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On Some Fixed Point Results in Metric Space

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Abstract: In this paper, we discuss about fixed point theory in metric space, Hausdorff metric space under contraction conditions and its generalized forms are important as an extension of famous Banach Contraction Principles and for its applications to other disciplines.

Keywords: Hausdorff metric, Nadler contraction, Nadler's Fixed Point.

1. Introduction

Historically, the concept of fixed point was initiated by H. Poincare in 1886. The concept of metric space was introduced by M. Frechet in 1906 which furnished the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of distance appears. The first fixed point theorem is due to L. E. J. Brower in 1912. Fixed point theory has played an important role in the problems of non-linear functional analysis which is the blend of analysis, topology and algebra. A fixed point theorem is one which ensures the existence of a fixed point of a mapping T under suitable assumptions both on X and T. As has been already pointed out, many non-linear equations can be solved using fixed point theorems. In fact, fixed point theorems have applications in nonlinear integral, differential equations, game theory, optimization theory and boundary value problems etc. In this article, we provide brief introduction on chronological development of fixed point theory in metric space and its generalized forms.

2. Main Results

In 1979, Nadler take some initiative to investigate on the existence of fixed point theory for some multivalued contraction mappings in the metric spaces and using the concept of Hausdorff metric, he established multivalued version of the Banach contraction principle. Let (X, d) be a metric space. We denote by 2^X the collection of all nonempty subsets of X, $Cl(X)$ the collection of all nonempty closed subsets of X, $CB(X)$ the collection of all nonempty closed bounded subsets of X,and H the Hausdorff metric [7] on $CB(X)$, that is, $H(A,B) = \max\left\{\sup\limits_{y\in A} d\left(x,B\right), \sup\limits_{y\in B} d(y,A)\right\}$, for all $A, B \in CB(X)$, where $d(x,A) = \sup\limits_{y\in A} d(x,y)$. In the metric space $(CB(X), H)$, $\lim\limits_{n\to\infty} A_n = A$, means that $\lim\limits_{n\to\infty} H(A_n, A) = 0$. Let $A_1, A_2 \in CB(X)$. Then, for each $x \in A_1$ and $\varepsilon > 0$, there is $y \in A_2$ such that $d(x,y) \leq H(A_1, A_2) + \varepsilon$.

Example 2.1 ([2]). *Let $X = R$, $A = [0,1]$, $B = [2,4]$. Then, $\sup\limits_{a\in A} d\left(a,B\right) = 2, \sup\limits_{b\in B} d\left(b,A\right) = 3$, and $H\left(A,B\right) = 3$*

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In 1969, using the concept of Hausdorff metric, Nadler proved the following theorem on the existence of fixed points for multivalued mappings, known as Nadler Contraction Principle (NCP).

Theorem 2.2 ([12]). *Let (X, d) be a complete metric space. Then, each contraction mapping $T : X \to CB(X)$ has a fixed point.*

In 1989, Mizoguchi and Takahashi generalized Nadler's fixed point theorem as follows;

Theorem 2.3 ([11]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a mapping such that, for each $x, y \in X$, $H(T(x), T(y)) \leq k(d(x, y))d(x, y)$, where k is a function from $[0, \infty)$ to $[0, 1)$, satisfying $\lim_{r \to t^+} \sup k(r) < 1$, for every $t \geq 0$. Then, T has a fixed point.*

Remark 2.4 ([2]). *In the original statement, the domain of the function k is $(0, \infty)$. However, both are equivalent because $d(x, y) = 0 \Rightarrow H(T(x), T(y)) = 0$. Also, it is noted that the stronger condition assumed on k implies that $k(t) < h$ for some $0 < h < 1$. Thus with this condition, one may get that the map T is a contraction over a region for which $d(x, y)$ is sufficiently small.*

In 2008, Kikkawaa and Suzuki generalized the Nadles's fixed point theorem as follows:

Theorem 2.5 ([10]). *Define a strictly increasing function η from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by $\eta(h) = \frac{1}{1+h}$, let (X, d) be a complete metric space and let $T : X \to CB(X)$. Assume that there exists a fixed $h \in [0, 1)$ such that $\eta(h)\, d(x, T(x)) \leq d(x, y) \Rightarrow H(T(x), T(y)) \leq hd(x, y)$, for all $x, y \in X$. Then T has a fixed point.*

The Nadler's Theorem have been modified and generalized in successive years. But the nature of the Hausdorff metric is not used and the existence part of results can be proved without using the concept of a Hausdorff metric. For instance, recently, Feng and Liu obtained some interesting fixed point results for multivalued mappings and extended Nadler's result [2] as follows.

