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On a Generalized K^h-Birecurrent Finsler Space

Research Article

Fahmi Yaseen Abdo Qasem[1] and Wafa'a Hadi Ali Hadi[2]*

1 Department of Mathematics, Faculty of Education-Aden, University of Aden, Khormaksar, Aden, Yemen.

2 Department of Mathematics, Community College-Aden, Dar Saad, Aden, Yemen.

Abstract: In the present paper, a Finsler space whose curvature tensor K^i_{jkh} satisfies $K^i_{jkh|\ell|m} = a_{\ell m} K^i_{jkh} + b_{\ell m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$, $K^i_{jkh} \neq 0$, where $a_{\ell m}$ and $b_{\ell m}$ are non-zero covariant tensor fields of second order called recurrence tensor fields, is introduced, such space is called as a generalized K^h-birecurrent Finsler space. The associate tensor K_{jrkh} of Cartan's fourth curvature tensor K^i_{jkh}, the torsion tensor H^i_{kh}, the deviation tensor K^i_h, the Ricci tensor K_{jk}, the vector H_k and the scalar curvature K of such space are non-vanishing. Under certain conditions, a generalized K^h-birecurrent Finsler space becomes Landsberg space. Some conditions have been pointed out which reduce a generalized K^h-birecurrent Finsler space $F_n(n > 2)$ into Finsler space of scalar curvature.

Keywords: Finsler space, Generalized K^h-birecurrent Finsler space, Ricci tensor, Landsberg space, Finsler space of scalar curvature.

1. Introduction

H.S. Ruse [3] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as Riemannian space of recurrent curvature. This idea was extended to n-dimensional Riemannian and non-Riemannian space by A.G. Walker [1], Y.C.Worg [9], Y.C. Worg and K. Yano [10] and others. This idea was extended to Finsler spaces by A.Moor [2] for the first time. Due to different connections of Finsler space, the recurrent of Cartan's fourth curvature tensor K^i_{jkh} have been discussed by N.S.H.Hussien [5], birecurrent of Cartan's fourth curvature tensor K^i_{jkh} have been discussed by M.A.A.Ali [4]. P.N.Pandey, S.Saxena and A.Goswami [7] interduced a generalized H-recurrent Finsler space. Let F_n be an n-dimensional Finsler space equipped with the metric function a $F(x, y)$ satisfying the request conditions [3]. The vectors y_i, y^i and the metric tensor g_{ij} satisfies the following relations

$$\begin{aligned}
&a)\ \ y_i y^i = F^2\\
&b)\ \ g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i\\
&c)\ \ y_{i|k} = 0 \qquad\qquad (1)\\
&d)\ \ y^i_{|k} = 0\\
&e)\ \ g_{ij|k} = 0 \ \text{ and}\\
&f)\ \ g^{ij}_{|k} = 0.
\end{aligned}$$

* E-mail: wf_hadi@yahoo.com

The unit vector $\imath^i$ and the associate vector $\imath_i$ is defined by

$$a)\ \imath^i = \frac{y^i}{F} \qquad b)\ \imath_i = g_{ij}\imath^j = \dot{\partial}_i F = \frac{y_i}{F}. \tag{2}$$

The process h- covariant differentiation commute with the partial differentiation with respect to y^j according to

$$\begin{aligned}
&a)\ \dot{\partial}_j \left(X^i_{|k}\right) - \left(\dot{\partial}_j X^i\right)_{|k} = X^r \left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk},\\
&b)\ P^r_{jk} = \left(\dot{\partial}_j \Gamma^{*r}_{hk}\right) y^h = \Gamma^{*r}_{jhk} y^h,\\
&c)\ \Gamma^{*i}_{jkh} y^h = G^i_{jkh} y^h = 0,\\
&d)\ P^i_{jk} y^j = 0,\\
&e)\ g_{ir} P^i_{kh} = P_{rkh}.
\end{aligned} \tag{3}$$

