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# $\eta$-Ricci Solitons on $\beta$-Kenmotsu Manifolds

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**Abstract:** $\eta$-Ricci solitons on $\beta$-Kenmotsu manifold satisfying certain curvature conditions $R(\xi, X).S = 0$, $S(\xi, X).R = 0$, $W_2(\xi, X).S = 0$ and $S(\xi, X).W_2 = 0$. We proved that in $\beta$-Kenmotsu manifold $(M, \phi, \xi, \eta, g)$. Then the existence of an $\eta$-Ricci solitons implies that M is Einstein manifold and if the Ricci curvature tensor satisfies, $S(\xi, X).R = 0$, then Ricci solitons M is steady. If the condition $\mu = 0$, then $\lambda = 0$, which shows that $\lambda$ is steady.

**MSC:** 53C25, 53C15.

**Keywords:** $\beta$-Kenmotsu manifolds, $\eta$-Ricci solitons, $W_2$ curvature tensor.



## 1. Introduction

In 1982, Hamilton [20, 21] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman [30] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij} \tag{1}$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci solution if it moves only by a one parameter group of diffeomorphism and scaling. The concept is named after Gegorio Ricci-Curbastro. In 2015, A.M. Blaga have obtained some results on $\eta$-Ricci solitons satisfying certain curvature conditions. Ricci solitons were introduced by R. S. Hamilton as natural generalization of Einstein metric [20, 21], and have studied in many contexts on $\alpha$-Sasakian manifold [4, 37], trans-sasakian etc. Ricci soliton firstly appeared in the paper of G. Calvaruso and D. Perrone [15]. Recently C.L. Bejan and M. Crasmareanu [13, 14], defined with Ricci soliton an $\alpha$-Sasakian manifold [4]. In [17], Cho and Kimura studied Ricci solitons of real hyper surfaces in non-flat complex space from and they defined $\eta$-Ricci soliton, which satisfies the equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{2}$$

where $\lambda$ and $\mu$ are real constants. The $\eta$-Ricci solitons are studied on Hopt hyper surfaces in the paper [14]. Also it may be noted that a generalized of $\eta$-Ricci Einstein geometry is provided by Ricci soliton for this frame work are studied.

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In 1970, Pokhariyal and Mishra [32], have introduced new curvature tensor called $W_2$-curvature tensor in a Riemannian manifold and studied some properties. Further, Pokhariyal [32], has studied some properties of this curvature tensor in a Sasakian manifold. Matsumoto, Ianus and Mihai [22, 23], Ahmet Yildiz and U. C. De [1, 2] and Venkatesha, C. S. Bagewadi, and K. T. Pradeep Kumar have studied $W_2$-curvature in P-Sasakian ,Kenmotsu and Lorentzian para-Sasakian manifolds respectively. The concept of Ricci solitons was introduced by Hamilton [20, 21] in the Year 1982. In differential geometry, a Ricci solitons is a special type of Riemannian metric such metric evalve under Ricci flow only be symmetries of the flow and ofter arise as limits of dilations of Singularities in the Ricci flow [21, 14, 33]. They can be viewed as generalizations of Einstein metrics. They can viewed as fixed point of the Ricci flow, as a dynamical system on the space of Riemmanian metrics moduls diffeomorphisms ans scallings. Ricci solitons have been studied in many contexts: on Kahler manifolds [18], on contact and Lorentzian manifolds [12, 14, 37, 83, 10, 14, 23], on Sasakian [6], $\alpha$-Sasakian manifolds [4, 37] and K-contact manifolds [34], on Kenmotsu [1, 2, 26, 29, 35], and f-Kenmotsu manifolds [13], etc. In Para contact geometry, Ricci solitons firstly appeared in the papers of G. Calvaruso and D. Perrone [15], Recently, C. L. Bejan and M. Crasmareanu studied Ricci solitons on 3-dimensional normal Para contact manifolds [8, 34]. As a generalizations of Ricci solitons, the notion of $\eta$-Ricci solitons introduced by J. T. Cho and M. Kimura [17], which was treated by C. Calin and Crasmareanu on Hopf Hypersurfaces in complex space forms [16,14].

