstract: In this article, we investigated and study the p-k-generalized Mittag-Leffler type function and its basic properties including recurrence relation, usual differentiation and integration, Euler(Beta) transform, Laplace transform, Whittaker transform and Hankel transform. We also addressed their special cases.

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1. Introduction and Preliminaries

Mittag-Leffler functions play an essential role in determining the solutions of integral equations and fractional differential which are associated with an extensive variety of problems in diverse areas of mathematics, engineering, numerical analysis and mathematical physics. Solutions of fractional differential equations are represented in the form of Mittag–Leffler functions. Special functions including gamma function, beta function, Mittag–Leffler function, hypergeometric function, Wright function are very important in the study of geometric function theory, applied mathematics, physics, statistics and many other subjects. The extensions of Mittag–Leffler function is an interesting topic for researchers in which the classical notions linked with predefined Mittag–Leffler functions are investigated in more general prospect, see [1, 3, 9]. Gosta Mittag-Leffler (1846-1927) [8] presented an entire function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, z, \alpha \in \mathbb{C}; Re(\alpha) > 0, \tag{1}$$

which is called Mittag-Leffler function.

The p-k Pochhammer symbol ${}_p(x)_{n,k}$ was defined by Gehlot [5] as

$${}_p(x)_{n,k} = \left(\frac{xp}{k}\right)\left(\frac{xp}{k}+p\right)\left(\frac{xp}{k}+2p\right)....\left(\frac{xp}{k}+(n-1)p\right), where\ x \in \mathbb{C}, k, p \in \mathbb{R}^+, n \in \mathbb{N} \tag{2}$$

The p-k Gamma function ${}_p\Gamma_k(x)$ was defined by Gehlot [5] as

$${}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt, \quad x \in \frac{\mathbb{C}}{k\mathbb{Z}^-}, k, p \in \mathbb{R}^+, Re(x) > 0, \tag{3}$$

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and

$$ {}_p\Gamma_k(x+k) = \frac{xp}{k} {}_p\Gamma_k(x), $$

The p-k Beta function ${}_pB_k(x,y)$ was also defined by Gehlot [5] as

$$ {}_pB_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}\Big(1-t\Big)^{\frac{y}{k}-1} dt, x, y \in \mathbb{C}, k > 0, Re(x) > 0, Re(y) > 0, \tag{4} $$

and

$$ {}_pB_k(x,y) = \frac{{}_p\Gamma_k(x)\,{}_p\Gamma_k(y)}{{}_p\Gamma_k(x+y)}. $$

The well known Gamma function $\Gamma(n)$ is defined as

$$ \Gamma(n) = \int_0^\infty e^{-t}t^{n-1}, n \in \mathbb{C}, Re(n) > 0. \tag{5} $$

The Beta function B(m,n) is defined as

$$ B(m,n) = \int_0^1 t^{n-1}(1-t)^{m-1}dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m, n \in \mathbb{C}, Re(m) > 0, Re(n) > 0. \tag{6} $$

The Fox- Wright function ${}_p\psi_q(z)$ was defined by Srivastava and Karlsson [6] as

$$ {}_p\psi_q\left[\begin{matrix} (a_1, A_1), ..., (a_p, A_p); \\ \\ (b_1, B_1), ..., (b_q, B_q); \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + A_i n)}{\prod_{j=1}^{q}\Gamma(b_j + B_j n)} \frac{z^n}{n!}, \tag{7} $$

where, $z, a_i, b_j, A_i, B_j, \in \mathbb{C}, Re(a_i) > 0, Re(A_i) > 0, i = 1, ...p, Re(b_j) > 0, Re(B_j) > 0, j = 1, ..., q$ and $1 + Re\Big(\sum\limits_{j=1}^{q}(B_j) - \sum\limits_{i=1}^{p}(A_i)\Big) \geq 0$. The following established identities are needed in our sequel.

Identity I. Let $x \in \mathbb{C}/k\mathbb{Z}^-$, and $k, p \in \mathbb{R}^+$, then the following result holds

$$ {}_p\Gamma_k(x) = \Big(\frac{p}{k}\Big)^{\frac{x}{k}}\Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k}\Gamma\Big(\frac{x}{k}\Big). \tag{8} $$

Identity II. Let $x \in \mathbb{C}/k\mathbb{Z}^-$; $n, q \in \mathbb{N}$; $k, p \in \mathbb{R}^+$ and $Re(x) > 0$, then the following result holds

$$ {}_p(x)_{nq,k} = \Big(\frac{p}{k}\Big)^{nq}(x)_{nq,k} = (p)^{nq}\Big(\frac{x}{k}\Big)_{nq}. \tag{9} $$

1.1. A New Generalized Mittag-Leffler Type Function

In this section, we establish and describe a new generalized Mittag-Leffler type function, called p-k-generalized Mittag-Leffler function ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$ and list some of its special cases.

Definition 1.1. *Let $p, k \in \mathbb{R}^+, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\mu) > 0, Re(\nu) > 0, Re(\rho) > 0, Re(\sigma) > 0$, and $q, r > 0, q \leq Re(\alpha) + r$, the p-k-generalized Mittag-Leffler function denoted by ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$ and is defined as*

$$ {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n + \beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}}, \tag{10} $$

where ${}_p(x)_{nq,k}$ is the p-k Pochhammer symbol is defined in equation (2) and ${}_p\Gamma_k(x)$ is the p-k Gamma function defined in equation (3).