The tensor H^i_{jkh} satisfies the relation

$$H^i_{jkh} y^j = H^i_{kh}. \tag{4}$$

$$H^i_{jkh} = \dot{\partial}_j H^i_{kh}. \tag{5}$$

The torsion tensor H^i_{kh} satisfies

$$H^i_{kh} y^h = H^i_k, \tag{6}$$

$$K^i_{jkh} y^j = H^i_{kh}, \tag{7}$$

$$H_{jk} = H^i_{jki}, \tag{8}$$

$$H_k = H^i_{ki}, \quad \text{and} \tag{9}$$

$$H = \frac{1}{n-1} H^i_i. \tag{10}$$

where H_{jk} and H are called h-Ricci tensor [6] and curvature scalar respectively. Since contraction of the indices does not affect the homogeneity in y^i, hence the tensors H_{rk}, H_r and the scalar H are also homogeneous of degree zero, one and two in y^i respectively. The above tensors are also connected by

$$H_{jk} y^j = H_k, \tag{11}$$

$$H_{jk} = \dot{\partial}_j H_k, \tag{12}$$

$$H_k y^k = (n-1)\, H. \tag{13}$$

The tensors H^i_h, H^i_{kh} and H^i_{jkh} also satisfy the following :

$$H^i_{kh} = \dot{\partial}_k H^i_h, \tag{14}$$

$$g_{ij} H^i_k = g_{ik} H^i_j. \tag{15}$$

The associate tensor K_{ijkh} of Cartan's fourth curvature tensor K^i_{jkh} is given by

$$K_{ijkh} = g_{rj} K^r_{ikh}. \tag{16}$$

The necessary and sufficient condition for a Finsler space F_n $(n > 2)$ to be a Finsler space of scalar curvature is given by

$$H_h^i = F^2 R(\delta_h^i - \imath^i \imath_h). \tag{17}$$

A Finsler space F_n is said to be Landsberg space if satisfies

$$y_r G_{jkh}^r = -2C_{jkh|m} y^m = -2P_{jkh} = 0. \tag{18}$$

The Ricci tensor K_{jk} of the curvature tensor K_{jkh}^i, the tensor K_k^i and the scalar K are given by

$$\begin{aligned} &a)\ \ K_{jki}^i = K_{jk}, \\ &b)\ \ g^{jk} K_{jk} = K, \\ &c)\ \ g^{ij} K_{jk} = K_k^i. \end{aligned} \tag{19}$$

2. Generalized K^h-Birecurrent Finsler Space

Let us consider a Finsler space F_n whose Cartan's fourth curvature tensor K_{jkh}^i satisfies

$$K_{jkh|\ell}^i = \lambda_\ell K_{jkh}^i + \mu_\ell(\delta_k^i g_{jh} - \delta_h^i g_{jk}),\ K_{jkh}^i \neq 0, \tag{20}$$

where λ_ℓ and μ_ℓ are non-zero covariant vector fields and called the recurrence vector fields. Such space called it as a generalized K^h-recurrent Finsler space. Differentiating (20) covariantly with respect to x^m in the sense of Cartan and using (1e), we get

$$K_{jkh|\ell|m}^i = \lambda_{\ell|m} K_{jkh}^i + \lambda_\ell K_{jkh|m}^i + \mu_{\ell|m}(\delta_k^i g_{jh} - \delta_h^i g_{jk}). \tag{21}$$

Using (20) in (21) we get

$$K_{jkh|\ell|m}^i = (\lambda_{\ell|m} + \lambda_\ell \lambda_m) K_{jkh}^i + (\lambda_\ell \mu_m + \mu_{\ell|m})(\delta_k^i g_{jh} - \delta_h^i g_{jk}),$$

which can be written as

$$K_{jkh|\ell|m}^i = a_{\ell m} K_{jkh}^i + b_{\ell m}(\delta_k^i g_{jh} - \delta_h^i g_{jk}), K_{jkh}^i \neq 0, \tag{22}$$

Where $a_{\ell m} = \lambda_{\ell|m} + \lambda_\ell \lambda_m$ and $b_{\ell m} = \lambda_\ell \mu_m + \mu_{\ell|m}$ are non-zero covariant tensor fields of second order and called recurrence tensor fields.