Theorem 2.6 ([6]). *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a mapping such that for any fixed constants $h, b \in (0, 1)$, $h < b$, and for each $x \in X$, there is $y \in T(x)$ satisfying the following conditions:*

(i) $bd(x, y) \leq d(x, T(x))$, and

(ii) $d(y, T(y)) \leq hd(x, y)$

Then, $Fix\,(T) \neq \emptyset$ provided the real-valued function g on X, $g(x) = d(x, T(x))$ is lower semicontinuous.

Remark 2.7 ([2]). *Above Theorem 1.4. generalizes Nadler's fixed point result. Indeed, if T satisfies the condition of Nadler's result, then the lower semicontinuity of function $f(x) = d(x, T(x))$ follows from the contraction condition. Further, since $T(x)$ is closed and bounded, so there exists $y \in T(x)$ such that $bd(x, y) \leq d(x, T(x))$, for $b \in (0, 1)$. Also, $d(y, T(y)) \leq H(T(x), T(y)) \leq hd(x, y)$, for $h \in (0, 1)$.*

Theorem 2.8. *Let (X, d) be a complete metric space and let p be a w-distance on X. Let $T : X \to Cl(X)$ be a mapping such that for a fixed constant $h \in [0, 1)$ and for any $x, y \in X$, $u \in T(x)$, there is $v \in T(y)$ satisfying $p(u, v) \leq hp(x, y)$. Then, there exists $x_0 \in X$ such that $x_0 \in T(x_0)$ and $p(x_0, x_0) = 0$.*

Theorem 2.9 ([14]). *Let (X, d) be a complete metric space and let p be a τ-distance on X. Let $T : X \to Cl(X)$ be a mapping such that for a fixed constant $h \in [0, 1)$ and for any $x, y \in X$, $u \in T(x)$, there is $v \in T(y)$ satisfying $p(u, v) \leq hp(x, y)$. Then, T has a fixed point.*

Theorem 2.10 ([5]). *Let (X, d) be a complete metric space and $T : X \to R \cup \{+\infty\}$ be a proper, bounded below and lower semicontinuous functional. Let $\varepsilon > 0$ and $\hat{x} \in X$ be given such that $T(\hat{x}) \leq \inf_X T + \varepsilon$. Then, for a given $\lambda > 0$, there exists $\overline{x} \in X$, such that*

(a) $T(\overline{x}) \leq T(\hat{x})$

(b) $d(\hat{x}, \overline{x}) \leq \lambda$

(c) $T(\overline{x}) < T(x) + \frac{\varepsilon}{\lambda} d(x, \overline{x})$, *for all* $x \in X \backslash \{\overline{x}\}$.

In 1990, Aubin and Frankowska established some form of Ekeland's variational principle.

Theorem 2.11 ([3]). *Let (X, d) be a complete metric space and $T : X \to R \cup \{+\infty\}$ be a proper, bounded below and lower semicontinuous functional. Let $\hat{x} \in dom(T)$ and $e > 0$ be fixed. Then, there exists $\overline{x} \in X$ such that*

(a) $T(\overline{x}) - T(\hat{x}) + ed(\hat{x}, \overline{x}) \leq 0$

(b) $T(\overline{x}) < T(x) + ed(x, \overline{x})$,

for all $x \in X \backslash \{\overline{x}\}$.