In [36], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of a plane sections containing is a constant, say C. He showed that they can be divided into three classes: (1). Homogeneous normal contact Riemannian manifolds with $c > 0$: (2). Global Riemannian Products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if c=0 and: (3). A wared product space $\mathbb{R}\times_f$ if $c > 0$. It is know that the manifolds of class (1) are characterized by admitting a Sasakian structure. In the Gray-Hervella classification of almost Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [8, 12, 18, 19]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [5, 24, 25], if the product manifold $M \times \mathbb{R}$ belongs to the class $W_2$. The class $C_6 \otimes C_5$ [9]. coincides with the class of the trans-Sasakian structures of type $(\alpha, \beta)$ in fact, in [24]. Local nature of the two subclasses, namely, $C_5$ and $C_6$, structures of trans-Sasakian structures are characterized completely.We note that trans-Sasakian structure of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymlectic, $\beta$-kenmotsu [3], and $\alpha$-Sasakian [4], Respectively. In [24], it is proved that trans-Sasakian structures are generalized quasi-Sasakian Thus, trans-Sasakian structures also provide a lorge class of generalized quasi-Sasakian structures. An almost contact metric structure $(\phi, \xi, \eta, g)$ on M is called a trans-Sasakian structure [24, 25], if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ [24], where J is the almost complex structure on $M \times \mathbb{R}$, defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f, \eta(X)\frac{d}{dt}), \tag{3}$$

for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$. and G is the product metric on $M \times \mathbb{R}$. this may be expressed by the condition.

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y) - \eta(Y)\phi X), \tag{4}$$

for some smooth functions $\alpha$ and $\beta$ on M, and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. A trans-Sasakian structure of type $(\alpha, \beta)$ is $\alpha$-Sasakian if $\beta = 0$, and $\alpha$ a non zero constant [5]. If $\alpha = 1$, then $\alpha$-Sasakian manifold is a Sasakian manifold. Motivated by the above studied the object of the present paper is to study $\eta$-Ricci solitons on $\alpha$-Sasakian manifold which satisfy certain curvature properties $R(\xi, X).S = 0$, $S(\xi, X).R = 0$, respectively. Remark that in [26], H. G. Nagaraja and Premalatha have obtained some results on $\eta$-Ricci soliton satisfying conditions of the following type $W_2(\xi, X).S = 0$, and $S(\xi, X).W_2 = 0$, we also proved that $\alpha$-Sasakian manifold of constant curvature supporting an $\eta$-Ricci soliton is locally isometric to sphere. Then $\xi$ is (i) concurrent and (ii) Killing vector field. During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci sol itons to solve the long standing Poincare conjecture posed in 1904. There after Ricci solitons in contact metric manifolds have been studied by various authors such as Bejan and Crasmareanu [7], Blaga [10, 11], Nagaraja and Premalatta [26], I. T. Cho and M. Kirmura [17], U. C. De, and many others.

## 2. Preliminaries

A $(2n+1)$-dimemensional smooth manifold M is said to be an almost contact metric manifold. If it admits a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ Riemannian metric $g$ which satisfy

$$\phi.\xi = 0, \quad \eta\left(\phi X\right) = 0, \quad \phi^2 X = X + \eta(X)\xi, \tag{5}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{6}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta\left(X\right)\eta\left(Y\right), \tag{7}$$

for any vector field $X, Y \in M$, an almost contact metric manifold $M^{2n+1}\left(\phi, \xi, \eta, g\right)$ is said to be $\beta$-Kenmotsu manifold M. If the following condition holds.

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \tag{8}$$

and

$$(\nabla_X \phi)(Y) = \beta[g(\phi X, Y) - \eta(Y)\phi X], \tag{9}$$

where $\nabla$ denotes the Riemannian connection of $g$. If $\beta = 1$, then a operator of covariant differentiation with respect to the Lorentzian Metric $g$. Then $\beta$-Kenmotsu manifold is called Kenmotsu manifold and if $\beta$ is constant then it is called homothetic Kenmotsu manifold. In a $\beta$ -Kenmotsu manifold [1]. The following relation holds.

$$(\nabla_X \eta)(Y) = \beta[g(\phi X, Y) - \eta(Y)\phi X], \tag{10}$$

$$R(X, Y)\xi = -\beta^2\left[\eta\left(X\right)Y - \eta\left(Y\right)X\right] + \left[(X\beta)\left(Y - \eta\left(Y\right)\xi\right) - (Y\beta)\left(X - \eta(X)\xi\right)\right], \tag{11}$$

$$R(\xi, X)Y = (\beta^2 + \xi\beta)[\eta\left(Y\right)X - g\left(X, Y\right)\xi], \tag{12}$$

$$\eta(R(X, Y)Z) = \beta^2[g\left(X, Z\right)\eta\left(Y\right) - g\left(Y, Z\right)\eta(X)] - (X\beta)(g\left(Y, Z\right) - \eta(Y)\eta(Z)) + (Y\beta)(g\left(X, Z\right) - \eta(Z)\eta(X)), \tag{13}$$

$$S(X, \xi) = -(2n\beta^2 + \xi\beta)\eta\left(X\right) - (2n-1)\left(X\beta\right), \tag{14}$$