Definition 2.1. *If Cartan's fourth curvature tensor K_{jkh}^i of a Finsler space satisfying the condition (22), where $a_{\ell m}$ and $b_{\ell m}$ are non-zero covariant tensor fields of second order, the space will be called generalized K^h- birecurrent Finsler space, we shall denote such space briefly by $GK^h - BR - F_n$.*

However, if we start from condition (22), we cannot obtain the condition (20), we may conclude

Theorem 2.2. *Every generalized K^h- recurrent Finsler space is generalized K^h- birecurrent Finsler space, but the converse need not be true.*

Transvecting (22) by the metric tensor g_{ir}, using (1e) and (16), we get

$$K_{jrkh|\ell|m} = a_{\ell m} K_{jrkh} + b_{\ell m} \left(g_{kr}\ g_{jh} - g_{hr}\ g_{jk}\right). \tag{23}$$

Transvecting (22) by y^j, using (1d) and (7) we get

$$H^i_{kh|\ell|m} = a_{\ell m} H^i_{kh} + b_{\ell m} \left(\delta^i_k y_h - \delta^i_h y_k\right). \tag{24}$$

Further transvecting (24) by y^k, using (1d), (6) and (1a), we get

$$H^i_{h|\ell|m} = a_{\ell m} H^i_h + b_{\ell m} \left(y^i y_h - \delta^i_h F^2\right) \tag{25}$$

Thus we have

Theorem 2.3. *In $GK^h - BR - F_n$, the associate tensor K_{jrkh} of Cartan's fourth curvature tensor K^i_{jkh}, the torsion tensor H^i_{kh} and the deviation tensor H^i_h are non- vanishing.*

Contracting the indices i and h in equations (22), (24) and (25), using (19a), (9), (10) and (1a), we get

$$K_{jk|\ell|m} = a_{\ell m} K_{jk} + (1-n) b_{\ell m} g_{jk}. \tag{26}$$

$$H_{k|\ell|m} = a_{\ell m} H_k + (1-n) b_{\ell m} y_k. \tag{27}$$

$$H_{|\ell|m} = a_{\ell m} H - b_{\ell m} F^2. \tag{28}$$

Transvecting (26) by g^{ij}, using (1f), (19c), we get

$$K^i_{k|\ell|m} = a_{\ell m} K^i_k + (1-n) b_{\ell m} \delta^i_k. \tag{29}$$

Transvecting (26) by g^{jk}, using (1f) and (19b), we get

$$K_{|\ell|m} = a_{\ell m} K + (1-n) b_{\ell m}. \tag{30}$$

Thus, we conclude

Theorem 2.4. *In $GK^h - BR - F_n$, the Ricci tensor K_{jk}, the curvature vector H_k, the scalar curvature H the deviation tensor K^i_k and the scalar curvature tensor K are non- vanishing.*

Differentiating (24) partially with respect to y^j, using (5) and (1b), we get

$$\dot\partial_j \left(H^i_{kh|\ell|m}\right) = \left(\dot\partial_j a_{\ell m}\right) H^i_{kh} + a_{\ell m} H^i_{jkh} + \left(\dot\partial_j b_{\ell m}\right)\left(\delta^i_k y_h - \delta^i_h y_k\right) + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}). \tag{31}$$

Using commutation formula exhibited by (1.3a) for $(H^i_{kh|\ell})$ in (31), we get

$$\begin{aligned}\left\{\dot\partial_j \left(H^i_{kh|\ell}\right)\right\}_{|m} + H^r_{kh|\ell}\left(\dot\partial_j \Gamma^{*i}_{rm}\right) - H^i_{rh|\ell}\left(\dot\partial_j \Gamma^{*r}_{km}\right) - H^i_{rk|\ell}\left(\dot\partial_j \Gamma^{*r}_{hm}\right) - H^r_{kh|r}\left(\dot\partial_j \Gamma^{*i}_{m\ell}\right) - \dot\partial_r\left(H^i_{kh|\ell}\right) P^r_{jm} \\ = \left(\dot\partial_j a_{\ell m}\right) H^i_{kh} + a_{\ell m} H^i_{jkh} + \left(\dot\partial_j b_{\ell m}\right)\left(\delta^i_k y_h - \delta^i_h y_k\right) + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).\end{aligned} \tag{32}$$