Special cases of ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$

On giving some specific values to the parameters $p, k, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma$, we can obtain certain Mittag-Leffler functions defined before

(a). For $p = k$ in equation (10) reduces to known result of Bairwa and Sharma[10],

$$\begin{aligned}{}_kE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) &= \sum_{n=0}^{\infty} \frac{{}_k(\mu)_{\rho n,k}}{{}_k\Gamma_k(\alpha n+\beta)} \frac{{}_k(\gamma)_{qn,k}}{{}_k(\nu)_{\sigma n,k}} \frac{z^n}{{}_k(\delta)_{rn,k}} \\ &= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\gamma)_{qn,k} z^n}{\Gamma_k(\alpha n+\beta)(\nu)_{\sigma n,k}(\delta)_{rn,k}} \\ &= E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z).\end{aligned} \tag{11}$$

(b). Putting $p = k = 1$ in equation (10) reduces to known result of Khan and Ahmed [7],

$$\begin{aligned}E^{\mu,\rho,\gamma,q}_{1,\alpha,\beta,\nu,\sigma,\delta,r}(z) &= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,1}(\gamma)_{qn,1} z^n}{\Gamma_1(\alpha n+\beta)(\nu)_{\sigma n,1}(\delta)_{rn,1}} \\ &= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{rn}} \\ &= E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,r}(z).\end{aligned} \tag{12}$$

(c). Putting $p = k = 1$, $\mu = \nu, \rho = \sigma, r = 1, \delta = 1$, in equation (10) reduces to known result of Shukla and Prajapati [3],

$$\begin{aligned}E^{\mu,\rho,\gamma,q}_{1,\alpha,\beta,\mu,\rho,1,1}(z) &= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,1}(\gamma)_{qn,1} z^n}{\Gamma_1(\alpha n+\beta)(\mu)_{\rho n,1}(1)_{1n,1}} \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} \\ &= E^{\gamma,q}_{\alpha,\beta}(z).\end{aligned} \tag{13}$$

(d). On taking $p = k = 1$, $\mu = \nu, \rho = \sigma, q = 1, r = 1, \delta = 1$, in equation (10) reduces to known result of Prabhakar [12],

$$\begin{aligned}E^{\mu,\rho,\gamma,1}_{1,\alpha,\beta,\mu,\rho,1,1}(z) &= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,1}(\gamma)_{1n,1} z^n}{\Gamma_1(\alpha n+\beta)(\mu)_{\rho n,1}(1)_{1n,1}} \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta) n!} \\ &= E^{\gamma}_{\alpha,\beta}(z).\end{aligned} \tag{14}$$

(e). On taking $p = k = 1$, $\mu = \nu, \rho = \sigma, q = r = \delta = \gamma = 1$, in equation (10) reduces to known result of Wiman [4],

$$\begin{aligned}E^{\mu,\rho,1,1}_{1,\alpha,\beta,\mu,\rho,1,1}(z) &= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,1}(1)_{1n,1} z^n}{\Gamma_1(\alpha n+\beta)(\mu)_{\rho n,1}(1)_{1n,1}} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \\ &= E_{\alpha,\beta}(z).\end{aligned} \tag{15}$$

(f). Assigning $p = k = 1$, $\mu = \nu, \rho = \sigma, q = r = \delta = \gamma = \beta = 1$, equation (10) reduces to Mittag-Leffler function $E_\alpha(z)$ defined by Gosta Mittag-Leffler [8],

$$\begin{aligned} E^{\mu,\rho,1,1}_{1,\alpha,1,\mu,\rho,1,1}(z) &= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,1}(1)_{1n,1} z^n}{\Gamma_1(\alpha n+1)(\mu)_{\rho n,1}(1)_{1n,1}} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \\ &= E_\alpha(z). \end{aligned} \tag{16}$$

2. Basic Properties

In this section, we discussed some basic properties including recurrence relation, usual differentiation and integration, Euler(Beta) transform, Laplace transform,Whittaker transform and Hankel transform.

Theorem 2.1. *The elegant relationship between p-k-generalized Mittag-Leffler type function ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$ and generalized Mittag-Leffler type function has been defined as follows*

$${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = kp^{\frac{-\beta}{k}} E^{\frac{\mu}{k},\rho,\frac{\gamma}{k},q}_{\frac{\alpha}{k},\frac{\beta}{k},\frac{\nu}{k},\sigma,\frac{\delta}{k},r}(p^{\rho+q-\sigma-r-\frac{\alpha}{k}} z). \tag{17}$$

Proof. From the definition of p-k-generalized Mittag-Leffler function given by the equation (10) we get

$${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}},$$

On using equations (8) and (9), we have

$$\begin{aligned} {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) &= kp^{-\frac{\beta}{k}} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{k}\right)_{\rho n} \left(\frac{\gamma}{k}\right)_{qn} \left(p^{\rho+q-\sigma-r-\frac{\alpha}{k}} z\right)^n}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)\left(\frac{\nu}{k}\right)_{\sigma n}\left(\frac{\delta}{k}\right)_{rn}} \\ {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) &= kp^{-\dfrac{\beta}{k}} E^{\dfrac{\mu}{k},\rho,\dfrac{\gamma}{k},q}_{\dfrac{\alpha}{k},\dfrac{\beta}{k},\dfrac{\nu}{k},\sigma,\dfrac{\delta}{k},r} \left(p^{\rho+q-\sigma-r-\frac{\alpha}{k}} z\right). \end{aligned}$$

This completes the proof of the above theorem. $\square$

Theorem 2.2 (Recurrence Relation)**.** *If $p, k \in \mathbb{R}^+, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\mu) > 0, Re(\nu) > 0, Re(\rho) > 0, Re(\sigma) > 0$, and $q, r > 0, q \le Re(\alpha) + r$, then*

$${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \beta\, {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\nu,\sigma,\delta,r}(z) + \alpha z \frac{d}{dz} {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\nu,\sigma,\delta,r}(z). \tag{18}$$

Proof. In the R.H.S. of equation (18) and making use of the series form of ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$ given by the equation (10), we have

$\beta\, {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\nu,\sigma,\delta,r}(z) + \alpha z \frac{d}{dz} {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\nu,\sigma,\delta,r}(z)$

$$\begin{aligned} &= \beta \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}} + \alpha z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}} \\ &= \beta \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}} + \alpha n \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}} \end{aligned}$$

$$= \beta \sum_{n=0}^{\infty} \frac{(\alpha n+\beta)}{{}_p\Gamma_k(\alpha n+\beta+k)} \frac{{}_p(\mu)_{\rho n,k}}{{}_p(\nu)_{\sigma n,k}} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\delta)_{rn,k}} z^n$$
$$= \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}}$$
$$= {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z).$$

This completes the proof of the above theorem. □

Special Cases

Remark 2.3. *On Setting $p = k$, in equation (18), takes the following result, we get*

$$E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \beta E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\nu,\sigma,\delta,r}(z) + \alpha z \frac{d}{dz} E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\nu,\sigma,\delta,r}(z),$$

which is the similar result as deduced by Bairwa and Sharma [10].