$$S(\xi, \xi) = -(2n\beta^2 + \xi\beta), \tag{15}$$

For any vector field X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2). If the Ricci tensor of an almost contact Riemannian manifold M is of the form

$$S = ag + b\eta \otimes \eta, \tag{16}$$

for some functions a and b on M, then M is said to be an $\eta$-Einstein Manifold. In Pokhariyal and Mishra have defined the curvature tensor $W_2$ given by

$$W_2\left(X, Y, U, V\right) = R\left(X, Y, U, V\right) + \frac{1}{n-1}[g(X, U)S(Y, V) - g(X, U)S(Y, V)], \tag{17}$$

where S is a Ricci tensor of type (0,2). Consider an $\beta$-Kenmotsu manifold satisfying $W_2 = 0$, (17), then we have

$$R(X,Y,U,V) = \frac{1}{n-1}[g(X,U)S(Y,V) - g(X,U)S(Y,V)], \tag{18}$$

Putting $Y = V = \xi$, in above equation (18), then using equations (13) and (16), we obtain

$$S(X,U) = (2n\beta^2 + \xi\beta)[g(X,U) - \eta(U)\eta(X)],$$

Thus M is an Einstein manifold.

**Theorem 2.1.** *If $\eta$-Ricci solitons on $\beta$-Kenmotsu manifold M, the condition $W_2 = 0$, holds, then M is an Einstein manifold.*

**Definition 2.2.** *If $\eta$-Ricci solitons $\beta$-Kenmotsu manifold is called $W_2$-semi-symmetric if it satisfies*

$$R(X,Y).W_2 = 0, \tag{19}$$

*where $R(X,Y)$ is to be considered as a derivation of tensor algebra at each point of the manifold for tangent vector X and Y. In an $\beta$-Kenmotsu manifold the $W_2$-curvature tensor satisfies the condition*

$$\eta(W_2(X,Y)Z) = 0, \tag{20}$$

## 3. $\eta$-Ricci Solitons on $\beta$-Kenmotsu Manifolds

Let $(M,\phi,\xi,\eta,g)$ be a $\beta$-Kenmotsu manifold. Consider the equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta = 0, \tag{21}$$

where $L_\xi$ is the Lie derivative operator along the vector field $\xi, S$ is Ricci tensor of the metric $g$, $\lambda$ and $\mu$ are real constants. Now, from above equations (21) and (8), we get

$$2S(X,Y) = -g(\nabla_X\xi,Y) - g(X,\nabla_Y\xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{22}$$

Now, from above equations (22) and (8), we get

$$S(X,Y) = (\lambda+\beta)g(X,Y) + (\beta-\mu)\eta(X)\eta(Y), \tag{23}$$

for any $X,Y \in \chi(M)$, or the data $(g,\xi,\lambda,\mu)$ which satisfy the equation (21) is said to be an $\eta$-Ricci soliton on $\beta$-Kenmotsu manifold M, in Particular, if $\mu = 0$, $(g,\xi,\lambda)$ is a Ricci solitons [26] and it is called shrinking, steady or expanding according as $\lambda$ is negative, zero or positive respectively.for this proposition we get the (0,2), tensor field

$$\Omega(X,Y) = g(X,\phi X),$$

Is symmetric and satisfies

$$\Omega(\phi X,Y) = \Omega(X,\phi Y), \Omega(\phi X,\phi Y) = \Omega(X,Y)$$

$$(\nabla_X\Omega)(Y,Z) = \eta(Y)g(X,Z) + \eta(Z)g(X,Y) + 2\eta(X)\eta(Y)\eta(Z)$$

**Remark 3.1.** *$\eta$-Ricci-Solitons on $\beta$-Kenmotsu manifold $(M, \phi, \xi, \eta, g)$ we deduce that.*

**Proposition 3.2.** *$\eta$-Ricci-Solitons on $\beta$-Kenmotsu manifold $(M, \phi, \xi, \eta, g)$ the Para-contact from $\eta$ is closed and the Nijenhuis tensor field of structural endomorphism $\phi$ identically vanishes.*

*(1). The 1-form $\eta$ is closed indeed, from the above equation (9), we get*

$$(d\eta)(X, Y) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi) \tag{24}$$

*Now, taking from the above equations (37) and (11), we get $(d\eta)(X, Y) = 0$.*