Again applying the commutation formula exhibited by (1.3a) for (H^i_{kh}) in (32) and using (5), we get

$$\begin{aligned}&\left\{H^i_{jkh|\ell}+H^r_{kh}\left(\dot\partial_j\Gamma^{*i}_{r\ell}\right)-H^i_{rh}\left(\dot\partial_j\Gamma^{*r}_{k\ell}\right)-H^i_{rk}\left(\dot\partial_j\Gamma^{*r}_{h\ell}\right)-H^i_{rkh}P^r_{j\ell}\right\}_{|m}+H^r_{kh|\ell}\left(\dot\partial_j\Gamma^{*i}_{rm}\right)-H^i_{rh|\ell}\left(\dot\partial_j\Gamma^{*r}_{km}\right)-H^i_{rk|\ell}\left(\dot\partial_j\Gamma^{*r}_{hm}\right)\\&-H^i_{kh|r}\left(\dot\partial_j\Gamma^{*r}_{m\ell}\right)-\left\{H^i_{rkh|\ell}+H^s_{kh}\left(\dot\partial_r\Gamma^{*i}_{s\ell}\right)-H^i_{sh}\left(\dot\partial_r\Gamma^{*s}_{k\ell}\right)-H^i_{sk}\left(\dot\partial_r\Gamma^{*s}_{h\ell}\right)-H^i_{skh}P^s_{r\ell}\right\}P^r_{jm}\\&=\left(\dot\partial_j a_{\ell m}\right)H^i_{kh}+a_{\ell m}H^i_{jkh}+\left(\dot\partial_j b_{\ell m}\right)\left(\delta^i_k y_h-\ \delta^i_h y_k\right)+b_{\ell m}(\delta^i_k g_{jh}-\delta^i_h g_{jk}).\end{aligned}\tag{33}$$

This shows that

$$H^i_{jkh|\ell|m}=a_{\ell m}H^i_{jkh}+b_{\ell m}(\delta^i_k g_{jh}-\delta^i_h g_{jk}).\tag{34}$$

if and only if

$$\begin{aligned}&\left\{H^r_{kh}\left(\dot\partial_j\Gamma^{*i}_{r\ell}\right)-H^i_{rh}\left(\dot\partial_j\Gamma^{*r}_{k\ell}\right)-H^i_{rk}\left(\dot\partial_j\Gamma^{*r}_{h\ell}\right)-H^i_{rkh}P^r_{j\ell}\right\}_{|m}+H^r_{kh|\ell}\left(\dot\partial_j\Gamma^{*i}_{rm}\right)-H^i_{rh|\ell}\left(\dot\partial_j\Gamma^{*r}_{km}\right)\\&-H^i_{rk|\ell}\left(\dot\partial_j\Gamma^{*r}_{hm}\right)-H^i_{kh|r}\left(\dot\partial_j\Gamma^{*r}_{m\ell}\right)-\left\{H^i_{rkh|\ell}+H^s_{kh}\left(\dot\partial_r\Gamma^{*i}_{s\ell}\right)-H^i_{sh}\left(\dot\partial_r\Gamma^{*s}_{k\ell}\right)-H^i_{sk}\left(\dot\partial_r\Gamma^{*s}_{h\ell}\right)-H^i_{skh}P^s_{r\ell}\right\}P^r_{jm}\\&=\left(\dot\partial_j a_{\ell m}\right)H^i_{kh}+\left(\dot\partial_j b_{\ell m}\right)\left(\delta^i_k y_h-\ \delta^i_h y_k\right).\end{aligned}\tag{35}$$

Contracting the i and h in (33) and using (8), we get

$$\begin{aligned}&H_{jk|\ell|m}+\left\{H^r_{kp}\left(\dot\partial_j\Gamma^{*p}_{r\ell}\right)-H_r\left(\dot\partial_j\Gamma^{*r}_{k\ell}\right)-H^p_{rk}\left(\dot\partial_j\Gamma^{*r}_{p\ell}\right)-H_{rk}P^r_{j\ell}\right\}_{|m}+H^r_{kp|\ell}\left(\dot\partial_j\Gamma^{*p}_{rm}\right)-H_{r|\ell}\left(\dot\partial_j\Gamma^{*r}_{km}\right)\\&-H^p_{rk|\ell}\left(\dot\partial_j\Gamma^{*r}_{pm}\right)-\quad H_{k|r}\left(\dot\partial_j\Gamma^{*r}_{m\ell}\right)-\left\{H_{rk|\ell}+H^s_{kp}\left(\dot\partial_r\Gamma^{*p}_{s\ell}\right)-H_s\left(\dot\partial_r\Gamma^{*s}_{k\ell}\right)-H^p_{sk}\left(\dot\partial_r\Gamma^{*s}_{p\ell}\right)-H_{sk}P^s_{r\ell}\right\}P^r_{jm}\\&=\left(\dot\partial_j a_{\ell m}\right)H_k+a_{\ell m}H_{jk}+(1-n)\left(\dot\partial_j b_{\ell m}\right)y_k+(1-n)\,d_{\ell m}g_{jk}.\end{aligned}\tag{36}$$