The Converse of Ekeland's Variational Principle as follows:

Theorem 2.12. *A metric space (X, d) is complete if for every functional $T : X \to R \cup \{+\infty\}$ which is proper, bounded below and lower semicontinuous on X and for every given $\varepsilon > 0$, there exists $\overline{x} \in X$ such that $T(\overline{x}) \leq \inf_X T + \varepsilon$, and $T(\overline{x}) \leq T(x) + \varepsilon d(x, \overline{x})$, for all $x \in X$.*

Using Ekeland's variational principle, Mizoguchi and Takahashi derived the following Caristi and Kirk's theorem:

Theorem 2.13 ([11]). *Let (X, d) be a complete metric space and $T : X \to X$ be a set-valued map with nonempty values such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$d(x, y) + \varphi(y) \leq \varphi(x),$$

where $\varphi : X \to R \cup \{+\infty\}$ is a proper, lower semi continuous and bounded below functional. Then, T has a fixed point, that is, there exists $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$.

Using Ekeland's variational principle, Mizoguchi and Takahashi derived a theorem, known as Clarke's fixed point theorem as follows:

Theorem 2.14 ([11]). *Let (X, d) be a complete metric space and $T : X \to X$ be a directional contraction mapping. Then, T has a fixed point.*

Theorem 2.15 (Takahashi's Minimization Theorem [2]). *Let X be a complete metric space and $T : X \to R \cup \{+\infty\}$ be a proper, bounded below and lower semi continuous functional. Suppose that, for each $\hat{x} \in X$ with $\inf_X T(x) < T(\hat{x})$, there exists $z \in X$, such that $z \neq \hat{x}$ and $T(z) + d(\hat{x}, z) \leq T(\hat{x})$. Then, there exists $\overline{x} \in X$ such that $T(\overline{x}) = \inf_{X \in X} T(x)$, that is, $\overline{x}$ is a solution of optimization problem.*

In 1991, Takahashi characterizes in metric space X as follows:

Theorem 2.16 ([15])**.** *A metric space X is complete if for every uniformly continuous function $T : X \to R \cup \{+\infty\}$ and every $\hat{x} \in X$ with $\inf_X T < T(\hat{x})$, there exists $z \in X$ such that $z \neq \hat{x}$ and $T(z) + d(\hat{x}, z) \leq T(\hat{x})$, then, there exists $\overline{x} \in X$ such that $T(\overline{x}) = \inf_{x \in X} T(x)$.*

Daffer, Kaneko and Li and Hamel establish an application of Takahashi's Minimization Theorem to prove the existence of weak sharp minima for a class of lower semi continuous [2] functions as, let (X, d) be a complete metric space and $T : X \to R \cup \{+\infty\}$ be a lower semicontinuous function. We define

$$m = \inf\{T(x) : x \in X\} \ \ and \ \ M = \{y \in X : T(y) = m\} \tag{1}$$

Then, T is said to have weak sharp minima if $d(x, M) \leq T(x) - m$, For all $x \in X$.

Theorem 2.17. *Let (X, d) be a complete metric space and $T : X \to R \cup \{+\infty\}$ be a proper, bounded below and lower semicontinuous functional. Suppose that, for each $\hat{x} \in X$ with $\inf_X T < T(\hat{x})$, there exists $z \in X$ such that, $z \neq \hat{x}$ and $d(\hat{x}, z) \leq T(\hat{x}) - T(z)$. Then, M as defined above relation (1) is nonempty and T has weak sharp minima.*

The Ekeland's variational principle used to study the existence of a solution of Ekeland's Principle as follows:

Theorem 2.18 ([2])**.** *Let K be a nonempty closed subset of a complete metric space (X, d) and $T : K \times K \to R$ be a bifunction. Assume that $\varepsilon > 0$ and the following assumptions are satisfied:*

(1) For all $x \in K, L = \{y \in K : T(x, y) + \varepsilon d(x, y) \leq 0\}$ is closed;

(2) $T(x, x) = 0$, for all $x \in K$;

(3) $T(x, y) \leq T(x, z) + T(z, y)$, for all $x, y, z \in K$.

If $\inf_{y \in K} T(x_0, y) \in KT(x_0, y) > -\infty$ for some $x_0 \in K$, then there exists $\overline{x} \in K$ such that

(a) $T(x_0, \overline{x}) + \varepsilon d(x_0, \overline{x}) \leq 0$,

(b) $T(\overline{x}, x) + \varepsilon d(\overline{x}, x) > 0$, for all $x \in K, x \neq \overline{x}$.

