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Numerical Solution of Non Linear Differential Equation by Using Shooting Techniques

Research Article

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Abstract: Many problems that occur in physics and engineering can be modelled by linear or nonlinear differential equations. In this paper we find the solution of Blasius type equations which are nonlinear ordinary differential equations on a semi-infinite interval. The Blasius equation is a third-order non-linear ordinary differential equation. The non-linear mathematical model of the problem prohibits the use of the analytical methods. A numerical solution is the single approach for these problems. The two-point boundary problem was solved by a Runge-Kutta method and shooting method. Matlab functions make numerical solution of the mathematical models of the fluid flow relatively simple and quick solutions are presented for Blasius equations with additional computations based on the numerical results obtained by the Matlab function. Numerical study on boundary layer equation due to stationary flat plate, Matlab is the mathematical programming that used to solve the boundary layer equation applied toolbox method. The numerical results show a good agreement with the exact solution of Blasius equation and consistent with prior published result. The accuracy of the proposed method is higher than other approximation analytical solutions; hence suggest that proposed method is efficient and practical.

Keywords: Boundary layer, Blasius flow, Newton method, RungeKutta method, Shooting Technique.

1. Introduction

1.1. Blasius Equation

If a fluid flows past a solid, a fluid layer is formed adjacent to the boundary of the solid. This layer is called a boundary layer and strong viscous effects exist within this layer. Consider a uniform flow over a flat surface, $y = 0$, $x \geq 0$, $-\infty < z < \infty$ Equations of the flow in the boundary layer are the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

and the reduced Navier-Stokes equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

where u and v are respectively the components of the velocity vector and v represents the viscosity of the fluid. Boundary conditions are

$$u(x,0) = 0 \qquad x \geq 0 \tag{3a}$$

$$v(x,0) = 0 \qquad x \geq 0 \tag{3b}$$

$$u(x,y) \to U \text{ as } y \to \infty \tag{3c}$$

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where U is the constant speed of the flow outside the boundary Layer. Define a stream function $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y}\ ,\ v = -\frac{\partial \psi}{\partial x} \tag{4}$$

then equation (1) is satisfied identically and equation (2) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = v\frac{\partial^3 \psi}{\partial y^3} \tag{5}$$

Blasius used a similarity transformation to reduce (1.5) to an ordinary differential equation. A similarity transformation is based on the symmetry analysis of a differential equation [11?]. When a symmetry property of a differential equation is identified it can be exploited to achieve a simplification. If it is an ordinary differential equation then usually the order of the equation can be reduced. If it is a partial differential equation then usually the dependent and independent variables can be combined to achieve a reduction of order or a reduction of the partial differential equation to an ordinary differential equation. In the case of (5) symmetry analysis leads to the following transformation [11]

$$\eta = a\frac{y}{\sqrt{x}}$$
$$\psi(x,y) = b\sqrt{x}\,f(\eta)$$

where a and b are constants and are chosen to make η and $f(\eta)$ dimensionless. They are taken as

$$a = \sqrt{\frac{U}{\nu}}$$
$$b = \sqrt{\nu U}$$

With this choice, η is called the dimensionless similarity variable and $f(\eta)$ is called the dimensionless stream function. Now

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= -\frac{U}{2}\frac{y}{x}f'(\eta) + \frac{1}{2}\sqrt{\nu U}\frac{f(\eta)}{\sqrt{x}} \\
\frac{\partial \psi}{\partial y} &= Uf'(\eta) \\
\frac{\partial^2 \psi}{\partial y^2} &= Uf''(\eta)\frac{a}{\sqrt{x}} \\
\frac{\partial^2 \psi}{\partial x \partial y} &= -\frac{U}{2}\sqrt{\frac{U}{\nu}}\frac{y}{x^{\frac{3}{2}}}f''(\eta) = -\frac{U}{2x}\eta f''(\eta) \\
\frac{\partial^3 \psi}{\partial y^3} &= \frac{U^2}{\nu x}f'''(\eta).
\end{aligned}$$

A substitution of the above derivatives in equation (5) reduces it to

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2}f(\eta)\frac{d^2 f}{d\eta^2} = 0 \tag{6}$$