This shows that

$$H_{jk|\ell|m}=\ a_{\ell m}H_{jk}+(1-n)\,d_{\ell m}g_{jk}.\tag{37}$$

if and only if

$$\begin{aligned}&\left\{H^r_{kp}\left(\dot\partial_j\Gamma^{*p}_{r\ell}\right)-H_r\left(\dot\partial_j\Gamma^{*r}_{k\ell}\right)-H^p_{rk}\left(\dot\partial_j\Gamma^{*r}_{p\ell}\right)-H_{rk}P^r_{j\ell}\right\}_{|m}+H^r_{kp|\ell}\left(\dot\partial_j\Gamma^{*p}_{rm}\right)-H_{r|\ell}\left(\dot\partial_j\Gamma^{*r}_{km}\right)\\&-H^p_{rk|\ell}\left(\dot\partial_j\Gamma^{*r}_{pm}\right)-\quad H_{k|r}\left(\dot\partial_j\Gamma^{*r}_{m\ell}\right)-\left\{H_{rk|\ell}+H^s_{kp}\left(\dot\partial_r\Gamma^{*p}_{s\ell}\right)-H_s\left(\dot\partial_r\Gamma^{*s}_{k\ell}\right)-H^p_{sk}\left(\dot\partial_r\Gamma^{*s}_{p\ell}\right)-H_{sk}P^s_{r\ell}\right\}P^r_{jm}\\&=\left(\dot\partial_j a_{\ell m}\right)H_k+(1-n)\left(\dot\partial_j b_{\ell m}\right)y_k.\end{aligned}\tag{38}$$

Thus, we have

Theorem 2.5. *In GK^h-BR-F_n, Berwald curvature tensor H^i_{jkh} and Ricci curvature tensor H_{jk} are non-vanishing if and only if conditions (35) and (38) hold, respectively.*

Differentiating (27) partially with respect to y^j, using (**??**) and (1b), we get

$$\dot\partial_j\left(H_{k|\ell|m}\right)=\left(\dot\partial_j a_{\ell m}\right)H_k+a_{\ell m}H_{jk}+(1-n)\left(\dot\partial_j b_{\ell m}\right)y_k+(1-n)b_{\ell m}\ g_{jk}\ .\tag{39}$$

Using the commutation formula exhibited by (1. 3a) for $(H_{k|\ell})$ and using (12), we get

$$\begin{aligned}\left(\dot\partial_j H_{k|\ell}\right)_{|m}-H_{r|\ell}\left(\dot\partial_j\Gamma^{*r}_{km}\right)-H_{k|r}\left(\dot\partial_j\Gamma^{*r}_{\ell m}\right)-\left(\dot\partial_r H_{k|\ell}\right)P^r_{jm}=\left(\dot\partial_j a_{\ell m}\right)H_k+a_{\ell m}H_{jk}+(1-n)\left(\dot\partial_j b_{\ell m}\right)y_k\\(1-n)b_{\ell m}g_{jk}.\end{aligned}\tag{40}$$

Again using commutation formula exhibited by (3a) for (H_k) in (40), we get

$$\left\{(\dot\partial_j H_k)_{|\ell} - H_r\left(\dot\partial_j \Gamma^{*r}_{\ell k}\right) - (\dot\partial_r H_k)P^r_{j\ell}\right\}_{|m} - H_{r|\ell}\left(\dot\partial_j \Gamma^{*r}_{km}\right) - H_{k|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right) - \left\{(\dot\partial_r H_k)_{|\ell} - H_s\left(\dot\partial_r \Gamma^{*s}_{\ell k}\right) - (\dot\partial_s H_k)P^s_{r\ell}\right\} P^r_{jm}$$
$$= \left(\dot\partial_j a_{\ell m}\right) H_k + \quad a_{\ell m} H_{jk} + (1-n)\ \left(\dot\partial_j b_{\ell m}\right) y_k + (1-n) b_{\ell m} g_{jk}\ . \tag{41}$$