Theorem 2.19 ([1])**.** *Let (X, d) be a complete metric space. Let $T : X \times X \to R \cup \{+\infty\}$ be a bifunction such that it is lower semi continuous in the second argument and the following conditions hold:*

(1) $T(x, x) = 0$ for all $x \in X$;

(2) $T(x, y) \leq T(x, z) + T(z, y)$ for all $x, y, z \in X$.

Assume that there exists $\overline{x} \in X$ such that, $\inf_{x \in X} T(\overline{x}, x) > -\infty$. Let $\overline{S} = \{x \in X : T(\overline{x}, x) + d(\overline{x}, x) \leq 0\}$. From (1) it follows that $\hat{x} \in \hat{S} \neq 0)$. Then, the following statements are as

(a) Extended Ekeland's Variational Principle. There exists $\hat{x} \in \hat{S}$ such that $T(\overline{x}, x) + d(\overline{x}, x) > 0$, for all $x \neq \overline{x}$.

(b) Extended Takahashi's Minimization Theorem.

$$\text{Assume that} \begin{cases} \text{for every } \hat{x} \in \hat{S} \text{ with } \inf_{x \in X} T(\hat{x}, x) < 0 \text{ there exists} \\ x \in X \text{ such that } T(\hat{x}, x) + d(\hat{x}, x) \leq 0, \text{ for all } x \neq \hat{x}. \end{cases}$$

Then, there exists $\overline{x} \in \hat{S}$ such that $T(\overline{x}, x) \geq 0$ for all $x \in X$.

(c) *Caristi-Kirk Fixed Point Theorem*

Let $T : X \to X$ *be a set-valued map such that* $\begin{cases} \text{for every } \hat{x} \in \hat{S} \text{ there exists} \\ x \in T(\hat{x}) \text{ satisfying } T(\hat{x}, x) + d(\hat{x}, x) \leq 0, \end{cases}$

Then, there exists $\overline{x} \in \widehat{S}$ *such that* $\overline{x} \in T(\overline{x})$.

(d) *Oettli–Théra Theorem*

Let $D \subset X$ *have the property that* $\begin{cases} \text{for every } \hat{x} \in \hat{S} \backslash D \text{ there exists} \\ x \in X \text{ such that } T(\hat{x}, x) + d(\hat{x}, x) \leq 0, \text{ for all } x \neq \hat{x}. \end{cases}$

Then, there exists $\overline{x} \in \hat{S} \cap D$.

As an application of the Theorem 1.15 is as follows;

Theorem 2.20 ([2]). *Let* K *be a nonempty closed subset of a complete metric space* (X, d) *and* $T : K \times K \to R$ *be a bifunction. Assume that* $\varepsilon > 0$ *and the following assumptions are satisfied:*

(i). *For all* $x \in K, L = \{y \in K : T(x, y) + \varepsilon d(x, y) \leq 0\}$ *is closed,*

(ii). $T(x, x) = 0$, *for all* $x \in K$;

(iii). $T(x, y) \leq T(x, z) + T(z, y)$, *for all* $x, y, z \in K$;

(iv). *for every* $x \in K$ *with* $\inf_{y \in K} T(x, y) < 0$

There exists $z \in K$ *such that* $z \neq x$ *and* $T(x, z) + \varepsilon d(x, z) \leq 0$. *Then the solution set of Ekeland's Principals is nonempty, That is, there exists* $\overline{x} \in K$ *such that* $T(\overline{x}, y) \geq 0$, *for all* $y \in K$.

Theorem 2.21. *Let* K *be a nonempty closed subset of a complete metric space be* (X, d), $T : K \to R$ *a lower bounded function. If for every* $x \in K$ *with* $\inf_{y \in K} T(y) < T(x)$, *there exists* $z \in K$ *such that* $z \neq x$, *and* $T(z) + \varepsilon d(x, z) \leq T(x)$, *then there exists* $\overline{x} \in K$ *such that* $T(\overline{x}) \leq T(y)$ *for all* $y \in K$.