*(2). The Nijenhuis tensor field associated to $\phi$.*

$$N_\phi\,(X, Y) = 0,$$

In [3] and [10] the authors proved that on a $\eta$-Ricci soliton on $\beta$-Kenmotsu manifold $(M, \phi, \xi, \eta, g)$ tensor field satisfies. Now, using equations (23) and (13) in above equation $Y = \xi$, we get

$$S(X, \xi) = (-\lambda + 2\beta + \mu)\eta\,(X) \tag{25}$$

Now, using equations (25) and (18), in above equation, we get

$$(n-1)\beta^2 = (\lambda + 2\beta - \mu)$$

**Theorem 3.3.** *$\eta$-Ricci Soliton on $\beta$-Kenmotsu structure on the manifold M. Let $(\phi, \xi, \eta, g)$ and $(g, \xi, \lambda, \mu)$ be an $\beta$-Kenmotsu structure on M. Then*

*(1). If the manifold M and Let $(M, g)$ has cyclic tensor*

$$\nabla_X S\,(Y, Z) + \nabla_Y S\,(Z, X) + \nabla_Z S\,(X, Y) = 0$$

*(2). If the manifold $(M, g)$ has cyclic $\eta$-recurrent Ricci tensor*

$$\nabla_X S\,(Y, Z) + \nabla_Y S\,(Z, X) + \nabla_Z S\,(X, Y) = -(\beta - \mu)\beta[g\,(X, Y)\,\eta\,(Z) + g(X, Z)\eta(Y) + g(Y, Z)\eta\,(X) - 3\eta\,(X)\,\eta\,(Y)\,\eta(Z)]$$

*Proof.* Replacing the expansion of S form

$$\nabla_X S(Y, Z) = (\beta - \mu)\nabla_X g\,(Y, Z) + (\beta - \mu)\,[\eta\,(Z)\,\nabla_X \eta\,(Y) + \eta\,(Y)\,(\nabla_X \eta)\,(Y)] \tag{26}$$

Taking from above equations (26) and (9), we get

$$\nabla_X S(Y, Z) = (\beta - \mu)\,\beta\,[g\,(X, Y)\,\eta(Z) - 2\eta\,(X)\,\eta\,(Y)\,\eta\,(Z) + g\,(X, Z)\,\eta(Y)] \tag{27}$$

By cyclic permutation X, Y and Z, we get

$$\nabla_Y S(Z, X) = (\beta - \mu)\beta[g(Y, Z)\eta\,(X) - 2\eta\,(X)\,\eta\,(Y)\,\eta\,(Z) + g\,(Y, X)\,\eta(Z)] \tag{28}$$

and

$$\nabla_Z S(X,Y) = (\beta-\mu)\beta[g(Z,X)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z) + g(Z,Y)\eta(X)] \tag{29}$$

Adding from above equations (27), (28) and (29), we get

$$\nabla_X S(Y,Z) + \nabla_Y S(Z,X) + \nabla_Z S(X,Y) = (\beta-\mu)\beta[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 3\eta(X)\eta(Y)\eta(Z)]$$

□

**Corollary 3.4.** *On a $\eta$-Ricci soliton $\beta$-Kenmotsu manifold. $(M,\phi,\xi,\eta,g)$ having cyclic Ricci tensor or cyclic $\eta$-recurrent Ricci tensor, then is no Ricci solitons with potential vector field $\xi$.*

**Proposition 3.5.** *Let $(\phi,\xi,\eta,g)$ be an $\eta$-Ricci soliton $\beta$-Kenmotsu structure on the manifold $M$ and let $(g,\xi,\lambda,\mu)$ be an $\eta$-Ricci soliton on $M$. If the Manifold $(M,g)$ is Ricci symmetric $\nabla S = 0$, then $\beta = 0$, or $\beta = \mu$.*

*Proof.* If $\nabla s = 0$, taking from above equation (23), we get

$$(\nabla_X S)(Y,Z) = (\beta-\mu)[\eta(Z)(\nabla_X\eta)(Y) + \eta(Y)(\nabla_X\eta)(Z)] \tag{30}$$

Now, from above equations (30) and (9), we get

$$(\nabla_X S)(Y,Z) = (\beta-\mu)\beta[g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z) + g(X,Z)\eta(Y)] \tag{31}$$

Taking from above equation (31), $Z = \xi$, we get

$$(\nabla_X S)(Y,\xi) = (\beta-\mu)\beta[-g(X,Y) - \eta(X)\eta(Y)], \tag{32}$$

where $(\nabla_X S)(Y,\xi) = 0$, $\beta = 0$, or $\beta = \mu$. □

**Corollary 3.6.** *If a $\eta$-Ricci soliton on $\beta$-Kenmotsu manifold $(M,\phi,\xi,\eta,g)$ is Ricci symmetric or has $\eta$-recurrent Ricci tensor, then $M$ is no Ricci solitons with the potential vector field $\xi$.*

In what follows we shall consider $\eta$-Ricci soliton requiring for the curvature to satisfy $R(\xi,X).S = 0$ and $S.(\xi,X).R = 0$, respectively.