Remark 2.4. *Putting $p = k = 1$, in equation (18), takes the following result, we get*

$$E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,r}(z) = \beta E^{\mu,\rho,\gamma,q}_{\alpha,\beta+1,\nu,\sigma,\delta,r}(z) + \alpha z \frac{d}{dz} E^{\mu,\rho,\gamma,q}_{\alpha,\beta+1,\nu,\sigma,\delta,r}(z)$$

which is the similar result as deduced by Khan and Ahmed [7].

Remark 2.5. *Putting $p = k = 1$, $\mu = \nu, \rho = \sigma, r = 1, \delta = 1$, equation (18) reduces to the following result, we have*

$$E^{\gamma,q}_{\alpha,\beta}(z) = \beta E^{\gamma,q}_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E^{\gamma,q}_{\alpha,\beta+1}(z)$$

which is the similar result as deduced by Shukla and Prajapati [3].

Remark 2.6. *The equation (18) reduces to the following form on taking p=k=1, $\mu = \nu, \rho = \sigma$, r=1, q=1, and $\delta = 1$, we have*

$$E^{\gamma}_{\alpha,\beta}(z) = \beta E^{\gamma}_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E^{\gamma}_{\alpha,\beta+1}(z)$$

which is the similar result as given by Prabhakar [12].

Theorem 2.7 (Differential Formula)**.** *If $p, k \in \mathbb{R}^+, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\mu) > 0, Re(\nu) > 0, Re(\rho) > 0, Re(\sigma) > 0,$ and $q, r > 0, q \leq Re(\alpha) + r, m \in \mathbb{N}$, then*

$$\left(\frac{d}{dz}\right)^m {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \frac{(1)_m\, {}_p(\mu)_{\rho m,k}}{{}_p(\nu)_{\sigma m,k}} \frac{{}_p(\gamma)_{qm,k}}{{}_p(\delta)_{rm,k}} \times \sum_{n=0}^{\infty} \frac{{}_p(\mu+\rho mk)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\alpha n+\beta)} \frac{{}_p(\gamma+qmk)_{qn,k}}{{}_p(\nu+\sigma mk)_{\sigma n,k}} \frac{(1+m)_n}{{}_p(\delta+pmk)_{rn,k}} \frac{(z)^n}{n!}. \tag{19}$$

Proof. In the L. H. S. of the equation (19), and making use of the series form of ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$, given by the equation (10), we have

$$\left(\frac{d}{dz}\right)^m {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}}$$
$$= \sum_{n=m}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}} \frac{n!}{(n-m)!}$$

$$= \sum_{n=m}^{\infty} \frac{(1)_m(1+m)_n}{{}_p\Gamma_k(\alpha n+\alpha m+\beta)} \frac{{}_p(\mu)_{\rho(n+m),k}}{{}_p(\nu)_{\sigma(n+m),k}} \frac{{}_p(\gamma)_{q(n+m),k}}{{}_p(\delta)_{r(n+m),k}} \frac{z^n}{(n)!},$$

on using the relation ${}_p(x)_{n+j,k} = {}_p(x)_{j,k} \times {}_p(x+jk)_{n,k}$, we have

$$\left(\frac{d}{dz}\right)^m {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \frac{{}_p(\mu)_{\rho m,k}}{{}_p(\nu)_{\sigma m,k}} \frac{{}_p(\gamma)_{qm,k}}{{}_p(\delta)_{rm,k}} \times \sum_{n=0}^{\infty} \frac{{}_p(\mu+\rho mk)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\alpha n+\beta)} \frac{{}_p(\gamma+qmk)_{qn,k}}{{}_p(\nu+\sigma mk)_{\sigma n,k}} \frac{(1)_m(1+m)_n}{{}_p(\delta+pmk)_{rn,k}} \frac{(z)^n}{n!}.$$

This completes the proof of the above theorem. □

Special Cases

Remark 2.8. *Taking $p = k$, equation (19), reduces to the following result, we have*

$$\left(\frac{d}{dz}\right)^m E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z) = \frac{(1)_m(\mu)_{\rho m,k}}{(\nu)_{\sigma m,k}} \frac{(\gamma)_{qm,k}}{(\delta)_{rm,k}} \times \sum_{n=0}^{\infty} \frac{(\mu+\rho mk)_{\rho n,k}}{\Gamma_k(\alpha n+\alpha n+\beta)} \frac{(\gamma+qmk)_{qn,k}}{(\nu+\sigma mk)_{\sigma n,k}} \frac{(1+m)_n}{(\delta+pmk)_{rn,k}} \frac{(z)^n}{n!},$$

which is the similar result as deduced by Bairwa and Sharma [10].

Remark 2.9. *Taking $p = k = 1$, equation (19), reduces to the following result, we get*

$$\left(\frac{d}{dz}\right)^m E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,r}(z) = \frac{(1)_m(\mu)_{\rho m}}{(\nu)_{\sigma m}} \frac{(\gamma)_{qm}}{(\delta)_{sm}} \times \sum_{n=0}^{\infty} \frac{(\mu+\rho mk)_{\rho n}}{\Gamma(\alpha n+\alpha n+\beta)} \frac{(\gamma+qmk)_{qn}}{(\nu+\sigma mk)_{\sigma n}} \frac{(1+m)_n}{(\delta+pmk)_{rn}} \frac{(z)^n}{n!},$$

which is the similar result as deduced by Khan and Ahmed [7].

