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Finite Product Topologies Modulo an Ideals

Research Article

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Abstract: Given a topological space (X,τ) and an ideal $\Im$ in X, a finer topology τ^* in X can be associated with τ and $\Im$. Given two topological spaces (X,τ_1), (Y,τ_2) and ideals $\Im$, ϑ in X, Y respectively, an ideal $\Im \times \vartheta$ in $X \times Y$, called the product ideal of $\Im$ and ϑ, in $X \times Y$. We investigate inclusion relations between $\tau_1^* \times \tau_2^*$ and $(\tau_1 \times \tau_2)^*$ and the conditions under which $\tau_1^* \times \tau_2^* = (\tau_1 \times \tau_2)^*$ and we extend the theorem for finite case.

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1. Introduction

In 1945, R.Vaidyanathaswamy [10] introduced the concept of ideal topological spaces. Havashi [1] defined the local function and studied some topological properties using local function in ideal topological spaces in 1964. Since then many mathematicians studied various topological concepts in ideal topological spaces. The first unified and extensive study on these τ^*-topologies was done by Jankovic and Hamlett in [2] and proofs for the facts stated above may be found in [2]. The initial important articles on topological spaces are [5] and [3], a thesis [4] and a book that includes ideal is [9]. For given ideals on coordinate topological spaces of a finite product space, an ideal is introduced on the product space and the relations between two *-topologies are studied and we extend the theorem for finite case.

2. Preliminaries

Given a nonempty set X, a collection $\Im$ of subsets of X is called an ideal if

(i). A $\in \Im$ and $B \subseteq A$ implies $B \in \Im$ (heredity)

(ii). A $\in \Im$ and $B \in \Im$ implies $A \cup B \in \Im$ (finite additivity).

If $X \notin \Im$, then $\Im$ is called a proper ideal. An ideal $\Im$ is called a σ-ideal if the following holds:
If $\{A_n : n = 1, 2, \dots\}$ is a countable sub collection of $\Im$, then $\cup\{A_n : n = 1, 2, \dots\} \in \Im$. The notation $(X, \tau, \Im)$ denotes a nonempty set X, a topology τ on X and an ideal $\Im$ on X. Given a point $x \in X$, $\aleph(x)$ denotes the neighborhood system of x. i.e. $\aleph(x) = \{U \in \tau : x \in U\}$. Given a space $(X, \tau, \Im)$ and a subset A of X , we define

$$A^*(\Im,\tau) = \{x \in X : U \cap A \notin \Im, \quad \text{for every} \quad U \in \aleph(x)\}.$$

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We simply write A^* for A^* $(\Im, \tau)$, where there are only one ideal $\Im$ and only one topology τ under consideration. A Kuratowski closure operator cl^* for a topology $\tau^*(\Im)$ finer than τ is defined as follows: $cl^*(A) = A \cup A^*$, for all $A \in \wp(X)$. A basis $\beta(\Im, \tau)$ for $\tau^*(\Im)$ can be described as follows: $\beta(\Im, \tau) = \{U - I : U \in \tau, I \in \Im\}$ and we will write β for $\beta(\Im, \tau)$ and τ^* for $\tau^*(\Im)$, when τ and $\Im$ are fixed.

We shall use $cl(A)$, $int(A)$ to denotes closure and interior of a subset A respectively in a topological space (X, τ) and $cl^*(A)$, $int^*(A)$ will denotes closure and interior of A respectively with respect to τ^*. The symbol $\wp(X)$ will denote collection of all subsets of X. Let (X, τ) be a topological space with ideal $\Im$ on X and let A and B be subsets of X, then

(i). $A \subset B \Rightarrow A^* \subset B^*$.

(ii). $(A \cup B)^* = A^* \cup B^*$.

(iii). $A^* = cl(A^*) \subseteq cl(A)$.

(iv). $\tau = \tau^*$ if and only if $\Im$ contains the collection of all closed sets of (X, τ).

3. Definition and Basic Results on Product Ideal

Let us start with a natural definition for ideal on product space.

