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# Asymptotic Stability in Impulsive Functional Differential Equations

**Research Article**

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**Abstract:** In this paper conditions on Lyapunov functionals $V(t,\emptyset)$ are improved which ensure asymptotic stability and uniform asymptotic stability.

**Keywords:** Stability, Asymptotic Stability, Impulsive Functional Differential Equation, Lyapunov functional.


## 1. Introduction

Consider the impulsive functional differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \neq t_k \;\; t \geq t_0 \\ \Delta x = I_k(t, (x_t^-)), & t = t_k, \quad k \in Z^+. \end{cases} \tag{1}$$

Where $f : J \times PC \to R^n$, $\Delta x = x(t) - x(t^-)$, $t_0 < t_1 < \ldots t_k < t_{k+1} < \ldots$, With $t_k \to \infty$ as $k \to \infty$ and $I_k : J \times S(\rho) \to R^n$, where $J = [t_0, \infty)$, $S(\rho) = \{x \in R : |x| < \rho\}$. $PC = PC([-\tau, 0], R^n)$ denotes the space of piecewise right continuous functions $\varphi : [-\tau, 0] \to R^n$ with sup-norm $\|\varphi\|_\infty = \sup\limits_{-\tau \leq s \leq 0} |\varphi(s)|$ and the norm $\|\varphi\|_2 = (\int\limits_{-\tau}^{0} |\varphi(s)|^2 ds)^{1/2}$, where $\tau$ is a positive constant, $\|.\|$ is a norm in $R^n$. $x_t \in PC$ is defined by $x_t(s) = x(t+s)$ for $-\tau \leq s \leq 0.x'(t)$ denotes the right-hand derivative of $x(t)$. $Z^+$ is the set of all positive integers,

Let $f(t, 0) = 0$ and $J(0) = 0$, then $x(t) = 0$ is the zero solution of (1). Set $PC(\rho) = \{\varphi \in PC : \|\varphi\|_\infty < \rho\}$, $\forall \rho > 0$.

**Definition 1.1.** *Let $\sigma$ be the initial time, $\forall\ \sigma \in R$, the zero solution of (1) is said to be*

*(a). stable if , for each $\sigma \geq t_0$ and $\varepsilon > 0$, there is a $\delta = \delta(\sigma, \varepsilon) > 0$ such that, for $\varphi \in PC(\delta)$, a solution $x(t, \sigma, \varphi)$ satisfies $|x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0$.*

*(b). uniformly stable if it is stable and $\delta$ in the definition of stability is independent of $\sigma$.*

*(c). asymptotically stable if it is stable and, for each $t_0 \in R_+$, there is an $\eta = \eta(t_0) > 0$ such that, for $\varphi \in PC(\eta)$, $x(t, \sigma, \varphi) \to 0$ as $t \to \infty$.*

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*(d). uniformly asymptotically stable if it is uniformly stable and there is an $\eta > 0$ and , for each $\varepsilon > 0$, a $T = T(\varepsilon) > 0$ such that, for $\varphi \in PC(\eta)$, $|x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0 + T$*

**Definition 1.2.** *A functional $V(t, \varphi) : J \times PC(\rho) \to R_+$ belong to class $v_o(.)$ (a set of Liapunov like functional if*

*(a). $V$ is continuous on $[t_{k-1}, t_k) \times PC(\rho)$ for each $k \in Z_+$, and for all $\varphi \in PC(\rho)$ and $k \in Z_+$, the limit $\lim\limits_{(t,\varphi)\to(t_k{}^-,\varphi)} V(t, \varphi) = V(t_k{}^-, \varphi)$ exists.*

*(b). $V$ is locally Lipchitzian in $\varphi$ in each set in $PC(\rho)$ and $V(t, 0) = 0$. The set $\mathfrak{R}$ is defined by $\mathfrak{R} = \{W \in C(R_+, R_+) :$ strictly increasing and $W(0) = 0$.*

## 2. Main Results

**Theorem 2.1.** *Suppose that $V \in v_o(.)$, $W_1 \in \mathfrak{R}$ and*

*(a). There is an $0 < H_1 < H$ and for each $0 \leq \gamma < H_1$, there is a $\mu = \mu(\gamma) \geq 0$ with $\mu(\gamma) > 0$ for $\gamma > 0$ such that if $|x(t)| \geq \gamma$ and $\|x_t\| < H_1$*

*(b). $V'(t, x_t) \leq -\mu |f(t, x_t)| - \lambda(t) W_1(\inf\{|x(s)| : t - h \leq s \leq t\})$ Where $\lambda : R_+ \to R_+$ is continuous with $\int\limits_0^\infty \lambda(s) ds = \infty$*

*(c). $V(t_k, x + I_k(t_k, x)) - V(t_k{}^-, x) \leq 0$. Then the zero solution of (1) is asymptotically stable.*

*Proof.* For given $0 < \varepsilon < H_1$ and $t_0 \in R_+$, choose a $0 < \delta < \varepsilon/2$ such that $\varphi \in PC(\delta)$ implies $V(t_0, \varphi) < \mu\left(\frac{\varepsilon}{2}\right)\varepsilon/2$. Fix $\varphi \in PC(\delta)$ and let $x(t) = x(t, t_0, \varphi)$ be a solution of (1) on $[t_0, t_0 + \beta]$. Suppose, for contradiction, that there is a $t_0 < t_1 < t_0 + \beta$ with $|x(t_1)| = \varepsilon$ and $|x(t)| < \varepsilon$ on $[t_0, t_1)$.