Theorem 2.22. *Let* K *be a nonempty compact subset of a metric space* (X, d), $T : K \times K \to R$ *be a bifunction and* $\{t_n\}$ *be a decreasing sequence of positive real numbers such that* $t_n \to 0$. *Assume that*

(i). $L = \{y \in K : T(x, y) + t_n d(x, y) \leq 0\}$ *is closed for every* $x \in K$ *and for all* $n \in N$,

(ii). $T(x, x) = 0$, *for all* $x \in K$,

(iii). $T(x, y) \leq T(x, z) + T(z, y)$, *for all* $x, y, z \in K$,

(iv). $U = \{y \in K : T(y, x) + t_n d(y, x)\}$ *is closed for every* $x \in K$ *and for all* $n \in N$.

If $\inf_{y \in K} T(x_0, y) > -\infty$ for some $x_0 \in K$, then the set of solutions of Ekeland's Principles is nonempty. If $X = R^n$, then Euclidean space and $c = 0$, then

Theorem 2.23 ([4]). *Let* K *be a nonempty closed subset of* (X, d) *and* $\{t_n\}$ *be a decreasing sequence of positive real numbers such that* $t_n \to 0$. *Suppose that* $T : K \times K \to R$ *satisfies conditions (i) - (iv) of above Theorem 1.19. If* $\inf_{y \in K} T(x_0, y) - \infty$ *for some* $x_0 \in K$ *and* $\exists B(c, r) : \forall \;\; x \in K \backslash K_r, \;\; \exists \; y \in K$ *satisfying* $d(y, c) < d(x, y)$ *and* $F(x, y) \leq 0$ *hold. Then the set of solutions of Ekeland's Principles is nonempty.*

Theorem 2.24 ([2]). *Let (X, d) be a complete metric space and let $\{D_n\}$ be a decreasing sequence $(D_{n+1} \subseteq D_n)$ of nonempty closed subsets of X such that the diameter of $D_n, \delta(D_n) \to 0$ as $n \to \infty$. Then, the intersection $\bigcap_{n=1}^{\infty}$ contains exactly one point.*

Theorem 2.25 ([1]). *Assume that (X, d) is complete and uniformly convex. Let $C \subset X$ be nonempty, convex and closed. Let $x \in X$ be such that $d(x, C) < \infty$. Then there exists a unique best approximate of x in C, that is, there exists a unique $x_0 \in C$ such that $d(x, x_0) = d(x, C)$.*

Recall that a hyperbolic metric space (X, d) is said to have the property (R).

Theorem 2.26 ([7]). *If (X, d) is complete and uniformly convex, then (X, d) has the property (R).*

Lemma 2.27 ([9]). *Let (X, d) be a uniformly convex metric space. Assume that there exists $R \in [0, +\infty)$ such that $\lim_{n\to\infty} d(x_n, a) \le R$, $\lim_{n\to\infty} d(y_n, a) \le R$, and $\lim_{n\to\infty} d\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) = R$. Then, $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

A metric version of the parallelogram identity as follows;

Theorem 2.28 (Parallelogram Inequality [2, 9]). *Let (X, d) be uniformly convex. Fix $a \in X$. For each $0 < r$ and for each $\varepsilon > 0$ denote $\Psi(r, \varepsilon) = \inf\left\{\frac{1}{2}d^2(a, x) + \frac{1}{2}d^2(a, y) - d^2\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right)\right\}$, Where the infimum is taken over all $x, y \in M$ Such that $d(a, x) \le r, d(a, y) \le r$, and $(x, y) \ge r\varepsilon$. Then, $\Psi(r, \varepsilon) > 0$ for any $0 < r$ and for each $\varepsilon > 0$. Moreover, for a fixed $r > 0$, we have*

(a). $\Psi(r, 0) = 0$;

(b). $\Psi(r, \varepsilon)$ *is a nondecreasing function of* ε;

(c). If $\lim_{n\to\infty} \Psi(r, t_n) = 0$, *then* $\lim_{n\to\infty} t_n = 0$.