Remark 2.10. *On setting $p = k = 1$, $\mu = \nu, \rho = \sigma, r = 1$, and $\delta = 1$, equation (19), given as*

$$\left(\frac{d}{dz}\right)^m E^{\gamma,q}_{\alpha,\beta}(z) = (\gamma)_{qm} \sum_{n=0}^{\infty} \frac{(\gamma+qmk)_{qn}}{\Gamma(\alpha n+\alpha m+\beta)} \frac{z^n}{n!}$$

$$= (\gamma)_{qm} E^{\gamma+qm,q}_{\alpha,\beta+m\alpha}(z),$$

which is the similar result as deduced by Shukla and Prajapati [3].

Remark 2.11. *Putting $p = k = 1$, $\mu = \nu, \rho = \sigma$, r=1, q=1, and $\delta = 1$, equation (19), takes the following form, we get*

$$\left(\frac{d}{dz}\right)^m E^{\gamma}_{\alpha,\beta}(z) = (\gamma)_m \sum_{n=0}^{\infty} \frac{(\gamma+mk)_n}{\Gamma(\alpha n+\alpha m+\beta)} \frac{z^n}{n!}$$

$$= (\gamma)_m E^{\gamma+m}_{\alpha,\beta+m\alpha}(z),$$

which is the similar result as given by Prabhakar [12].

Theorem 2.12 (Integral Representation)**.** *If $p, k \in \mathbb{R}^+, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\mu) > 0, Re(\nu) > 0, Re(\rho) > 0, Re(\sigma) > 0,$ and $q, r > 0, q \leq Re(\alpha) + r,$ then*

$$\frac{1}{{}_p\Gamma_k(\eta)} \int_0^1 u^{\frac{\beta}{k}-1}\left(1-u\right)^{\frac{\eta}{k}-1} {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}\left(zu^{\frac{\alpha}{k}}\right)du = k\ {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+\eta,\nu,\sigma,\delta,r}(z). \tag{20}$$

Proof. In the L. H. S. of equation (20), and making use of the series form of ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$, given by the equation (10), we get

$$\frac{1}{{}_p\Gamma_k(\eta)} \int_0^1 u^{\frac{\beta}{k}-1}\left(1-u\right)^{\frac{\eta}{k}-1} {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}\left(zu^{\frac{\alpha}{k}}\right)du = \frac{1}{{}_p\Gamma_k(\eta)} \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}} \int_0^1 u^{\frac{\alpha n+\beta}{k}-1}\left(1-u\right)^{\frac{\eta}{k}-1}$$

$$= \frac{k}{{}_p\Gamma_k(\eta)} \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}} \frac{{}_p\Gamma_k(\alpha n+\beta)\,{}_p\Gamma_k(\eta)}{{}_p\Gamma_k(\alpha n+\beta+\eta)}$$

$$= k \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta+\eta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{z^n}{{}_p(\delta)_{rn,k}}$$

$$= k\ {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+\eta,\nu,\sigma,\delta,r}(z).$$

This completes the proof of the above theorem. □

Special Cases

Remark 2.13. *The equation (20) reduces to the following form on taking $p = k$, we get*

$$\frac{1}{\Gamma_k(\eta)} \int_0^1 u^{\frac{\beta}{k}-1} \Big(1-u\Big)^{\frac{\eta}{k}-1} E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}\Big(zu^{\frac{\alpha}{k}}\Big)du = kE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+\eta,\nu,\sigma,\delta,r}(z)$$

which is the similar result as obtained by Bairwa and Sharma [10].

Remark 2.14. *Taking $p = k = 1$, equation (20), reduces to the following result, we have*

$$\frac{1}{\Gamma(\eta)} \int_0^1 u^{\beta-1} \Big(1-u\Big)^{\eta-1} E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,r}\Big(zu^{\alpha}\Big)du = kE^{\mu,\rho,\gamma,q}_{\alpha,\beta+\eta,\nu,\sigma,\delta,r}(z)$$

which is the similar result as deduced by Khan and Ahmed [7].

Remark 2.15. *Taking $p = k = 1$, $\mu = \nu, \rho = \sigma, r = 1$, and $\delta = 1$, equation (20), reduces to the following result, we get*

$$\frac{1}{\Gamma(\eta)} \int_0^1 u^{\beta-1} \Big(1-u\Big)^{\eta-1} E^{\gamma,q}_{\alpha,\beta}\Big(zu^{\alpha}\Big)du = kE^{\gamma,q}_{\alpha,\beta+\eta}(z)$$

which is the similar result as given by Shukla and Prajapati [3].

Remark 2.16. *Putting $p = k = 1$, $\mu = \nu, \rho = \sigma$, q=r=1 and $\delta = 1$, equation (20), takes the following form, we get*

$$\frac{1}{\Gamma(\eta)} \int_0^1 u^{\beta-1} \Big(1-u\Big)^{\eta-1} E^{\gamma}_{\alpha,\beta}\Big(zu^{\alpha}\Big)du = kE^{\gamma}_{\alpha,\beta+\eta}(z)$$

which is the similar result as deduced by Prabhakar [12].

Theorem 2.17 (Euler (Beta) Transform)**.** *If $p, k \in \mathbb{R}^+, a, b, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma, \eta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\mu) > 0, Re(\nu) > 0, Re(\rho) > 0, Re(\sigma) > 0$, and $q, r > 0, q \le Re(\alpha) + r$, then*

$$\int_0^1 z^{a-1}\Big(1-z\Big)^{b-1} {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(xz^{\eta})dz = kp^{-\frac{\beta}{k}} \frac{\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)\Gamma(b)}{\Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)}$$
$$\times\ {}_4\psi_4\left[\begin{matrix}\Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(a,\eta),(1,1);\\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(a+b,\eta);\end{matrix}\ \Big(p^{\rho+q-\sigma-r-\frac{\alpha}{k}}\Big)x\right]. \tag{21}$$