Then there is a $t_0 < t_2 < t_1$ such that $|x(t_2)| = \varepsilon/2$ and $\varepsilon/2 < |x(t)| < \varepsilon$ on $(t_2, t_1)$, so that

$$\int_{t_2}^{t_1} |f(s, x_s)| ds = |x(t_1) - x(t_2)| = \varepsilon/2$$

Condition (c) yields $V'(t, x_t) \leq 0$, $t_0 \leq t_{k-1} \leq t < t_k$, $k \in Z^+$. And so $V(t)$ is decreasing on the interval of the form $[t_{k-1}, t_k)$, then from condition (c), we have

$$V(t_k) - V\left(t_k{}^-\right) = V\left(t_k, x\left(t_k{}^-\right) + I_k\left(t_k, x\left(t_k{}^-\right)\right)\right) - V(t_k{}^-, x\left(t_k{}^-\right)) \leq 0$$

Thus $V(t)$ is non increasing on $[t_0, \infty)$. Thus we have,

$$0 \leq V(t_1, x_{t_1}) \leq V(t_2, x_{t_2}) - \mu\left(\frac{\varepsilon}{2}\right)\int_{t_2}^{t_1} |f(s, x_s)| ds + \sum_{t_2 < t < t_1}\left[V(t_k) - V\left(t_k{}^-\right)\right]$$
$$\leq V(t_0, \varphi) - \mu\left(\frac{\varepsilon}{2}\right)\varepsilon/2 < 0$$

A contradiction. Therefore, $|x(t)| < \varepsilon$ on $[t_0, t_0 + \beta)$, which implies $\beta = \infty$. This proves the stability.

For a given $t_0 \in R_+$, set $\eta = \eta(t_0) = \delta\left(\frac{H_1}{2}, t_0\right) > 0$, where $\delta$ is that in the definition of stability. For a given $\varphi \in PC(\eta)$, let $x(t) = x(t, t_0, \varphi)$ ba a solution.

To show asymptotic stability, suppose $x(t) \nrightarrow 0$ as $t \to \infty$. Then, for some $\varepsilon_0 > 0$, there is a sequence $\{t_i\}$ such that $|x(t_i)| \geq \varepsilon_0$ and $t_i \to \infty$. If there is an $i \geq 1$ such that $|x(t)| \geq \varepsilon_0/2$ for $t \geq t_i$, then $\inf\{|x(s)| : t - h \leq s \leq t\} \geq \varepsilon_0/2$ for $t \geq t_i + h$, and we have

$$0 \leq V(t, x_t) \leq V(t_i + h, x_{t_i+h}) - W_1\left(\frac{\varepsilon_0}{2}\right)\int_{t_i+h}^{t} \lambda(s) ds + \sum_{t \geq t_i +}\left[V(t_k) - V\left(t_k{}^-\right)\right] \to -\infty \text{ as } t \to \infty.$$

A contradiction. Thus we may conclude that there is a sequence $\{T_i\}$ with $t_i < T_i < t_{i+1}$, $|x(T_i)| = \varepsilon_0/2$ and $|x(t)| > \varepsilon_0/2$ on $[t_i, T_i)$ so that

$$V(t_{i+1}, x_{t_i+1}) \le V(T_i, x_{T_i}) \le V(t_i, x_{t_i}) - \mu\left(\frac{\varepsilon_0}{2}\right)\varepsilon_0/2$$

And we obtain

$$\begin{aligned} 0 \le V(t_n, x_{t_n}) &= V(t_1, x_{t_1}) + \sum_{i=2}^{n}\left(V(t_i, x_{t_i}) - V\left(t_{i-1}, x_{t_{i-1}}\right)\right) \\ &\le V(t_0, x_{t_0}) - \frac{(n-1)\,\mu\left(\frac{\varepsilon_0}{2}\right)\varepsilon_0}{2} + \sum_{t_i<t_k<t_{i+1}}\left[V(t_k) - V\left(t_k{}^-\right)\right] \to -\infty \end{aligned}$$

As $n \to \infty$; a contradiction. Therefore, $x(t) \to 0$ as $t \to \infty$. This completes the proof of asymptotic stability. □

**Theorem 2.2.** *Suppose that $V \in v_0(.)$, $W_1, W_2 \in \mathfrak{R}$ and*

*(a). $V(t, \varphi) \le W_1(\|\varphi\|)$*

*(b). there is an $0 < H_1 < H$ and for each $0 \le \gamma < H_1$, there is a $\mu = \mu(\gamma) \ge 0$ with $\mu(\gamma) > 0$ for $\gamma > 0$ such that if $|x(t)| \ge \gamma$ and $\|x_t\| < H_1$*

