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Fuzzy Supermodular Lattice

Research Article

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Abstract: In this Paper, Fuzzy Supermodular Lattice Definition of Fuzzy Supermodular Lattice-Characterization theorem are given.

Keywords: Fuzzy Modular Lattice, Fuzzy Distributive Lattice, Fuzzy Supermodular Lattice.

1. Introduction

The Concept of Fuzzy Lattice was already introduced by N.Ajmal [1], S.Nanda [3] and L.R.WilCox [6] explained modularity in the theory of Lattices, Iqbalunnisa and W.B.Vasantha [7] explained by Supermodular Lattices, G.Gratzer [2], M.Mullai and B.Chellappa [4] explained Fuzzy L-ideal and V.Vinoba and K.Nithya [5] Explained fuzzy modular pairs in Fuzzy Lattice and Fuzzy Modular Lattice. A few of definitions and results are listed that the fuzzy Supermodular lattice using in this paper we explain fuzzy Supermodular lattice, Definition of fuzzy Supermodular lattice, Characterization theorem of Fuzzy Supermodular Lattice and some examples are given.

Definition 1.1. *A Fuzzy lattice L is said to be Fuzzy supermodular lattice, if* $\mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d) = \mu(a) \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(c) \wedge \mu(d) \wedge \mu(a \vee b)] \vee [\mu(d) \wedge \mu(b) \wedge \mu(a \vee c)]$, *for all* $\mu(a)$, $\mu(b)$, $\mu(c)$*inL.*

2. Main Section

Example 2.1. *The Fuzzy lattice* M_4 *of figure is not a Fuzzy supermodular lattice.*

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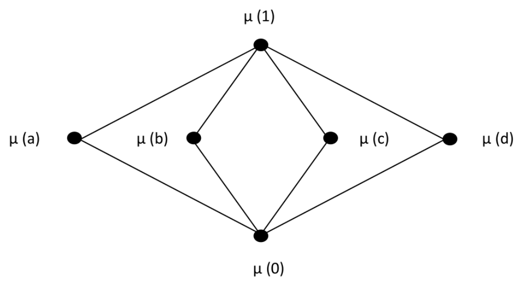

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Here

$$\begin{aligned}\mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d) &\geq \min\{\mu(a \vee b), \mu(a \vee c), \mu(a \vee d)\}\\ &\geq \min\{\mu(1), \mu(1), \mu(1)\}\\ &= \mu(1).\end{aligned}$$

$\mu(a) \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(c) \wedge \mu(d) \wedge \mu(a \vee b)] \vee [\mu(d) \wedge \mu(b) \wedge \mu(a \vee c)]$

$$\begin{aligned}&\geq \mu(a) \min\{\mu(b) \wedge \mu(c) \wedge \mu(a \vee d), \mu(c) \wedge \mu(d) \wedge \mu(a \vee b), \mu(d) \wedge \mu(b) \wedge \mu(a \vee c)\}\\ &\geq \mu(a) \min\{\mu(o \wedge l)\mu(o \wedge l), \mu(o \wedge l)\}\\ &\geq \mu(a) \min\{\mu(o)\mu(o), \mu(o)\}\\ &= \mu(a) \vee \mu(o)\\ &= \mu(a).\end{aligned}$$

$\mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d) \neq \mu(a) \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(c) \wedge \mu(d) \wedge \mu(a \vee b)] \vee [\mu(d) \wedge \mu(b) \wedge \mu(a \vee c)]$. *Hence M_4 is not a Fuzzy Supermodular lattice.*

Example 2.2. *The Fuzzy lattice $M_{3,3}$ of figure is not a Fuzzy supermodular lattice.*

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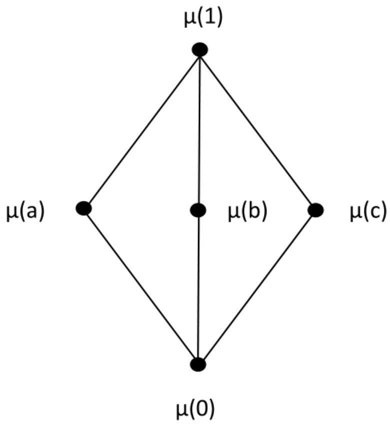


Here

$$\begin{aligned}\mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d) &\geq \min\{\mu(a \vee b), \mu(a \vee c), \mu(a \vee d)\}\\ &\geq min\{\mu(1), \mu(1), \mu(1)\}\\ &= \mu(1).\end{aligned}$$