Theorem 2.29 ([9]). *Let (X, d) be hyperbolic metric space which is 2-uniformly convex. Let C be a non-empty, closed, convex and bounded subset of X. Let $T : C \to C$ be uniformly Lipschitzian with $\lambda(T) < \left(\frac{1+\sqrt{1+8c_M N(X)^2}}{2}\right)^{\frac{1}{2}}$. Then, T has a fixed point in C.*

Theorem 2.30 ([2]). *Let (X, d) be a nonempty bounded metric space which possesses a convexity structure which is compact and normal. Then, every non expansive mapping $T : X \to X$ has a fixed point.*

Theorem 2.31. *Let A be a nonempty admissible subset of Hilbert space l^∞. Then, every non expansive mapping $T : A \to A$ has a fixed point.*

We have the following Hahn–Banach theorem.

Theorem 2.32 (Hahn–Banach Theorem [2]). *Let X be a real vector space, Y be a linear subspace of X, and ρ a semi norm on X. Let T be a linear functional defined on Y such that $T(y) \le \rho(y)$, for all $y \in Y$. Then there exists a linear functional g defined on X, which is an extension of T (that is, $g(y) = T(y)$, for all $y \in Y$), which satisfies $g(x) \le \rho(x)$, for all $x \in X$.*

The Hahn–Banach extension theorem is closely related to an intersection property of the closed balls combined with some kind of metric convexity. Hyper convexity captures these ideas.

Theorem 2.33 ([1]). *Let $(M_\propto, d_\propto)_{\propto\in\Gamma}$ be a collection of hyperconvex metric spaces. Consider the product space $\mathcal{M} = \prod_{\propto\in\Gamma} M_\alpha$. Fix a $(a_\alpha) \in \mathcal{M}$ and consider the subset M of $\mathcal{M}$ defined by $M = \{x_\alpha\} \in \mathcal{M}; \sup_{\alpha\in\Gamma} d_\alpha(x_\alpha, a_\alpha) < \infty$. Then, (M, d_∞) is a hyperconvex metric space where d_∞ is defined by $d_\infty((x_\propto), (y_\propto)) = \sup_{\alpha\in\Gamma} d_\alpha(x_\alpha, y_\alpha)$, for any $(x_\alpha), (y_\alpha) \in M$.*

It is clear that hyperconvex metric spaces are complete. We have the following Theorem;

Theorem 2.34 ([2]). *Suppose A is a bounded subset of a hyperconvex metric space X. Set $cov(A) = \bigcap\{B : B \text{ is a ball and } B \supseteq A\}$. Then,*

(a). $cov(A) = \{B(x, r_x(A)) : x \in X\}$.

(b). $r_x\,(cov\,(A)) = r_x\,(A)$, for any $x \in X$.

(c). $R(cov(A)) = R(A) = \frac{1}{2}diam(A)$.

(d). $diam(cov(A)) = diam(A)$.

Theorem 2.35. *Let H be a metric space. The following statements are equivalent:*

(a). H is hyperconvex;

(b). H is injective.

This fundamental result may be stated in terms of retractions as follows:

Theorem 2.36. *Let H be a metric space. The following statements are equivalent:*

(a). H is hyperconvex;

(b). For every metric space M which contains H metrically, there exists a nonexpansive retraction $T : M \to H$;

(c). For any point ω not in H, there exists a non-expansive retraction $T : H \cup \{\omega\} \to H$.

Khamsi introduced a new concept called 1-local retract.

Theorem 2.37 ([8]). *Let T be a bounded metric space. Let $(H_\beta)_{\beta\in\Gamma}$ be a decreasing family of nonempty hyperconvex subsets of X, where Γ is totally ordered. Then $\bigcap\limits_{\beta\in\Gamma} H_\beta$.*

Is not empty and is hyperconvex. It is clear that hyperconvex hulls are not unique. But they do enjoy some kind of uniqueness.