Using (12) and (37) in (41), we get

$$\left\{-H_r\left(\dot\partial_j \Gamma^{*r}_{\ell k}\right) - (H_{kr})P^r_{j\ell}\right\}_{|m} - H_{r|\ell}\left(\dot\partial_j \Gamma^{*r}_{km}\right) - H_{k|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right)$$
$$- \left\{H_{kr|\ell} - H_s\left(\dot\partial_r \Gamma^{*s}_{\ell k}\right) - H_{ks}P^s_{r\ell}\right\} P^r_{jm} = \left(\dot\partial_j a_{\ell m}\right) H_k + (1-n)\ \left(\dot\partial_j b_{\ell m}\right) y_k. \tag{42}$$

Transvecting (42) by y^k, using (1d), (**??**), (3b) and (1a), we get

$$-2H_{r|\ell}P^r_{jm} - (n-1)\, H_{|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right) \ = \ (n-1)\left(\dot\partial_j a_{\ell m}\right) H - (n-1)\left(\dot\partial_j b_{\ell m}\right) F^2.$$

Which can be written as

$$(\dot\partial_j b_{\ell m}) = \frac{\left(\dot\partial_j a_{\ell m}\right) H}{F^2}. \tag{43}$$

if and only if

$$-2H_{r|\ell}P^r_{jm} - (n-1)\, H_{|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right) \ = 0. \tag{44}$$

If the tensor $a_{\ell m}$ is independent of y^i, the equation (43) shows that the tensor $b_{\ell m}$ is also independent of y^i. Conversely, if the tensor $b_{\ell m}$ is independent of y^i, we get $H\dot\partial_j a_{\ell m} = 0$. In view of Theorem 2.3, the condition $H\dot\partial_j a_{\ell m} = 0$ implies $\dot\partial_j a_{\ell m} = 0$, i.e. the covariant tensor $a_{\ell m}$ is also independent of y^i. This leads to

Theorem 2.6. *The covariant tensor $b_{\ell m}$ is independent of the directional arguments if the covariant tensor $a_{\ell m}$ is independent of directional arguments if and only if conditions* (44) *and* (38) *hold.*

Suppose the tensor $a_{\ell m}$ is not independent of y^i, then (42) and (43) together imply

$$\left\{-H_r\left(\dot\partial_j \Gamma^{*r}_{\ell k}\right) - (H_{kr})P^r_{j\ell}\right\}_{|m} - H_{r|\ell}\left(\dot\partial_j \Gamma^{*r}_{km}\right) - H_{k|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right) - \left\{H_{kr|\ell} - H_s\left(\dot\partial_r \Gamma^{*s}_{\ell k}\right) - H_{ks}P^s_{r\ell}\right\} P^r_{jm}$$
$$= \left(\dot\partial_j a_{\ell m}\right)\left[H_k - \frac{(n-1)}{F^2} H y_k\right]. \tag{45}$$

Transvecting (45) by y^m and using (1d), (3c) and (3d), we get

$$\left\{-H_r\left(\dot\partial_j \Gamma^{*r}_{\ell k}\right) - (H_{kr})P^r_{j\ell}\right\}_{|m} y^m = \left(\dot\partial_j a_\ell - a_{j\ell}\right)(H_k - \frac{(n-1)}{F^2} H y_k). \tag{46}$$

where $a_{\ell m} y^m = \ a_\ell$, if

$$\left\{-H_r\left(\dot\partial_j \Gamma^{*r}_{\ell k}\right) - (H_{kr})P^r_{j\ell}\right\}_{|m} y^m = 0, \tag{47}$$

Equation (46) implies at least one of the following conditions

$$a)\ \ a_{j\ell} = \dot\partial_j a_\ell, \quad b)\ \ H_k = \frac{(n-1)}{F^2} H y_k \tag{48}$$

Thus, we have

Theorem 2.7. *In $GK^h - BR - F_n$ for which the covariant tensor $a_{\ell m}$ is not independent of the directional arguments and if conditions (47) and (38), (44) hold, at least one of the conditions (48a) and (48b) hold.*