**Theorem 3.7.** *If $(\phi,\xi,\eta,g)$ to $\eta$-Ricci soliton on $\beta$-Kenmotsu structure on the manifold $(M,g,\xi,\lambda,\mu)$ is an $\eta$-Ricci soliton on $\beta$-Kenmotsu manifold is satisfying condition $R(\xi,X).S = 0$, then $\beta = 0$.*

*Proof.* Let us suppose that $R(\xi,X).S = 0$. Then we have

$$S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z) = 0 \tag{33}$$

Replacing the expression of S from above equation (11) and from the symmetric of R, we get

$$\beta^2[\eta(Y)S(X,Z) - g(X,Y)S(\xi,Z) + \eta(Z)S(X,Y) - g(X,Z)S(Y,\xi)] = 0 \tag{34}$$

Now, from above equations (34) and (13), we get

$$\beta^2[\eta(Y)S(X,Z) + \eta(Z)S(X,Y) + (n-1)\beta^2\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\}] = 0 \tag{35}$$

Taking from above equation (35), in above equation X=Y=$\xi$,we get

$$\beta^2[S(\xi,Z)+\eta(Z)\,S(\xi,\xi)+(n-1)\beta^2\{g(\xi,\xi)\eta(Z)+g(\xi,Z)\}]=0. \tag{36}$$

Taking from above equation (36), in above equation Z=$\xi$, we get $\beta=0$. □

**Theorem 3.8.** *If $(\phi,\xi,\eta,g)$ is $\eta$-Ricci soliton on $\beta$-Kenmotsu structure on the manifold $(M,g,\xi,\lambda,\mu)$ is an $\eta$-Ricci soliton on $M$ and $R(\xi,X).S=0$, $\beta=0$, then $\lambda=0$, where is no Ricci soliton with potential vector field $\xi$. Ricci and $\eta$-Ricci soliton on $\beta$-Kenmotsu manifold satisfying $S(\xi,X).R=0$, then $\beta=0$.*

*Proof.* Let us suppose that $S(\xi,X).R=0$. Then we have

$$\begin{aligned} &S(X,R(Y,Z)W)\xi-S(\xi,R(Y,Z)W)X+S(X,Y)R(\xi,Z)W-S(\xi,Y)R(X,Z)W\\ &\quad+S(X,Z)R(Y,\xi)W-S(\xi,Z)R(Y,X)W+S(X,W)R(Y,Z)\xi-S(\xi,W)R(Y,Z)X=0, \end{aligned} \tag{37}$$

Taking inner product with $\xi$, we get

$$\begin{aligned} &S(X,R(Y,Z)W)\eta\,(X)\,S\,(\xi,R\,(Y,Z)\,W)+S\,(X,Y)\,\eta\,(R\,(\xi,Z)\,W)-S\,(\xi,Y)\,\eta\,(R\,(X,Z)\,W)+S\,(X,Z)\,\eta\,(R\,(Y,\xi)\,W)\\ &\quad-S\,(\xi,Z)\,\eta\,(R\,(Y,X)\,W)+S\,(X,W)\,\eta\,(R\,(Y,Z)\,\xi)-S\,(\xi,W)\,\eta\,(R\,(Y,Z)\,X)=0 \end{aligned} \tag{38}$$

Now, from above equations (38) and (10), we get

$$\begin{aligned} S(X,R(Y,Z)W)-\eta\,(X)\,S\,(\xi,R\,(Y,Z)\,W)&=\beta^2[S\,(X,Y)\,\{g(Y,Z)-\eta(Y)\eta(Z)\}+S\,(\xi,Y)\,\{g(X,W)\eta(Z)-g(Z,W)\eta(X)\}\\ &\quad-S\,(X,Z)\,\{g\,(Y,W)-\eta\,(Y)\,\eta\,(W)\}+S\,(\xi,Z)\,\{g\,(Y,W)\,\eta\,(X)-g\,(X,W)\,\eta\,(Y)\}\\ &\quad-S(\xi,W)\{g\,(Y,X)\,\eta\,(Z)-g\,(Z,X)\,\eta\,(Y)\}\\ &=0 \end{aligned} \tag{39}$$