Proof. In the L.H.S. of equation (21), and making use of the series form of ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$, given by the equation (10), we get

$$\int_0^1 z^{a-1}\Big(1-z\Big)^{b-1} {}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(xz^{\eta})dz = \int_0^1 z^{a-1}\Big(1-z\Big)^{b-1} \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{(xz^{\eta})^n}{{}_p(\delta)_{rn,k}}dz$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{(x)^n}{{}_p(\delta)_{rn,k}} \int_0^1 z^{a+\eta n-1}\Big(1-z\Big)^{b-1} dz$$

$$= \sum_{n=0}^{\infty} \frac{kp^{(\rho+q-\sigma-r)n}\Big(\frac{\mu}{k}\Big)_{\rho n}\Big(\frac{\gamma}{k}\Big)_{qn} x^n}{p^{\left(\frac{\alpha n+\beta}{k}\right)}\Gamma\Big(\frac{\alpha n}{k}+\frac{\beta}{k}\Big)\Big(\frac{\nu}{k}\Big)_{\sigma n}\Big(\frac{\delta}{k}\Big)_{rn}} \frac{\Gamma(a+\eta n)\Gamma(b)}{\Gamma(a+\eta n+b)}$$

$$= kp^{-\frac{\beta}{k}} \frac{\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)\Gamma(b)}{\Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma\Big(\frac{\mu}{k}+\rho n\Big)\Gamma\Big(\frac{\gamma}{k}+qn\Big)\Gamma(n+1)\Gamma(a+\eta n)\Big(p^{\rho+q-\sigma-r-\frac{\alpha}{k}}x\Big)^n}{\Gamma\Big(\frac{\alpha n}{k}+\frac{\beta}{k}\Big)\Gamma\Big(\frac{\nu}{k}+\sigma n\Big)\Gamma\Big(\frac{\delta}{k}+rn\Big)\Gamma(a+\eta n+b)(n!)}$$

$$= kp^{-\frac{\beta}{k}} \frac{\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)\Gamma(b)}{\Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)} \times {}_4\psi_4 \left[\begin{matrix} \Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(a,\eta),(1,1); \\ \\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(a+b,\eta); \end{matrix} \quad \Big(p^{\rho+q-\sigma-r-\frac{\alpha}{k}}\Big)x \right].$$

This completes the proof of the above theorem. □

Special Cases

Remark 2.18. *Taking $p=k$, equation (21) reduces to the following result, we have*

$$\int_0^1 z^{a-1}\Big(1-z\Big)^{b-1} E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(xz^\eta)dz = k^{1-\frac{\beta}{k}} \frac{\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)\Gamma(b)}{\Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)}$$

$$\times {}_4\psi_4 \left[\begin{matrix} \Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(a,\eta),(1,1); \\ \\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(a+b,\eta); \end{matrix} \quad \Big(k^{\rho+q-\sigma-r-\frac{\alpha}{k}}\Big)x \right],$$

which is the similar result as obtained by Bairwa and Sharma [10].

Remark 2.19. *Taking $p=k=1$, equation (21), reduces to the following result, we have*

$$\int_0^1 z^{a-1}\Big(1-z\Big)^{b-1} E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,r}(xz^\eta)dz = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(b)}{\Gamma(\mu)\Gamma(\gamma)} \times {}_4\psi_4 \left[\begin{matrix} (\mu,\rho),(\gamma,q),(a,\eta),(1,1); \\ \\ (\beta,\alpha),(\nu,\sigma),(\delta,r),(a+b,\eta); \end{matrix} \quad x \right],$$

which is the similar result as deduced by Khan and Ahmed [7].

Remark 2.20. *Taking $p=k=1$, $\mu=\nu, \rho=\sigma, r=1$, and $\delta=1$, equation (21), takes the following form, we get*

$$\int_0^1 z^{a-1}\Big(1-z\Big)^{b-1} E^{\gamma,q}_{\alpha,\beta}(xz^n)dz = \frac{\Gamma(b)}{\Gamma(\gamma)} {}_2\psi_2 \left[\begin{matrix} (\gamma,q),(a,\eta); \\ \\ (\beta,\alpha),(a+b,\eta); \end{matrix} \quad x \right],$$

which is the similar result as given by Shukla and Prajapati [3].

Remark 2.21. *On setting $p=k=1$, $\mu=\nu, \rho=\sigma$, q=1, r=1 and $\delta=1$, equation (21), takes the following form, we have*

$$\int_0^1 z^{a-1}\Big(1-z\Big)^{b-1} E^{\gamma}_{\alpha,\beta}(xz^\eta)dz = \frac{\Gamma(b)}{\Gamma(\gamma)} {}_2\psi_2 \left[\begin{matrix} (\gamma,1),(a,\eta); \\ \\ (\beta,\alpha),(a+b,\eta); \end{matrix} \quad x \right],$$

which is the similar result as deduced by Saxena [11].

Theorem 2.22 (Laplace Transform). *If $p,k\in\mathbb{R}^+,\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\sigma\in\mathbb{C}, Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(\delta)>0, Re(\mu)>0, Re(\nu)>0, Re(\rho)>0, Re(\sigma)>0, Re(r)>0, \left|\frac{x}{r^\eta}\right|$ and $q,r>0, q\le Re(\alpha)+r$, then*

$$\int_0^1 z^{a-1}e^{-sz}\,{}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(xz^\eta)dz = s^{-\alpha}kp^{-\frac{\beta}{k}}\frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)} \times {}_4\psi_3\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),(a,\eta),(1,1);\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\nu}{k},\sigma\right),\left(\frac{\delta}{k},r\right);\end{matrix}\ \frac{x}{s^\eta}\left(p^{\rho+q-\sigma-r-\frac{\alpha}{k}}\right)\right]. \tag{22}$$