Suppose (48b) holds equation (45) implies

$$\left\{-\frac{(n-1)}{F^2}Hy_r\dot{\partial}_j\Gamma^{*r}_{\ell k} - H_{kr}P^r_{j\ell}\right\}_{|m} - \left\{\frac{(n-1)}{F^2}Hy_r\right\}_{|\ell}\dot{\partial}_j\Gamma^{*r}_{km} \tag{49}$$

$$-\left\{\frac{(n-1)}{F^2}Hy_k\right\}_{|r}\dot{\partial}_j\Gamma^{*r}_{\ell m} - \quad H_{kr|\ell}P^r_{jm} - \frac{(n-1)}{F^2}Hy_s\left(\dot{\partial}_r\Gamma^{*s}_{\ell k}\right)P^r_{jm} - H_{ks}P^s_{r\ell}P^r_{jm} = 0.$$

Transvecting (49) by y^j, using (1d), (3b) and (3d), we get

$$\left\{\frac{(n-1)}{F^2}Hy_r\mathrm{P}^r_{\ell k}\right\}_{|m} + \left\{\frac{(n-1)}{F^2}Hy_r\right\}_{|\ell}P^r_{km} + \left\{\frac{(n-1)}{F^2}Hy_k\right\}_{|r}\mathrm{P}^r_{\ell m} = 0. \tag{50}$$

Thus, we have

Theorem 2.8. *In $GK^h - BR - F_n$, we have the identity (50) provided (48b).*

Transvecting (50) by the metric tensor g_{rj}, using (1e) and (3e), we get

$$\left\{\frac{(n-1)}{F^2}Hy_rP_{j\ell k}\right\}_{|m} + \left\{\frac{(n-1)}{F^2}Hy_r\right\}_{|\ell}P_{jkm} + \left\{\frac{(n-1)}{F^2}Hy_k\right\}_{|r}P_{j\ell m} = 0. \tag{51}$$

By using (1c), equation (51) can be written as

$$y_r(HP_{j\ell k})_{|m} + y_rH_{|\ell}P_{jkm} + y_kH_{|r}P_{j\ell m} = 0.$$

In view of Theorem 2.3, we have

$$P_{j\ell m} = 0. \tag{52}$$

if and only if

$$y_r(HP_{j\ell k})_{|m} + y_rH_{|\ell}P_{jkm} = 0. \tag{53}$$

Therefore the space is Landsberg space. Thus, we have

Theorem 2.9. *An $GK^h - BR - F_n$ is Landsberg space if and only if conditions (53), (48b), (38) and (44) hold good.*

If the covariant tensor $a_{j\ell} \neq \dot{\partial}_j a_\ell$, in view of Theorem 2.6, (48b) holds good. In view of this fact, we may rewrite Theorem 2.8 in the following form

Theorem 2.10. *An $GK^h - BR - F_n$ is necessarily Landsberg space if and only if conditions (53), (38), (44) and (48b) hold good and provided $a_{j\ell} \neq \dot{\partial}_j a_\ell$.*

Using (34) in (33), we get

$$\begin{aligned}
&\left\{H^r_{kh}\left(\dot{\partial}_j\Gamma^{*i}_{r\ell}\right) - H^i_{rh}\left(\dot{\partial}_j\Gamma^{*r}_{k\ell}\right) - H^i_{rk}\left(\dot{\partial}_j\Gamma^{*r}_{h\ell}\right) - H^i_{rkh}P^r_{j\ell}\right\}_{|m} + H^r_{kh|\ell}\left(\dot{\partial}_j\Gamma^{*i}_{rm}\right) - H^i_{rh|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{km}\right)\\
&- H^i_{rk|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{hm}\right) - \quad H^i_{kh|r}\left(\dot{\partial}_j\Gamma^{*r}_{m\ell}\right) - \left\{H^i_{rkh|\ell} + H^s_{kh}\left(\dot{\partial}_r\Gamma^{*i}_{s\ell}\right) - H^i_{sh}\left(\dot{\partial}_r\Gamma^{*s}_{k\ell}\right) - H^i_{sk}\left(\dot{\partial}_r\Gamma^{*s}_{h\ell}\right) - H^i_{skh}P^s_{r\ell}\right\}P^r_{jm}\\
&= \left(\dot{\partial}_ja_{\ell m}\right)H^i_{kh} + \left(\dot{\partial}_jb_{\ell m}\right)\left(\delta^i_ky_h - \ \delta^i_hy_k\right).
\end{aligned} \tag{54}$$