Now, from above equations (39) and (18), (13), we get

$$\begin{aligned} (n-2)\eta(R(Y,Z)W)-g\,(X,R\,(Y,Z)\,W)&=\beta^2[g\,(X,U)\,g(Y,Z)-g(X,Z)\eta\,(Y)\,\eta\,(Z)\\ &\quad+g\,(Y,Z)\,\eta\,(X)\,\eta\,(U)-2\eta\,(X)\,\eta\,(Y)\,\eta\,(Z)+g\,(X,Z)\,g\,(Y,W)-g\,(X,Z)\,\eta\,(Y)\,\eta\,(W)\\ &\quad+g\,(Y,W)\,\eta\,(X)\,\eta(Z)]+(n-1)[g\,(Y,W)\,\eta\,(X)\,\eta\,(Z)-g\,(X,W)\,\eta\,(Y)\,\eta\,(Z)\\ &\quad+g\,(Y,X)\,\eta\,(W)\,\eta\,(Z)-g\,(Z,X)\,\eta\,(Y)\,\eta\,(W)\\ &\quad-g\,(X,W)\,\eta\,(Y)\,\eta\,(Z)+g(Z,W)\eta(X)\eta(Y)] \end{aligned} \tag{40}$$

Taking from above equation (40), in above equation X=$\xi$, we get

$$\begin{aligned} (n-3)\eta(R(Y,Z)W)&=-\beta^2[g(Y,Z)\eta(U)+2g(Y,W)\eta\,(Z)-\eta\,(U)\,\eta\,(Y)\,\eta\,(Z)-\eta\,(Y)\,\eta\,(Z)\,\eta\,(W)-2\eta\,(Y)\,\eta(Z)]\\ &\quad+(n-1)[g\,(Y,W)\,\eta\,(Z)-2\eta\,(Y)\,\eta\,(Z)\,\eta\,(W)+g(Z,W)\eta(Y)], \end{aligned} \tag{41}$$

Now, from above equations (41) and (10), we get

$$\begin{aligned} (n-3)\beta^2\,[g\,(Y,W)\,\eta\,(Z)-g\,(Z,W)\,\eta\,(Y)]&=-\beta^2[2g\,(Y,Z)\,\eta\,(U)+2g\,(Y,W)\,\eta\,(Z)\\ &\quad-\eta\,(W)\,\eta\,(Y)\,\eta\,(Z)-\eta\,(Y)\,\eta\,(Z)\,\eta\,(W)-2\eta\,(Y)\,\eta\,(Z)]\\ &\quad+(n-1)\,[g\,(Y,W)\,\eta\,(Z)-2\eta\,(Y)\,\eta\,(Z)\,\eta\,(W)+g\,(Z,W)\,\eta\,(Y)] \end{aligned} \tag{42}$$

Now, from above equation (43) in Putting $Y=W=\xi$, we get $\beta=0$ or $[1-\eta\,(U)]\,\eta\,(Z)\neq 0$. □

If $\lambda = 0$, which shows that $\lambda$ is steady. Thus we can states as follows.

**Example 3.9.** *We consider 3-dimensional $\eta$-Ricci solitons on $\beta$-Kenmotsu manifold with the Schouten-van Kanpen connection we consider the 3-dimensional manifold $M = \{(X, Y, Z) \in R^3, U \neq 0\}$, where $(X, Y, Z)$ are the Standard coordinates in $R^3$. Let $(e_1, e_2, e_3)$ are linearly independent at each point of $M$. Let $g$ be the Riemannian metric $g$ defined by*

$$e_1 = U^2 \frac{d}{dX}, \qquad g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$
$$e_2 = U^2 \frac{d}{dY}, e_3 = \frac{d}{dU}, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

*Let $\eta$ be the 1-form defined by $\eta(U) = g(U, e_3)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0$, then using linearity of $\phi$ and $g$, we have $\eta(e_3) = 1$, $\phi^2 U = -U + \eta(U) e_3$, $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$, for any $U, W \in \chi(M)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.*

$$[e_1, e_2] = 0, [e_2, e_3] = -\frac{2}{U} e_2, [e_1, e_3] = -\frac{2}{Z} e_1$$

*The Riemannian connection of the metric tensor $g$ is given by the Koszul's formula which is*

$$2g(\nabla_X Y, U) = Xg(Y, U) + Yg(U, X) - 2g(X, Y) - g(X, [Y, U]) - g(Y, [X, U]) + g(U, [X, Y]), \tag{43}$$

*Using from above equation (43), we get $2g(\nabla_{e_1} e_3, e_1) = 2g(-\frac{2}{Z} e_1, e_1)$, $2g(\nabla_{e_1} e_3, e_2) = 0$, and $2g(\nabla e_1, e_3, e_3) = 0$. Hence $\nabla_{e_1} e_3 = \frac{2}{U} e_1$. Similarly $\nabla_{e_2} e_3 = -\frac{2}{U} e_2$, $\nabla_{e_3} e_3$=0, further yields*

$$\nabla_{e_1} e_2 = 0, \nabla_{e_1} e_1 = \frac{2}{U} e_3, \nabla_{e_2} e_2 = \frac{2}{U} e_3, \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_1 = 0, \tag{44}$$