$\mu(a) \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(c) \wedge \mu(d) \wedge \mu(a \vee b)] \vee [\mu(d) \wedge \mu(b) \wedge \mu(a \vee c)]$

$$\begin{aligned}&\geq \mu(a) min\{\mu(b) \wedge \mu(c) \wedge \mu(a \vee d), \mu(c) \wedge \mu(d) \wedge \mu(a \vee b), \mu(d) \wedge \mu(b) \wedge \mu(a \vee c)\}\\ &\geq \mu(a) min\{\mu(0 \wedge 1)\mu(0 \wedge 1), \mu(0 \wedge 1)\}\\ &\geq \mu(a) min\{\mu(0)\mu(0), \mu(0)\}\\ &= \mu(a) \vee \mu(0)\\ &= \mu(a).\end{aligned}$$

$\mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d) \neq \mu(a) \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(d) \wedge \mu(b) \wedge \mu(a \vee c)]$. *Hence* $M_{3,3}$ *is not a Fuzzy Supermodular lattice.*

Theorem 2.3. *Every Fuzzy distributive lattice is a Fuzzy supermodular lattice and the converse is not true.*

Proof. Given that L is a Fuzzy distributive lattice. To Prove that L is a Fuzzy supermodular lattice. Let $\mu(a)$, $\mu(b)$, $\mu(c)$, $\mu(d)$ in L be arbitrary. Then $\mu(a) \wedge [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(c) \wedge \mu(d) \wedge \mu(a \vee b)] \vee [\mu(d) \wedge \mu(b) \wedge \mu(a \vee c)]$

$$\geq \min\{\mu(a), \mu(b) \wedge \mu(c) \wedge \mu(a \vee d), \mu(c) \wedge \mu(d) \wedge \mu(a \vee b), \mu(d) \wedge \mu(b) \wedge \mu(a \vee c)\}$$

$$\geq \min\{[\mu(a) \vee \mu(b \wedge c)] \wedge \mu(a \vee d), \mu(c) \wedge \mu(d) \wedge \mu(a \vee b), \mu(d) \wedge \mu(b) \wedge \mu(a \vee b)\}, \text{ by associative law.}$$

$$\geq \min\{\mu(a \vee b) \wedge \mu(a \vee c)] \wedge \mu(a \vee d), \mu(c) \wedge \mu(d) \wedge \mu(a \vee b), \mu(d) \wedge \mu(b) \wedge \mu(a \vee c)\}, \text{ Since L is a Fuzzy distributive}$$

$$\geq \min\{\mu(a \vee b) \wedge \mu(a \vee c)] \wedge \mu(a \vee d), \mu(d) \wedge \mu(b) \wedge \mu(a \vee c)\},$$

Since $\mu(a \vee c) \geq \mu(c)$; $\mu(a \vee d) \geq \mu(d)$

$$\begin{aligned} \Rightarrow \mu(a \vee c) \wedge \mu(a \vee d) &\geq \min\{\mu(a \vee c), \mu(a \vee d)\} \\ &\geq \min\{\mu(c), \mu(d)\} \\ &\geq \mu(c \wedge d) \end{aligned}$$

$$\mu(a \vee c) \wedge \mu(a \vee d) \wedge \mu(a \vee b) \geq \mu(c \wedge d) \wedge \mu(a \vee b) = \mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d),$$

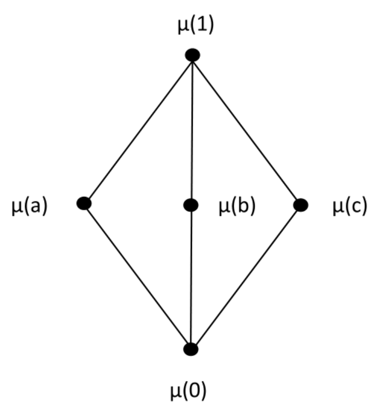
Since $\mu(a \vee b) \geq \mu(b)$; $\mu(a \vee d) \geq \mu(d)$

$$\Rightarrow \mu(a \vee b) \wedge \mu(a \vee d) \geq \mu(b \wedge d)$$

$$\Rightarrow \mu(a \vee b) \wedge \mu(a \vee d) \wedge \mu(a \vee c) \geq \mu(a \vee c) \wedge \mu(b \wedge d)$$

Thus $\mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d) = \mu(a) \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(c) \wedge \mu(d) \wedge \mu(a \vee b)] \vee [\mu(d) \wedge \mu(b) \wedge (a \vee c)]$, for all $\mu(a)$, $\mu(b)$, $\mu(c)$, $\mu(d)$ in L. Hence L is a Fuzzy Supermodular lattice.

The Converse is not true. That is fuzzy Supermodular lattice need not be Fuzzy distributive. Consider the Fuzzy lattice M_3 of figure.