Equation (6) is known as the Blasius equation. The boundary condition (3a) transforms to

$$f'(0) = 0 \tag{7a}$$

while (3b) becomes

$$f(0) = 0 \tag{7b}$$

and (1.3c) reduces to

$$f'(\eta) \to 1 \quad as \quad \eta \to \infty \tag{7c}$$

Equation (6) together with the boundary conditions (7a), (7b) and (7c) is called the Blasius problem. Several methods have been used for numerical solution of Falkner-Skan equations. Meksyn [3] solved the Falkner-Skan equation through analytical approximations. Asaithambi [4, 5] used finite difference method and piecewise linear functions for solving Falkner-Skan equation switch high accuracy. Recently Shi-JunLiao [6] applied the homotopy analysis to solve the Falkner-Skan equation. Khabibrakhmanov and D.Summers [7] used a spectral method with generalized Laguerre polynomials for solving the Blasius equation ($\beta = 0$). Moreover, the Blasius equation was solved by Rosales and Valencia [8] using Fourier series. Also, Vera and Valencia [9] solved the Falkner-Skan equation with heat transfer through an expansion in Fourier series. In this paper, write the Blasius equation as a first order differential system and obtain a numerical solution to the differential using 4^{th} order Runge- Kutta method by using a guess α and find out the solution.

1.2. Method of Solution

The non-linear differential equations (1) subject to the boundary conditions (2) constitute a two-point boundary value problem. In order to solve these equations numerically, we follow RungeKutta 4^{th} order with shooting technique. In this method it is most important to choose the appropriate finite values of $\eta \to \infty$. The solution process is repeated with another large value of $\eta \to \infty$ until two successive values of $f''(0)$ differ only after a desired digit signifying the limit of the boundary along η. The last value of $\eta \to \infty$ is chosen as appropriate value of the limit $\eta \to \infty$ for that particular set of parameters. The ordinary differential equation (1) was first converted into a set of three first-order simultaneous equations. To solve this system we require three initial conditions but we have only two initial conditions, $f(0)$ and $f'(0)$ on $f(\eta)$. The initial condition $f''(0)$ is not prescribed. However the values of $f'(\eta)$ is known at $\eta = 0$. Now we employ the numerical shooting technique based where this ending boundary condition is utilized to produce unknown initial conditions at $\eta = 0$ finally, the problem has been solved numerically using Runge-Kutta 4^{th} order.

2. Description of the Method

2.1. Reduction to a First Order System

The solve the falkner-skan equation numerically, the equation is reduced to a first order system by introducing the three auxiliary variables.

$$f = u_1 \frac{\partial f}{\partial \eta} = u_2 \text{ and } \frac{\partial^2 f}{\partial \eta^2} = u_3,$$

So that we have the following system of three coupled ODEs:

$$f_1(\eta, u_1, u_2, u_3) = u_1' = u_2$$

$$f_2(\eta, u_1, u_2, u_3) = u_2' = u_2$$

$$f_3(\eta, u_1, u_2, u_3) = u_3' = -u_1 u_3$$

The first order system can be written more compactly using vector notation.

$$\frac{\partial f}{\partial \eta} = f_1(\eta, u_1, u_2, u_3). \ i.e \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_3 \\ -u_1 u_3 \end{bmatrix}$$

it is important to note the ODE system is in normal form and then the boundary condition

$$u_1(0) = 0$$

$$u_2(0) = 0$$

$$u_2(\eta = \infty) = 1$$

Where $\eta = \infty$ is the unknown free boundary used to truncate the semi-infinite interval to a finite one. Which is to be determine as the part of the procedure in addition, an initial condition on the second derivatives is introduced to apply the Shooting Method,

$$\frac{\partial^2 f}{\partial \eta^2} = \alpha \text{ at } \eta = 0,$$

where α is the shooting angle. The shooting algorithm therefore consists of the following procedure:

(1). Starting from a relatively large value of α as the initial guess $f'(\eta)$ is evaluated by increasing η through steps of h from zero to η_m.

(2). If at some η, $f'(\eta) > 1$, then α is decreased and $f'(\eta)$ evaluated until $f'(\eta) < 1$ for some α. At this point, the asymptotic profile is bracketed.

(3). A new α is then determined by Newton Method.

(4). If $f'(\eta)$ does not cross unity from below as η increases from zero to η_m, then is checked for negativity. If negative, α is below its correct value and Newton Method again determines the next α.

(5). Finally, when the estimate for α is approximately within an order of magnitude of the desired error, Runge-Kutta 4^{th} order method can be used to the initial value problems.