*(c). $V'(t, x_t) = -\mu\,|f(t, x_t)| - \lambda(t)\,W_2(\inf\{|x(s)| : t-h \le s \le t\})$ where $\lambda : R_+ \to R_+$ is continuous with $\lim\limits_{S\to\infty}\int_t^{t+S}\lambda(s)\,ds = \infty$ uniformly with respect to $t \in R_+$*

*(d). $V(t_k, x + I_k(t_k, x)) - V(t_k{}^-, x) \le 0$. Then the zero solution of (1) is uniformly asymptotically stable.*

*Proof.* Uniform stability is obtained by the same argument for the stability in the proof of Theorem 1. For $0 < H_1 < H$, set $\eta = \delta(H_1) > 0$, where $\delta$ is that of uniform stability. For $t_0 \in R_+$ and $\varphi \in PC_\eta$, let $x(t) = x(t, t_0, \varphi)$ be a solution. For a given $0 < \varepsilon < H_1$, choose $\delta = \delta(\varepsilon) > 0$ of uniform stability. Find $S = S(\varepsilon) > 0$ with $\int_t^{t+S}\lambda(s)\,ds > W_1(\eta)/W_2(\delta/2)$ for $t \in R_+$, then there is an

$$s_i \in [t_0 + (i-1)(S+2h),\ t_0 + (i-1)(S+2h) + h + S]$$

With $|x(s_i)| < \delta/2$ for each $i \ge 1$. Otherwise there is an $i \ge 1$ such that $|x(s)| \ge \delta/2$ on

$$[t_0 + (i-1)(S+2h),\ t_0 + (i-1)(S+2h) + h + S],$$

that is, $\inf\{|x(\sigma)| : s-h \le \sigma \le s\} \ge \delta/2$ on

$$I_i \stackrel{\text{def}}{=} [t_0 + (i-1)(S+2h) + h,\ t_0 + (i-1)(S+2h) + h + S],$$

and this implies

$$\begin{aligned} 0 &\le V(t_0 + (i-1)(S+2h) + h + S, x_{t_0+(i-1)(S+2h)+h+S}) \\ &\le V\left(t_0 + (i-1)(S+2h) + h, x_{t_0+(i-1)(S+2h)+h}\right) - W_2(\delta/2)\int_{I_i}\lambda(s)\,ds + \sum_{t_k\in I_i}\left[V(t_k) - V t_k{}^-\right] \\ &< 0; \end{aligned}$$

a contradiction. Choose an integer $N = N(\varepsilon) \ge 1$ with $N\mu(\delta/2)\delta/2 > W_1(\eta)$. Suppose, for contradiction, that for each $1 \le i \le N$, $\|x_{t_0+i(S+2h)}\| \ge \delta$ then there is a

$$T_i \in [t_0 + (i-1)(S+2h) + h + S, t_0 + i(S+2h)]$$

with $|x(T_i)| \geq \delta$, and thus there is a $t_i \in (s_i, T_i)$ such that $|x(t_i)| = \delta/2$ and $|x(t)| > \delta/2$ on $(t_i, T_i]$ for each $1 \leq i \leq N$. Hence

$$\begin{aligned} V\left(t_0 + i\left(S+2h\right), x_{t_0+i(S+2h)}\right) &\leq V\left(T_i, x_{T_i}\right) \leq V\left(t_i, x_{t_i}\right) - \int_{t_i}^{T_i} \mu(\delta/2)\left|f\left(t, x_t\right)\right| dt + \sum_{t_i \leq t_k \leq T_i} \left[V\left(t_k\right) - V\left(t_k{}^-\right)\right] \\ &\leq V\left(t_0 + (i-1)\left(S+2h\right), x_{t_0+(i-1)(S+2h)}\right) - \mu(\delta/2)\delta/2, \end{aligned}$$

and we have

$$\begin{aligned} 0 &\leq V\left(t_0 + N\left(S+2h\right), x_{t_0+N(S+2h)}\right) \\ &= V\left(t_0,\ x_{t_0}\right) + \sum_{i=1}^{N} \left(V\left(t_0 + i\left(S+2h\right), x_{t_0+i(S+2h)}\right) - V\left(t_0 + (i-1)\left(S+2h\right), x_{t_0+(i-1)(S+2h)}\right)\right) \\ &\leq W_1\left(\eta\right) - N\mu(\delta/2)\delta/2 \ < 0, \end{aligned}$$

a contradiction. Thus, there is a $1 \leq i \leq N$ with $\|x_{t_0+i(S+2h)}\| < \delta$, which implies $|x(t)| < \varepsilon$ for $t \geq t_0 + i(S+2h)$. Set $T = T(\varepsilon) = N(S+2h)$, and we obtain $|x(t)| < \varepsilon$ for $t \geq t_0 + T$. This completes the proof of uniform asymptotic stability. □

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