*from (44), we see that the manifold satisfies $\nabla_X \xi = \beta[X - \eta(X)\xi]$, for $\xi = e_3$, where $\beta = -\frac{2}{U}$. Hence we conclude that $M$ is an $\eta$-Ricci solitons on $\beta$-Kenmotsu manifold. Also then $\beta = 0$, known that $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, $R(e_1, e_2)e_3 = 0$, $R(e_2, e_3)e_3 = -\frac{\epsilon}{\beta^2} e_2$, $R(e_1, e_2)e_3 = -\frac{\epsilon}{\beta^2} e_1$, $R(e_1, e_2) e_2 = -\frac{4}{\beta^2} e_1$, $R(e_3, e_2)e_2 = -\frac{\epsilon}{\beta^2} e_3$, $R(e_1, e_3)e_2 = 0$, $R(e_1, e_2)e_1 = \frac{4}{\beta^2} e_2$, $R(e_2, e_3) e_1 = 0$, $R(e_1, e_3)e_1 = \frac{6}{\beta^2} e_3$. The Schouten-Van Kampen connection on $M$ is given by*

$$\nabla_{e_1} e_3 = \left(-\frac{2}{U} - \beta\right) e_1, \nabla_{e_2} e_3 = \left(-\frac{2}{U} - \beta\right) e_2, \nabla_{e_3} e_3 = -\beta(e_3 - \xi), \nabla_{e_1} e_2 = 0,$$
$$\overline{\nabla}_{e_2} e_2 = \frac{2}{\beta}(e_3 - \xi), \nabla_{e_3} e_2 = 0, \nabla_{e_1} e_1 = \frac{2}{U}(e_3 - \xi), \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_1 = 0, \nabla_{e_i} e_j = 0.$$

*From (**??**), we can see that $\nabla_{e_i} e_j = 0$, $(1 \leq i, j \leq 3)$ for $\xi = e_3$ and $\beta = -\frac{2}{U}$. Hence $M$ is a 3-dimensional $\eta$-Ricci solitons on $\beta$-kenmotsu manifold with the Schouten-Van Kampen connection. Also using (**??**), it can be seen that $R = 0$. Thus the manifold $M$ is a flat manifold with respect to the Schouten-Van Kampen connection. Since a flat is a Ricci-flat manifold with respect to the Schouten-Van Kampen connection, the manifold $M$ is both a Projectively flat and a conharmonically flat 3-dimensional $\eta$-Ricci solitons on $\beta$-Kenmotsu manifold with respect to the Schouten-Van Kampen connection. So, from Theorem 1 and Theorem 2 is $\eta$-Einstein manifold with respect to the Levi-Civita connection.*

## References

[1] Y. Ahmet, U. C. De and A. B. Eftal, *On Kenmotsu manifolds satisfying certain curvature conditions*, SUT J. Math., 45(2)(2009), 89-101.

[2] Y. Ahmet and U. C. De, *On a type of Kenmotsu manifolds*, Differ. Geom. Dyn. Syst., 12(2010), 289-298.

[3] C. S. Bagewadi, D. G. Prakasha and N. S. Basavarajappa, *On Lorentzian $\beta$-Kenmotsu manifolds*, Int. Jour. Math. Analysis, 19(2)(2008), 919-927.

[4] G. Ingalahalli and C. S. Bagewadi, *Ricci solitons $\alpha$-Sasakian manifolds*, ISRN Geometry, 2012(2012), Article ID 421384, 13 Pages.

[5] C. S. Bagewadi and E. G. Kumar, *Note on Trans- Sasakian manifolds*, Tensor N. S., 65(1)(2004), 80-88.

[6] K. K. Baishya, *Ricci Solitons in Sasakian manifold*, Afrika Matimatika, 28(7-8)(2017), 1061-1066.

[7] C. L. Bejan and M. Crasmareanu, *Ricci solitons in manifolds with quasi-contact curvature*, Publ. Math. Debrecen, 78(1)(2011), 235-243.

[8] A. Bejancu and K. L. Duggal, *Real Hypersurface of indefinite Kahler manifolds*, Int. Math. Sci., 16(3)(1993), 545-556.

[9] C. L. Bejan and M. Crasmareanu, *Second order parallel tensor and Ricci solitons in 3-dimensional narmal Para-contact geometry*, Anal. Global Anal. Geom., (2014).

[10] A. M. Blaga, *$\eta$-Ricci solitons on Lorentzian Para-Sasakian manifolds*, Filomat, 30(2)(2016), 489-496.

[11] A. M. Blaga, *$\eta$-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl., 20(2015), 1-13.