Proof. In the L. H. S. of equation (22), and making use of the series form of ${}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(z)$, given by the equation (10), we have

$$\begin{aligned}
\int_0^1 z^{a-1}e^{-sz}\,{}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(xz^\eta)dz &= \int_0^1 z^{a-1}e^{-sz}\sum_{n=0}^{\infty}\frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}}\frac{(xz^\eta)^n}{{}_p(\delta)_{rn,k}}dz\\
&= \sum_{n=0}^{\infty}\frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}}\frac{(x)^n}{{}_p(\delta)_{rn,k}}\int_0^1 z^{a+\eta n-1}e^{-sz}dz\\
&= \sum_{n=0}^{\infty}\frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}}\frac{(x)^n}{{}_p(\delta)_{rn,k}}\frac{\Gamma(a+n\eta)}{s^{a+n\eta}}\\
&= s^{-a}\sum_{n=0}^{\infty}\frac{kp^{(\rho+q-\sigma-r)n}\left(\frac{\mu}{k}\right)_{\rho n}\left(\frac{\gamma}{k}\right)_{qn}x^n}{p^{\left(\frac{\alpha n+\beta}{k}\right)}\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\left(\frac{\nu}{k}\right)_{\sigma n}\left(\frac{\delta}{k}\right)_{rn}}\frac{\Gamma(a+\eta n)\Gamma(n+1)}{n!}\left(\frac{x}{s^\eta}\right)^n\\
&= s^{-a}kp^{-\frac{\beta}{k}}\frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}\\
&\times\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\mu}{k}+\rho n\right)\Gamma\left(\frac{\gamma}{k}+qn\right)\Gamma(n+1)\Gamma(a+\eta n)}{\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\Gamma\left(\frac{\nu}{k}+\sigma n\right)\Gamma\left(\frac{\delta}{k}+rn\right)(n!)}\left(\frac{xp^{\rho+q-\sigma-r-\frac{\alpha}{k}}}{s^\eta}\right)^n\\
&= s^{-\alpha}kp^{-\frac{\beta}{k}}\frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}\times{}_4\psi_3\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),(a,\eta),(1,1);\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\nu}{k},\sigma\right),\left(\frac{\delta}{k},r\right);\end{matrix}\ \frac{x}{s^\eta}\left(p^{\rho+q-\sigma-r-\frac{\alpha}{k}}\right)\right].
\end{aligned}$$

This completes the proof of the above theorem. □

Special Cases

Remark 2.23. *Taking $p=k$, equation (22) reduces to the following result, we have*

$$\int_0^1 z^{a-1}e^{-sz}E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(xz^\eta)dz = s^{-\alpha}k^{1-\frac{\beta}{k}}\frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}\times{}_4\psi_3\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),(a,\eta),(1,1);\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\nu}{k},\sigma\right),\left(\frac{\delta}{k},r\right);\end{matrix}\ \frac{x}{s^\eta}\left(k^{\rho+q-\sigma-r-\frac{\alpha}{k}}\right)\right],$$

which is the similar result as obtained by Bairwa and Sharma [10].

Remark 2.24. *Taking $p=k=1$, equation (22), reduces to the following result, we get*

$$\int_0^1 z^{a-1}e^{-sz}E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,r}(xz^\eta)dz = \frac{\Gamma(\nu)\Gamma(\delta)s^{-a}}{\Gamma(\mu)\Gamma(\gamma)}\,{}_4\psi_3\left[\begin{matrix}(\mu,\rho),(\gamma,q),(a,\eta),(1,1);\\ (\beta,\alpha),(\nu,\sigma),(\delta,r);\end{matrix}\ \frac{x}{s^\eta}\right],$$

which is the similar result as deduced by Khan and Ahmed [7].

Remark 2.25. *Taking* $p = k = 1$, $\mu = \nu, \rho = \sigma, r = 1$, *and* $\delta = 1$, *equation (22), take the following form, we get*

$$\int_0^1 z^{a-1} e^{-sz} E_{\alpha,\beta}^{\gamma,q}(xz^\eta)dz = \frac{s^{-a}}{\Gamma(\gamma)} {}_2\psi_1 \left[\begin{matrix} (\gamma, q), (a, \eta); & \\ & \dfrac{x}{s^\eta} \\ (\beta, \alpha); & \end{matrix} \right],$$

which is the similar result as deduced by Shukla and Prajapati [3].

Remark 2.26. *On setting* $p = k = 1$, $\mu = \nu, \rho = \sigma$, *q=1, r=1 and* $\delta = 1$, *equation (22), takes the following form, we have*

$$\int_0^1 z^{a-1} e^{-sz} E_{\alpha,\beta}^{\gamma}(xz^\eta)dz = \frac{s^{-a}}{\Gamma(\gamma)} {}_2\psi_1 \left[\begin{matrix} (\gamma, 1), (a, \eta); & \\ & \dfrac{x}{s^\eta} \\ (\beta, \alpha); & \end{matrix} \right],$$

which is the similar result as given by Saxena [11].

Theorem 2.27 (Whittaker Transform)**.** *If* $p, k \in \mathbb{R}^+, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma, \eta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\mu) > 0, Re(\nu) > 0, Re(\rho) > 0, Re(\sigma) > 0, Re(r) > 0, Re(\lambda + \psi) > -1/2$ *and* $q, r > 0, q \leq Re(\alpha) + r$, *then*

$$\int_0^\infty t^{\xi-1} e^{-\frac{\phi t}{2}} W_{\lambda,\psi}(\phi t) {}_pE_{k,\alpha,\beta,\nu,\sigma,\delta,r}^{\mu,\rho,\gamma,q}(\omega t^\eta)dz = \phi^{-\xi} kp^{-\frac{\beta}{k}} \frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$
$$\times {}_5\psi_4 \left[\begin{matrix} \left(\frac{\mu}{k}, \rho\right), \left(\frac{\gamma}{k}, q\right), (\frac{1}{2} + \psi + \xi, \eta), (\frac{1}{2} - \psi + \xi, \eta), (1, 1); & \\ & ; \left(\dfrac{\omega p^{\rho+q-\sigma-r-\frac{\alpha}{k}}}{\phi^\eta}\right) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), \left(\frac{\nu}{k}, \sigma\right), \left(\frac{\delta}{k}, r\right), (1 - \lambda + \xi, \eta); & \end{matrix} \right]. \tag{23}$$