2.2. Numerical Solution

A numerical solution of the Blasius problem usually uses the shooting method. In this method it is assumed that

$$f''(0) = \sigma \tag{8}$$

and the problem is solved with different values of σ. Such values of σ will lead to different values of $\frac{df}{d\eta}$ as $\eta \to \infty$. We seek that value of σ which will yield an f which satisfies

$$\lim \eta \to \infty \frac{df}{d\eta} = 1.$$

First accurate numerical solution was obtained by Howarth [2]. More recently Asaithambi [4], and Cortell [10] have also solved the Blasius problem by the shooting method. In practice it is impossible to carry out calculations up to infinity. Hence an $\eta = \eta_\infty$ is arbitrarily fixed and we demand that

$$\frac{df}{d\eta} = 1 \text{ when } \eta = \eta_\infty.$$

We solve the Blasius equation by the shooting method. We start with $\alpha = 0.1$ and find that $f'(\eta)$ between $\eta = 12$ and 13 are practically a constant=0.449287. With $\alpha = 0.6$. We find the slope for $12 < \eta < 13$ equal to 1.18352. This means the actual value should lie between 0.1 and 0.6. Therefore our next choice is $\frac{0.1+0.6}{2} = 0.35$ and so on. We present the results in Table 1 and 2.

α	$f'(12.5)$
0.1	0.449287
0.6	1.48352
0.35	1.03571
0.225	0.771459
0.2875	0.908413
0.31875	0.973101
0.334375	1.00465
0.326562	0.988937
0.330468	0.996808
0.3324218	1.00073
0.3314453	0.998771
0.3319335	0.999751
0.3321771	1.00024
0.3320553	0.999996
0.3321162	1.00012
0.3320857	1.00006
0.3320705	1.00003
0.3320629	1.00001
0.3320591	1

Table 1. **Sequence of values of α converging to σ**

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.0	0	0	0.332059
0.2	0.00664105	0.0664081	0.331985
0.4	0.02656	0.132765	0.331468
0.6	0.059735	0.198938	0.330081
0.8	0.106109	0.264711	0.327391
1.0	0.165573	0.329782	0.323009
1.2	0.23795	0.393778	0.31659
1.4	0.322983	0.456264	0.307867
1.6	0.420323	0.516759	0.296665
1.8	0.529521	0.574761	0.282932
2.0	0.650028	0.629769	0.266753
2.2	0.781197	0.681314	0.248352
2.4	0.922295	0.728985	0.228092
2.6	1.07251	0.772459	0.206455
2.8	1.23098	0.811513	0.184007
3.0	1.39682	0.846048	0.161359
3.2	1.5691	0.876085	0.139129
3.4	1.74696	0.901765	0.117876
3.6	1.92953	0.923334	0.0980867
3.8	2.11604	0.941122	0.0801258
4.0	2.30576	0.955522	0.0642341
4.2	2.49805	0.966961	0.0505193
4.4	2.69237	0.975875	0.0389731
4.6	2.88826	0.982687	0.0294829
4.8	3.08533	0.987793	0.0218711
5.0	3.28329	0.991546	0.0159068
5.2	3.48188	0.994249	0.0113414
5.4	3.68094	0.996159	0.00792786
5.6	3.88031	0.997481	0.00543169
5.8	4.0799	0.998379	0.00364835
6.0	4.27964	0.998976	0.00240198
6.2	4.47948	0.999366	0.00155022
6.4	4.67938	0.999615	0.000980419

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
6.6	4.87932	0.999711	0.000608191
6.8	5.07928	0.999867	0.000369599
7.0	5.27926	0.999925	0.000220207
7.2	5.47925	0.999959	0.000128431
7.4	5.67924	0.999979	0.0000737176
7.6	5.87924	0.99999	0.0000413441
7.8	6.07924	0.999996	0.0000227099
8.0	6.27924	1	0.0000122503
8.2	6.47924	1	6.4618×10^{-6}
8.4	6.67924	1	3.28575×10^{-6}

Table 2. **Numerical values of $f(\eta)$, $f'(\eta)$ and $f''(\eta)$**

η	Approximation solution	Numerical solution
0	0	0
0.4	0.0266	0.02656
0.8	0.1061	0.106109
1.2	0.2379	0.23795
2.0	0.6500	0.650028
2.8	1.2311	1.23098
3.0	1.396	1.39682
4.0	2.3058	2.30576
5.0	3.2827	3.2832
8.0	6.2793	6.2792

Table 3. **Comparisons the values of $f(\eta)$ for different authors**

3. Conclusion

In this study, we have considered the classical Blasius problem. This nonlinear differential equation is successfully solved by employing Runge-Kutta method with shooting method to obtain numerical solutions. It is found that present results are in good agreement compared to exact solutions by Blasius [1] for the values $f(\eta)$ and $f'(\eta)$ as well as the values of $f''(\eta)$ in comparison with Howarth [2] as shown in Table 1-3 and Figure 1-3. The numerical results strongly display the efficiency and accuracy of the proposed method in solving the nonlinear equation.