Transvecting (54) by y^k , using (1d), (1a), (3b), (4) and (6), we get

$$\left\{H_h^r\left(\dot{\partial}_j\Gamma_{r\ell}^{*i}\right)-H_r^i\left(\dot{\partial}_j\Gamma_{h\ell}^{*r}\right)-2H_{rh}^iP_{j\ell}^r\right\}_{|m}+H_{h|\ell}^r\left(\dot{\partial}_j\Gamma_{rm}^{*i}\right)-H_{rh|\ell}^i\left(\mathrm{P}_{jm}^r\right)-H_{r|\ell}^i\left(\dot{\partial}_j\Gamma_{hm}^{*r}\right)-\ H_{h|r}^i\left(\dot{\partial}_j\Gamma_{m\ell}^{*r}\right)$$
$$-\{H_{rh|\ell}^i+H_h^s\left(\dot{\partial}_r\Gamma_{s\ell}^{*i}\right)-H_s^i\left(\dot{\partial}_r\Gamma_{h\ell}^{*s}\right)-2H_{sh}^iP_{r\ell}^s\}P_{jm}^r=\left(\dot{\partial}_ja_{\ell m}\right)H_h^i+\left(\dot{\partial}_jb_{\ell m}\right)\left(y^iy_h-\delta_h^iF^2\ \right). \tag{55}$$

Substituting the value of $\dot{\partial}_jb_{\ell m}$ from (43) , in (55), we get

$$\left\{H_h^r\left(\dot{\partial}_j\Gamma_{r\ell}^{*i}\right)-H_r^i\left(\dot{\partial}_j\Gamma_{h\ell}^{*r}\right)-2H_{rh}^iP_{j\ell}^r\right\}_{|m}+H_{h|\ell}^r\left(\dot{\partial}_j\Gamma_{rm}^{*i}\right)-H_{rh|\ell}^i\left(\mathrm{P}_{jm}^r\right)-H_{r|\ell}^i\left(\dot{\partial}_j\Gamma_{hm}^{*r}\right)-\ H_{h|r}^i\left(\dot{\partial}_j\Gamma_{m\ell}^{*r}\right)$$
$$-\{H_{rh|\ell}^i+H_h^s\left(\dot{\partial}_r\Gamma_{s\ell}^{*i}\right)-H_s^i\left(\dot{\partial}_r\Gamma_{h\ell}^{*s}\right)-2H_{sh}^iP_{r\ell}^s\}P_{jm}^r\ =\ \left(\dot{\partial}_ja_{\ell m}\right)[H_h^i-H\left(\delta_h^i-\imath^{\ i}\imath_h\right)]. \tag{56}$$

if

$$\left\{H_h^r\left(\dot{\partial}_j\Gamma_{r\ell}^{*i}\right)-H_r^i\left(\dot{\partial}_j\Gamma_{h\ell}^{*r}\right)-2H_{rh}^iP_{j\ell}^r\right\}_{|m}+H_{h|\ell}^r\left(\dot{\partial}_j\Gamma_{rm}^{*i}\right)-H_{rh|\ell}^i\left(\mathrm{P}_{jm}^r\right)$$
$$-H_{r|\ell}^i\left(\dot{\partial}_j\Gamma_{hm}^{*r}\right)-\ H_{h|r}^i\left(\dot{\partial}_j\Gamma_{m\ell}^{*r}\right)-\{H_{rh|\ell}^i+H_h^s\left(\dot{\partial}_r\Gamma_{s\ell}^{*i}\right)-H_s^i\left(\dot{\partial}_r\Gamma_{h\ell}^{*s}\right)-2H_{sh}^iP_{r\ell}^s\}P_{jm}^r=0. \tag{57}$$

We have at least one of the following conditions :

$$a)\quad\left(\dot{\partial}_ja_{\ell m}\right)=0,\qquad b)\ \ H_h^i=H\left(\delta_h^i-\imath^i\imath_h\right). \tag{58}$$

Putting $H=F^2R$, the equation (57b) may be written as

$$H_h^i=F^2R\left(\delta_h^i-\imath^i\imath_h\right), \tag{59}$$

where $R\neq 0$. Therefore the space is a Finsler space of scalar curvature. Thus, we have

Theorem 2.11. *An GK^h-BR-F_n for $n>2$ admitting equation (57) holds is a Finsler space of scalar curvature provided $R\neq 0$, the covariant tensor $a_{\ell m}$ is not independent of directional arguments and condition (35) holds .*

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