Proof. In the L. H. S. of equation (23), and substituting $\phi t = v$ and making use of the series form of ${}_pE_{k,\alpha,\beta,\nu,\sigma,\delta,r}^{\mu,\rho,\gamma,q}(z)$, given by the equation (10), we have

$$\int_0^\infty (\frac{v}{\phi})^{\xi-1} e^{-\frac{v}{2}} W_{\lambda,\psi}(v) \sum_{n=0}^\infty \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n + \beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{1}{{}_p(\delta)_{rn,k}} \left(\frac{\omega v}{\phi}\right)^{n\eta} \frac{1}{\phi} dv$$
$$= \phi^{-\xi} kp^{-\frac{\beta}{k}} \frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\mu}{k} + \rho n\right)\Gamma\left(\frac{\gamma}{k} + qn\right)\left(p^{\rho+q-\sigma-r-\frac{\alpha}{k}}\right)^n}{\Gamma\left(\frac{\alpha n}{k} + \frac{\beta}{k}\right)\Gamma\left(\frac{\nu}{k} + \sigma n\right)\Gamma\left(\frac{\delta}{k} + rn\right)}$$
$$\times \left(\frac{\omega}{\phi^\eta}\right)^n \int_0^\infty v^{\xi+n\eta-1} W_{\lambda,\psi}(v)dv,$$

Using the formula given in [2]:

$$\int_0^\infty e^{-\frac{t}{2}} t^{\nu-1} W_{\lambda,\mu}(t)dt = \frac{\Gamma(\frac{1}{2} + \mu + \nu)\Gamma(\frac{1}{2} - \mu + \nu)}{\Gamma(1 - \lambda + \nu)}$$
$$= \phi^{-\xi} kp^{-\frac{\beta}{k}} \frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$
$$\times \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\mu}{k} + \rho n\right)\Gamma\left(\frac{\gamma}{k} + qn\right)\Gamma\left(\frac{1}{2} + \psi + \xi + n\eta\right)\Gamma\left(\frac{1}{2} - \psi + \xi + n\eta\right)\Gamma(n+1)}{\Gamma\left(\frac{\alpha n}{k} + \frac{\beta}{k}\right)\Gamma\left(\frac{\nu}{k} + \sigma n\right)\Gamma\left(\frac{\delta}{k} + rn\right)(1 - \lambda + \xi + n\eta)(n!)} \left(\frac{\omega p^{\rho+q-\sigma-r-\frac{\alpha}{k}}}{\phi^\eta}\right)^n$$
$$= \phi^{-\xi} kp^{-\frac{\beta}{k}} \frac{\Gamma\left(\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$

$$\times\,{}_5\psi_4\left[\begin{array}{c}\Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(\frac{1}{2}+\psi+\xi,\eta),(\frac{1}{2}-\psi+\xi,\eta),(1,1);\\ \\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(1-\lambda+\xi,\eta);\end{array}\Big(\frac{\omega p^{\rho+q-\sigma-r-\frac{\alpha}{k}}}{\phi^{\eta}}\Big)\right].$$

This completes the proof of the above theorem. □

Special Cases

Remark 2.28. *Taking $p = k$, equation (23) reduces to the following result, we have*

$$\int_0^\infty t^{\xi-1}e^{-\frac{\phi t}{2}}W_{\lambda,\psi}(\phi t)E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(\omega t^{\eta})dz=\phi^{-\xi}k^{1-\frac{\beta}{k}}\frac{\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)}{\Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)}$$

$$\times\,{}_5\psi_4\left[\begin{array}{c}\Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(\frac{1}{2}+\psi+\xi,\eta),(\frac{1}{2}-\psi+\xi,\eta),(1,1);\\ \\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(1-\lambda+\xi,\eta);\end{array}\Big(\frac{\omega k^{\rho+q-\sigma-r-\frac{\alpha}{k}}}{\phi^{\eta}}\Big)\right],$$

which is the similar result as obtained by Bairwa and Sharma [10].

Remark 2.29. *Taking $p = k = 1$, equation (23), reduces to the following result, we get*

$$\int_0^\infty t^{\xi-1}e^{-\frac{\phi t}{2}}W_{\lambda,\psi}(\phi t)E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,r}(\omega t^{\eta})dt=\phi^{-\xi}\frac{\Gamma\Big(\nu\Big)\Gamma\Big(\delta\Big)}{\Gamma\Big(\mu\Big)\Gamma\Big(\gamma\Big)}$$

$$\times\,{}_5\psi_4\left[\begin{array}{c}\Big(\mu,\rho\Big),\Big(\gamma,q\Big),(\frac{1}{2}+\psi+\xi,\eta),(\frac{1}{2}-\psi+\xi,\eta),(1,1);\\ \\ \Big(\beta,\alpha\Big),\Big(\nu,\sigma\Big),\Big(\delta,r\Big),(1-\lambda+\xi,\eta);\end{array}\Big(\frac{\omega}{\phi^{\eta}}\Big)\right],$$

which is the similar result as obtained by Khan and Ahmed [7].

Remark 2.30. *Taking $p = k = 1$, $\mu = \nu, \rho = \sigma$, and $r = 1$, equation (23), takes the following form, we get*

$$\int_0^\infty t^{\xi-1}e^{-\frac{\phi t}{2}}W_{\lambda,\psi}(\phi t)E^{\gamma,q}_{\alpha,\beta,\delta}(\omega t^{\eta})dt=\frac{\phi^{-\xi}\Gamma\Big(\delta\Big)}{\Gamma\Big(\gamma\Big)}\times{}_4\psi_3\left[\begin{array}{c}\Big(\gamma,q\Big),(\frac{1}{2}+\psi+\xi,\eta),(\frac{1}{2}-\psi+\xi,\eta),(1,1);\\ \\ \Big(\beta,\alpha\Big),\Big(\delta,s\Big),(1-\lambda+\xi,\eta);\end{array}\Big(\frac{\omega}{\phi^{\eta}}\Big)\right],$$

which is the similar result as deduced by Shukla and Prajapati [3].