Theorem 2.38. *Let H be a bounded hyperconvex metric space. Any non expansive map $T : H \to H$ has a fixed point. Moreover, the fixed point set of T, $Fix(T)$, is hyperconvex.*

Using Baillon's theorem, we get the following:

Theorem 2.39. *Let H be a bounded hyperconvex metric space. Any commuting family of non-expansive maps $\{T_i\}_{i\in I}$, with $T_i : H \to H$, has a common fixed point. Moreover, the common fixed point set $\bigcap\limits_{i\in I} Fix(T_i)$ is hyperconvex.*

In 2015, Panthi and Kumari established the following theorems;

Theorem 2.40 ([13]). *Let (X, d) be a complete metric space. Let $T : X \to X$ be continuous mapping satisfying the condition*

$$\begin{aligned} d(Tx,Ty) \le \alpha\frac{d(x,Tx)d(y,Ty)+d(x,Ty)d(y,Tx)}{d(x,y)} + \beta\frac{d(x,Ty)[d(x,Tx)+d(y,Ty)]}{d(x,y)+d(y,Ty)+d(y,Tx)} \\ + \gamma\frac{d(x,Tx)d(y,Tx)+d(y,Ty)d(x,Ty)}{d(x,Tx)+d(y,Tx)+d(y,Ty)+d(x,Ty)} + \kappa\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)} \\ + \delta[d(x,Tx)+d(y,Ty)] + \eta[d(y,Tx)+d(x,Ty] + \mu d(x,y) \end{aligned}$$

for all $x, y \in X$, $x \neq y$ and for $\alpha, \beta, \gamma, \kappa, \delta, \eta, \mu \in [0, 1)$ such that $2\alpha+2\kappa+4\delta+4\eta+2\mu < 2$, then T has a unique fixed point in X.

Remark 2.41 ([13]). *In Theorem 2.40, If*

(1). $\alpha = \beta = \gamma = \kappa = \delta = \eta = 0$ *then the theorem is reduced to Banach.*

(2). $\alpha = \beta = \gamma = \kappa = \eta = \mu = 0$ *then the theorem is reduced to Kannan.*

(3). $\alpha = \beta = \gamma = \kappa = \eta = 0$ *then the theorem is reduced to Chatterjee.*

(4). $\alpha = \beta = \gamma = \kappa = \delta = 0$ *then the theorem is reduced to Fisher.*

(5). $\alpha = \beta = \gamma = \kappa = 0$ *then the theorem is reduced to Riech.*

(6). $\alpha = \beta = \gamma = \delta = \eta = \mu = 0$ *then theorem is reduced to M. S. Khan.*

(7). $\kappa = 0$ *then the theorem is reduced to R. Bhardwas et.al.*

Theorem 2.42 ([13]). *Let T be a self mapping defined on a complete metric space (X, d) such that the first condition of Theorem 2.6 holds. If for some positive integer r, T^r is continuous then T has a unique fixed point in X.*

Theorem 2.43 ([13]). *Let S and T be mappings of a complete metric space (X, d) into itself. Suppose that there exists a non negative real number α and β such that $\alpha + 2\beta < 1$ and*

$$d(Tx, Sy) \leq \alpha \frac{d(x, Tx)d(x, Sy) + d(y, Sy)d(y, Tx)}{d(x, Sy) + d(y, Tx)} + \beta \max\{d(x, Tx) + d(y, Sy), d(y, Sy) + d(x, y), d(x, Tx) + d(x, y)\}$$

for all $x, y \in X$ then S and T have a unique common fixed point.

Theorem 2.44 ([13]). *Let T be continuous self map defined in a complete metric space (X, d) such that for some positive integer m, satisfies the condition*

$$\begin{aligned} d(T^m x, T^m y) \leq\ & \alpha \frac{d(x, T^m x)d(y, T^m y) + d(x, T^m y)d(y, T^m x)}{d(x, y)} + \beta \frac{d(x, T^m y)[d(x, T^m x) + d(y, T^m y)]}{d(x, y) + d(y, T^m y) + d(y, T^m x)} \\ & + \gamma \frac{d(x, T^m x)d(y, T^m x) + d(y, T^m y)d(x, T^m y)}{d(x, T^m x) + d(y, T^m x) + d(y, T^m y) + d(x, T^m y)} + \kappa \frac{d(x, T^m x)d(x, T^m y) + d(y, T^m y)d(y, T^m x)}{d(x, T^m y) + d(y, T^m x)} \\ & + \delta[d(x, T^m x) + d(y, T^m y)] + \eta[d(y, T^m x) + d(x, T^m y] + \mu d(x, y) \end{aligned}$$

for all $x, y \in X$, $x \neq y$ and for $\alpha, \beta, \gamma, \kappa, \delta, \eta, \mu \in [0, 1)$ such that $2\alpha + 2\kappa + 4\delta + 4\eta + 2\mu < 2$, if T^m is continuous then T has a fixed point in X.