[12] H. D. Cao, *Existence of gradient Kahler-Ricci Solitons*, Elliptic and Parabolic methods in Geometry (Minneapolis), (1996), 1-16.

[13] C. Calin and M. Crasmareanu, *From the Eisenhart Problem to the Ricci solitons in f-Kenmotsu manifold*, Bull. Malays. Math. Soc., 33(2010).

[14] C. Calin and M. Crasmareanu, *Eta Ricci solitons on Hopf hyper surfaces in complex space forms*, Revue Romaine de Mathematiques in complex space forms, 57(1)(2012), 55-63.

[15] G. Calvaruso and D. Perrone, *Ricci Solitons in three-dimensional Paracontact geometry*, arxiv: 1407.3458v1, (2014).

[16] B. Y. Chen and S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl., 19(2014), 13-31.

[17] J. T. Cho and M. Kimura, *Ricci solitons and real hyper-surfaces in a complex space form*, Tohoku Math. J., 61(2)(2009), 205-212.

[18] O. Chodosh and F. T. H. Fong, *Rotational symmetry of conical Kahler- Ricci Solitons*, arxiv:1304.0277v2, (2013).

[19] S. Dragomir and L. Ornea, *Locally conformal Kaehler geometry*, Progress in Mathemaics 155, Birkhauser Boston Inc., Boston M. A, (1998).

[20] R. S. Hamilton, *Three Manifolds with positive Ricci curvature*, J. Diff. Geom., 17(2)(1988), 255-306.

[21] R. S. Hamilton, *Eaternal Solutions of The Ricci flow*, J. Diff. Geom., 38(1993), 1-11.

[22] K. Matsumoto, S. Ianus and I. Mihai, *On P-Sasakian Manifold which Admit Certain Tensor fields*, Publicationes Mathematicae Debrecen, 33 (1986), 61-65.

[23] K. Matsumotso and I. Mihai, *On a certain Transformation in a Lorenzian Para-Sasakian Manifold*, Tensor N. S., 47(1988), 189-197.

[24] J. C. Marrero, *The Local structure of trans-Sasakian Manifolds*, Ann. Mat. Pura Appl., 162(4)(1992), 77-86.

[25] J. C. Marrero and Chinea, *On trans-Sasakian manifolds*, Proceedings of the Xivth Spanish-Portuquese conferences on Mathematics, I-III(1990).

[26] H. G. Nagaraja and C. R. Premalatha, *Ricci solitons in Kenmotsu manifolds*, Journal of Mathematical Analysis, 3(2012), 18-24.

[27] S. K. Pandey, G. Pandey, K. Tiwari and R. N. Singh, *On a semi-symmetric non-metric connection in an indefinite para Sasakian manifolds*, Journal of Mathematics and Computer Sciences, 12(2014), 159-172.

[28] R. L. Patel S. K. Pandey and R. N. Singh, *Ricci Solitons on ($\varepsilon$)-Para Sasakian manifolds* International Journal of

Mathematics And its Applications, 6(1-A)(2018), 73-82.

[29] R. L. Patel, S. K. Pandey and R. N. Singh, *On Ricci Solitons in ($\varepsilon$)-Kenmotsu manifolds*, International Journal of Engineering Sciences and Management Research, 5(2)(2018), 17-23.

[30] G. Perelman, *Ricci flow with surgery on three Manifolds*, http://ar Xiv.org/absMath, (2003).

[31] G. P. Pokhariyal, *Study of a new curvature tensor in Sasakian Manifold*, Tensor N.S., 36(2)(1982), 222-225.

[32] R. Sharma, *Certain results on K-contact and $(k,\mu)$-contact manifolds*, Journal of Geometry, 89(1-2)(2008), 138-147.

[33] N. Sesum, *Limiting behariour of Ricci flow*, arxiv: DG. Math. DG/0402194.

[34] A. A. Shaikh, U. C. De and T. Q. Binh, *On K-contact $\eta$-Einstein manifolds*, Inst. Math. Inform. Debrecen, (2001), 311-315.

[35] R. N. Singh, S. K. Pandey, G. Pandey and K. Tiwari, *On a semi-symmetric metric connection in an ($\varepsilon$)-Kenmotsu manifold*, Commun. Korean Math. Soc., 29(2)(2014), 331-343.

[36] S. Tanno, *The Automorphism Groups of Almost contact Riemannian Manifolds*, Tohoku Mathematical Sournal, 21(1)(1969), 21-38.

[37] A. Yildiz and C. Murathan, *On Lorentzian $\alpha$-Sasakian Manifolds*, Kyungpook Mathematical Journal, 45(1)(2005), 95-103.