Remark 2.31. *On setting $p = k = 1$, $\mu = \nu, \rho = \sigma$, q=1, r=1 and $\delta = 1$, equation (23), takes the following form, we get*

$$\int_0^\infty t^{\xi-1}e^{-\frac{\phi t}{2}}W_{\lambda,\psi}(\phi t)E^{\gamma}_{\alpha,\beta}(\omega t^{\eta})dt=\frac{\phi^{-\xi}}{\Gamma\Big(\gamma\Big)}\times{}_3\psi_2\left[\begin{array}{c}\Big(\gamma,1\Big),(\frac{1}{2}+\psi+\xi,\eta),(\frac{1}{2}-\psi+\xi,\eta);\\ \\ \Big(\beta,\alpha\Big),(1-\lambda+\xi,\eta);\end{array}\Big(\frac{\omega}{\phi^{\eta}}\Big)\right],$$

which is the similar result as deduced by Saxena [11].

Theorem 2.32 (Hankel Transform). *If* $p,k \in \mathbb{R}^+, \alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\sigma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\mu) > 0, Re(\nu) > 0, Re(\rho) > 0, Re(\sigma) > 0,$ *and* $q, r > 0, q \leq Re(\alpha) + r,$ *then*

$$\int_0^\infty z^{\eta-1} J_\mu(az)\,{}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(bz^\omega)dz = kp^{-\frac{\beta}{k}} \frac{2^{\eta-1}\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)}{a^\eta \Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)} \times {}_4\psi_4\left[\begin{matrix} \Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(\frac{\mu+\eta}{2},\frac{\omega}{2}),(1,1); \\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(1+\frac{\mu-\eta}{2},-\frac{\omega}{2}); \end{matrix} \; p^{\rho+q-\sigma-r-\frac{\alpha}{k}} b\Big(\frac{2}{a}\Big)^\omega\right]. \tag{24}$$

Proof. Taking L.H.S. of (24) and using (10), we have

$$\int_0^\infty z^{\eta-1} J_\mu(az)\,{}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(bz^\omega)dz = \sum_{n=0}^\infty \frac{{}_p(\mu)_{\rho n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{{}_p(\gamma)_{qn,k}}{{}_p(\nu)_{\sigma n,k}} \frac{b^n}{{}_p(\delta)_{rn,k}}, \int_0^\infty z^{\eta+\omega n-1} J_\mu(az)dz$$

Using the formula given in [2]:

$$\int_0^\infty t^{s-1} J_\nu(\alpha t)dt = \frac{2^{s-1}\alpha^{-s}\Gamma\Big(\frac{\nu+s}{2}\Big)}{\Gamma(1+\frac{\nu-s}{2})}, Re(\nu) < Re(s) < \frac{3}{2}, \alpha > 0,$$

we get,

$$\begin{aligned}\int_0^\infty z^{\eta-1} J_\mu(az)\,{}_pE^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(bz^\omega)dz &= kp^{-\frac{\beta}{k}} \frac{2^{\eta-1}\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)}{a^\eta \Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)} \\ &\times \sum_{n=0}^\infty \frac{\Gamma\Big(\frac{\mu}{k}+\rho n\Big)\Gamma\Big(\frac{\gamma}{k}+qn\Big)\Gamma\Big(\frac{\mu+\eta+\omega n}{2}\Big)\Gamma(n+1)\Big(p^{\rho+q-\sigma-r-\frac{\alpha}{k}} b\Big(\frac{2}{a}\Big)^\omega\Big)^n}{\Gamma\Big(\frac{\alpha n}{k}+\frac{\beta}{k}\Big)\Gamma\Big(\frac{\nu}{k}+\sigma n\Big)\Gamma\Big(\frac{\delta}{k}+rn\Big)\Big(1+\frac{\mu-\eta-\omega n}{2}\Big)(n!)} \\ &= kp^{-\frac{\beta}{k}} \frac{2^{\eta-1}\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)}{a^\eta \Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)} \\ &\times {}_4\psi_4\left[\begin{matrix} \Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(\frac{\mu+\eta}{2},\frac{\omega}{2}),(1,1); \\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(1+\frac{\mu-\eta}{2},-\frac{\omega}{2}); \end{matrix} \; p^{\rho+q-\sigma-r-\frac{\alpha}{k}} b\Big(\frac{2}{a}\Big)^\omega\right].\end{aligned}$$

This completes the proof of the above theorem. □

Special Case

Remark 2.33. *Taking* $p = k$*, equation (24) reduces to the following result, we have*

$$\int_0^\infty z^{\eta-1} J_\mu(az) E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\nu,\sigma,\delta,r}(bz^\omega)dz = k^{1-\frac{\beta}{k}} \frac{2^{\eta-1}\Gamma\Big(\frac{\nu}{k}\Big)\Gamma\Big(\frac{\delta}{k}\Big)}{a^\eta \Gamma\Big(\frac{\mu}{k}\Big)\Gamma\Big(\frac{\gamma}{k}\Big)} \times {}_4\psi_4\left[\begin{matrix} \Big(\frac{\mu}{k},\rho\Big),\Big(\frac{\gamma}{k},q\Big),(\frac{\mu+\eta}{2},\frac{\omega}{2}),(1,1); \\ \Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),\Big(\frac{\nu}{k},\sigma\Big),\Big(\frac{\delta}{k},r\Big),(1+\frac{\mu-\eta}{2},-\frac{\omega}{2}); \end{matrix} \; k^{\rho+q-\sigma-r-\frac{\alpha}{k}} b\Big(\frac{2}{a}\Big)^\omega\right].$$

3. Conclusion

In this paper, we have defined a new generalized Mittag-Leffler type function and also presented its basic properties including recurrence relation, usual differentiation and integration, Euler(Beta) transform, Laplace transform, Whittaker transform and Hankel transform. The obtained result provided extended forms of the known results earlier proved by many researchers.

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