Theorem 2.45 ([3]). *Let $\{T_n\}$ be a sequence of mappings of a complete metric space (X, d) into itself. Let x_n be a fixed point of $\{T_n\}$ $(n = 1, 2, ...)$ and suppose $\{T_n\}$ converges uniformly to T_0. If T_0 satisfies the condition*

$$d(T_0 x, T_0 y) \leq \alpha \frac{d(x, T_0 x)d(x, T_0 y) + d(y, T_0 y)d(y, T_0 x)}{d(x, T_0 y) + d(y, T_0 x)} + \beta \frac{d(x, T_0 y)[d(x, T_0 x) + d(y, T_0 y)]}{d(x, y) + d(y, T_0 y) + d(y, T_0 x)} + \gamma d(x, y)$$

for all $x, y \in X$, $x \neq y$ and for $\alpha, \beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma < 1$, then $\{x_n\}$ converges to the fixed point x_0 of T_0.

3. Conclusion

The fixed point theory in metric space, Hausdorff metric space under contraction conditions and its generalized forms are important as an extension of famous Banach Contraction Principles and for its applications to other disciplines. On the other hand, Ekeland's Variation Principles and its extended forms have significant role in the optimization process and in the classical results. Therefore, we find several extensions and generalizations of these results in the literature.

References

[1] S. Abian and A. B. Brown, *A theorem on partially ordered sets with applications to fixed point theorems*, Canad. J. Math., 13(1961), 78-82.

[2] S. Almezel, Q. A. Ansari and M. A. Khamsi, *Topics in Fixed Point Theory*, Springer International Publishing, Switzerland, (2014).

[3] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, Basel, Berlin, (1990).

[4] M. Bianchi and R. Pini, *A note on equilibrium problems with properly quasi monotone functions*, J. Global Optim., 20(2001), 67-76.

[5] K. Fan, *Fixed point and mini max theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci., 38(1952), 121-126.

[6] Y. Feng and S. Liu, *Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings*, J. Math. Anal. Appl., 317(2006), 103-112.

[7] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Series of Monographs and Text books in Pure and Applied Mathematics, Vol. 83, Marcel Dekker, New York, (1984).

[8] M. A. Khamsi, *One-local retract and common fixed point for commuting mappings in metric spaces*, Nonlinear Anal., 27(1996), 1307-1313.

[9] M. S. Khan, *A fixed Point theorem for metric spaces*, Riv. Mat. Univ. Parma, 4(1977), 53-57.

[10] M. Kikkawaa and T. Suzuk, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*, Nonlinear Anal., 69(2008), 2942-2949.

[11] N. Mizoguchi and W. Takahashi, *Fixed points theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., 141(1989), 177-188.

[12] S. B. Nadler, *Multivalued contraction mappings*, Pac. J. Math., 30(1969), 475-488.

[13] D. Panthi and P. S. Kumari, *Some fixed point results for single and two maps in complete metric space*, Int. Jr. Math. and Mathematical Sc., 1(2015), 18-30.

[14] D. Panthi and P. K. Jha, *A short survey on fixed point results and applications*, International Journal of Statistics and Applied Mathematics, 2(5)(2017), 25-34.

[15] D. Panthi, *Common Fixed Point Theorems for Compatible Mappings in Dislocated Metric Space*, International Journal of Mathematical Analysis, 9(45)(2015), 2235-2242.

[16] D. Panthi, *A Meir-Keeler Type Common Fixed Point Result in Dislocated Metric Space*, Journal of Nepal Mathematical Society, 1(2)(2018), 53-59.

[17] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Am. Math. Soc., 136(2008), 1861-1869.

[18] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings, in Fixed Point Theory and Applications*, Marseille, 1989, Pitman Res. Notes Math. Ser., 252 Longman Sci. Tech., Harlow, (1991), 397-406.