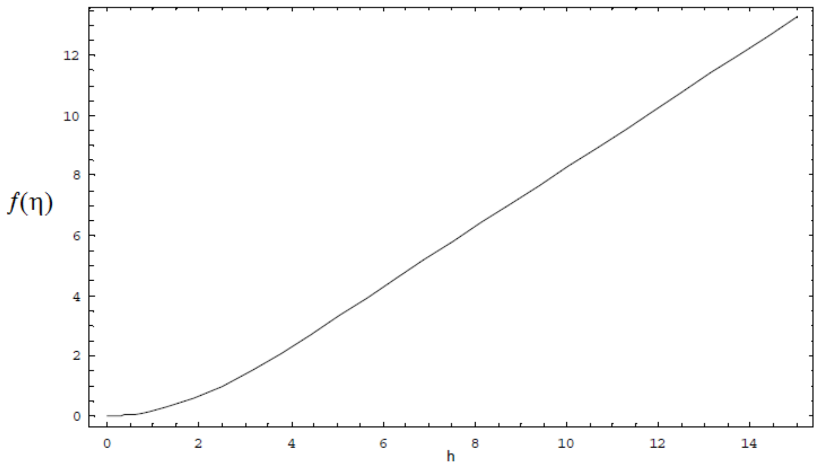

Fig1: $f(\eta)$ as a function of η

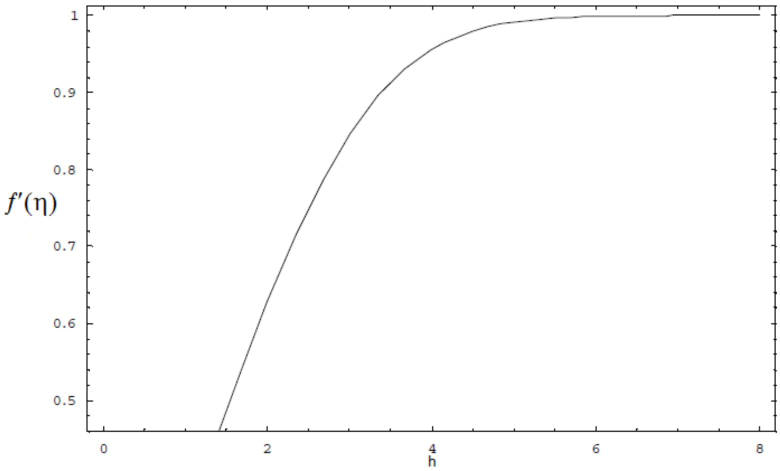

Fig2: $f'(\eta)$ as a function of η

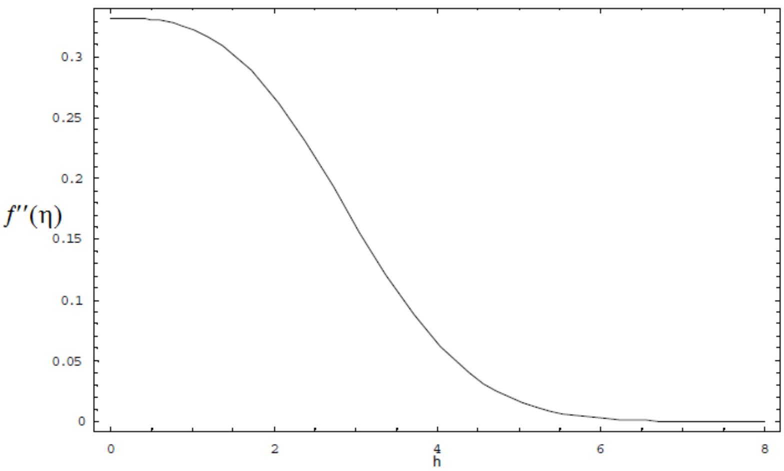

Fig3: $f''(\eta)$ as a function of η

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