This Fuzzy lattice is a Fuzzy Supermodular lattice but not Fuzzy distributive. Here

$$\begin{aligned} \mu(a \vee b) \wedge \mu(a \vee c) &\geq \min\{\mu(a \vee b), \mu(a \vee c)\} \\ &\geq \min\{\mu(l), \mu(l)\} \\ &= \mu(1) \end{aligned}$$

$$\begin{aligned}\mu(a)\vee\mu(b\wedge c) &\geq \min\{\mu(a),\mu(b\wedge c)\}\\ &\geq \min\{\mu(a),\mu(0)\}\\ &= \mu(a\vee 0)\\ &= \mu(a)\end{aligned}$$

Therefore $\mu(a\vee b)\wedge\mu(a\vee c)\neq\mu(a)\vee\mu(b\wedge c)$. Hence M_3 is not Fuzzy distributive. □

Theorem 2.4. *Every Fuzzy Supermodular lattice is Fuzzy modular lattice and the converse is not true.*

Proof. Given that L is a Fuzzy Supermodular lattice

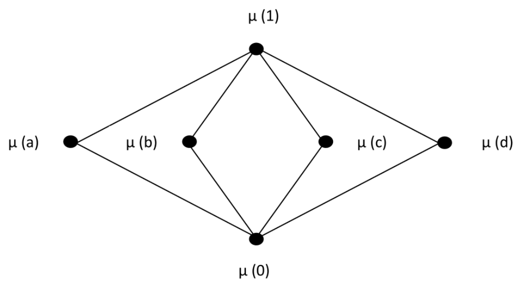
$$\Rightarrow \mu(a\vee b)\wedge\mu(a\vee c)\wedge\mu(a\vee d)=\mu(a)\vee[\mu(b)\wedge\mu(c)\wedge\mu(a\vee d)]\vee[\mu(c)\wedge\mu(d)\wedge\mu(a\vee b)]\vee[\mu(d)\wedge\mu(b)\wedge\mu(a\vee c)], \quad (1)$$

for all $\mu(a)$, $\mu(b)$, $\mu(c)$ in L. To Prove that L is a Fuzzy modular lattice. Put $d=c$ in (1). Then

$$\begin{aligned}LHS &= \mu(a\vee b)\wedge\mu(a\vee c)\wedge\mu(a\vee c)\\ &\geq \min\{\mu(a\vee b),\mu(a\vee c),\mu(a\vee c)\}\\ &\geq \min\{\mu(a\vee b),\mu(a\vee c)\}\\ &\geq \min\{\mu(a\vee c),\mu(a\vee b)\},\ \text{ by commutative law}\\ &= \mu(a\vee c)\wedge\mu(a\vee b).\\ RHS &= \mu(a)\vee[\mu(b)\wedge\mu(c)\wedge\mu(a\vee c)]\vee[\mu(c)\wedge\mu(c)\wedge\mu(a\vee b)]\vee[\mu(c)\wedge\mu(b)\wedge\mu(a\vee c)]\\ &\geq \min\{\mu(a),\mu(b)\wedge\mu(c)\wedge\mu(a\vee c),\mu(c)\wedge\mu(c)\wedge\mu(a\vee b),\mu(c)\wedge\mu(b)\wedge\mu(a\vee c)\}\\ &\geq \mu(a)\min\{\mu(b\wedge c),\mu(c)\wedge\mu(a\vee b),\mu(b\wedge c)\},\ \text{ by absorption law, idempotent law.}\\ &\geq \mu(a)\min\{\mu(b\wedge c),\mu(c)\wedge\mu(a\vee b)\}\\ &\geq \mu(a)\vee[\mu(c)\wedge\mu(a\vee b)]\end{aligned}$$

Since $\mu(b)\leq\mu(a\vee b)\Rightarrow\mu(b\wedge c)\leq\mu(a\vee b)\wedge\mu(c)$. Therefore $\mu(a\vee c)\wedge\mu(a\vee b)=\mu(a)\vee[\mu(c)\wedge\mu(a\vee b)]$, for all $\mu(a)$, $\mu(b)$, $\mu(c)$ in L. Hence L is a Fuzzy modular lattice.

The Converse need not be true. Consider the Fuzzy lattice M_4 of figure.



This Fuzzy lattice is a Fuzzy modular lattice but not Fuzzy supermodular. $\mu(a\vee b)\wedge\mu(a\vee c)\wedge\mu(a\vee d)\neq\mu(a)\vee[\mu(b)\wedge\mu(c)\wedge\mu(a\vee d)]\vee[\mu(c)\wedge\mu(d)\wedge\mu(a\vee b)]\vee[\mu(d)\wedge\mu(b)\wedge\mu(a\vee c)]\Rightarrow M_4$ is not Fuzzy supermodular lattice. □

Theorem 2.5. *A Fuzzy modular lattice L is Fuzzy supermodular if and only if it does not contain a fuzzy sublattice isomorphic to either M_4 or $M_{3,3}$.*

Proof. Assume that a Fuzzy modular lattice L is Fuzzy Supermodular. To Prove that L does not contain a Fuzzy sublattice isomorphic to either M_4 or $M_{3,3}$. Suppose L contains a Fuzzy sublattice isomorphic to M_4 or $M_{3,3}$.

$\Rightarrow$ L is not a Fuzzy supermodular lattice. This is a Contradiction. Hence L does not contain a Fuzzy sublattice isomorphic to either M_4 or $M_{3,3}$.

Conversely, Assume that L does not contain a Fuzzy sublattice isomorphic to either M_4 or $M_{3,3}$. To prove that the Fuzzy modular lattice is Fuzzy supermodular. Suppose a Fuzzy modular lattice L is not Fuzzy supermodular $\Rightarrow$ L contains a Fuzzy sublattice isomorphic to M_4 or $M_{3,3}$. This is a Contradiction. Hence a Fuzzy modular lattice L is a Fuzzy Supermodular lattice. □

Theorem 2.6. *A Fuzzy supermodular lattice L is Fuzzy distributive iff L does not contain a Fuzzy sublattice isomorphic to M_3.*

Proof. Assume that a Fuzzy supermodular lattice L is a Fuzzy distributive lattice. To Prove that L does not contain a Fuzzy sublattice isomorphic to M_3. Suppose L contain a Fuzzy sublattice isomorphic to $M_3 \Rightarrow$ L is not a Fuzzy distributive lattice. This is a Contradiction. Hence L does not contain a Fuzzy sublattice isomorphic to M_3.

Conversely, Assume that L is a Fuzzy supermodular lattice which does not contain a Fuzzy sublattice isomorphic to M_3. To Prove L is a Fuzzy distributive lattice. Suppose L is not a Fuzzy distributive lattice. Then L contains a Fuzzy sublattice isomorphic to M_3. This is a Contradiction. Hence L is a Fuzzy distributive lattice. □

Theorem 2.7. *Any homomorphic image of a Fuzzy supermodular lattice is Fuzzy supermodular.*

Proof. Given that L be any homomorphic image of a Fuzzy supermodular lattice. To prove that L is a Fuzzy supermodular lattice. Let $\mu(a)$, $\mu(b)$, $\mu(c)$, $\mu(d)$ in L be arbitrary. Then $\mu(a \vee b) \wedge \mu(a \vee c) \wedge \mu(a \vee d) \geq min\{\mu(a \vee b), \mu(a \vee c), \mu(a \vee d)\}$

$$\geq min\{\mu(f(a \vee b)), \mu(f(a \vee c)), \mu(f(a \vee d))\}$$

$$\geq \mu(f(a \vee b)) \wedge \mu(f(a \vee c)) \wedge \mu(f(a \vee d))$$

$$\geq \mu(f(a))min\{\mu(f(b)) \wedge \mu(f(c)) \wedge \mu(f(a \vee d)), \mu(f(c) \wedge (f(d)) \wedge \mu(f(a \vee b)), \mu(f(d)) \wedge \mu(f(b)) \wedge \mu(f(a \vee c))\}$$

$$\geq \mu(a)min\{\mu(b) \wedge \mu(c) \wedge \mu(a \vee d), \mu(c) \wedge \mu(d) \wedge \mu(a \vee b), \mu(d) \wedge \mu(b) \wedge \mu(a \vee c)$$

$$\geq \mu(a) \vee [\mu(b) \wedge \mu(c) \wedge \mu(a \vee d)] \vee [\mu(c) \wedge \mu(d) \wedge \mu(a \vee b)] \vee [\mu(d) \wedge \mu(b) \wedge \mu(a \vee c)]$$

L is a Fuzzy supermodular lattice. □

Theorem 2.8. *The class of Fuzzy supermodular lattice is an equational class of fuzzy lattices lying between the equational class of fuzzy modular lattices and the equational class of fuzzy distributive lattices.*

3. Conclusion

This paper is proved that Every Fuzzy distributive lattice is a Fuzzy supermodular lattice and the converse is not true, Every Fuzzy Supermodular lattice is Fuzzy modular lattice and the converse is not true, A Fuzzy modular lattice L is Fuzzy supermodular if and only if it does not contain a fuzzy sublattice isomorphic to either M_4 or $M_{3,3}$. A Fuzzy supermodular lattice L is Fuzzy distributive iff L does not contain a Fuzzy sublattice isomorphic to M_3, Any homomorphic image of a

Fuzzy supermodular lattice is Fuzzy supermodular and The class of Fuzzy supermodular lattice is an equational class of fuzzy lattices lying between the equational class of fuzzy modular lattices and the equational class of fuzzy distributive lattices.

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