Definition 3.1. *Let (X, τ_1) and (Y, τ_2) be two topological spaces. Let $\Im$ and ϑ be ideals on X and Y respectively. Define $\Im \times \vartheta = \{A \subset X \times Y : p_1(A) \in \Im \text{ and } p_2(A) \in \vartheta\}$, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are projections. Then $\Im \times \vartheta$ is called the product ideal of $\Im$ and ϑ in $X \times Y$.*

This collection $\Im \times \vartheta$ is an ideal on $X \times Y$. For, consider a set $A \in \Im \times \vartheta$ and a set $B \subseteq A$. Then $B \subseteq A \Rightarrow p_1(B) \subset p_1(A)$ and $p_1(A) \in \Im \Rightarrow p_1(B) \in \Im$. Similarly, $p_2(B) \in \vartheta$, so that $B \subseteq A \in \Im \times \vartheta \Rightarrow B \in \Im \times \vartheta$. If $A, B \in \Im \times \vartheta$ then $p_1(A)$, $p_1(B) \in \Im$ and $p_2(A)$, $p_2(B) \in \vartheta$ and hence $p_1(A \cup B) = p_1(A) \cup p_1(B) \in \Im$ and $p_2(A \cup B) = p_2(A) \cup p_2(B) \in \vartheta$; and this shows that $A \cup B \in \Im \times \vartheta$. Thus $\Im \times \vartheta$ is an ideal on $X \times Y$. Let the topology $(\tau_1 \times \tau_2)^*(\Im \times \vartheta)$ on $X \times Y$, obtained from the product topology $\tau_1 \times \tau_2$ on $X \times Y$ and the ideal $\Im \times \vartheta$, be denoted by $(\tau_1 \times \tau_2)^*$. The topology $(\tau_1 \times \tau_2)^*$ on $X \times Y$ is finer than $\tau_1 \times \tau_2$. There is another product topology $\tau_1^* \times \tau_2^*$ on $X \times Y$, obtained from τ_1^* and τ_2^*. Can we expect that $\tau_1^* \times \tau_2^* = (\tau_1 \times \tau_2)^*$?. We get one inclusion relation $(\tau_1 \times \tau_2)^* \subseteq \tau_1^* \times \tau_2^*$ in the next theorem. We shall show that this inclusion relation may be strict.

Theorem 3.2. *Let (X, τ_1) and (Y, τ_2) be two topological spaces with ideals $\Im$ and ϑ on X and Y respectively. Then we have $(\tau_1 \times \tau_2)^* \subseteq \tau_1^* \times \tau_2^*$.*

Proof. Let $A \subset X \times Y$. Assume that $A^* \subseteq A$ so that A is closed in $(\tau_1 \times \tau_2)^*$. Let $(x, y) \notin A$, so $(x, y) \notin A^*$. Let

$$B = \{(x_1, y_1) \in A : x_1 \neq x\}.$$

$$C = \{(x_1, y_1) \in A : y_1 \neq y\}.$$

Then $A = B \cup C$ and $A^* = (B \cup C)^* = B^* \cup C^*$. Now $(x, y) \notin A^*$ implies that there exist neighborhood U of x in X and neighborhood V of y in Y such that $(U \times V) \cap A \in \Im \times \vartheta$, so $p_1[(U \times V) \cap A] \in \Im$ and $p_2[(U \times V) \cap A] \in \vartheta$. Write $B_1 = p_1[(U \times V) \cap A]$ and $C_2 = p_2[(U \times V) \cap A]$.

Now U is neighborhood of x such that $U \cap B_1 \in \Im$ and hence $x \notin B_1^*$. Also V is a neighborhood of y such that $V \cap C_2 \in \vartheta$ and hence $y \notin C_2^*$. Write

$$U_1 = U - (B_1^* \cup B_1).$$

$$V_1 = V - (C_2^* \cup C_2).$$

Since U is open in τ_1, it is open in τ_1^* (as $\tau_1 \subset \tau_1^*$). Since $B_1^* \cup B_1$ is closed in τ_1^*, the set U_1 is open in τ_1^*. Similarly, V_1 is open in τ_2^*. We claim that $(U_1 \times V_1) \cap A = \varphi$. Now $(x_1, y_1) \in U_1 \times V_1$ implies $x_1 \notin B_1$, so there exist no y_0 such that $(x_1, y_0) \in B$, and so $(x_1, y_1) \notin B$.

Similarly, $(x_1, y_1) \notin C$. Therefore $(x_1, y_1) \notin B \cup C = A$ so $(U_1 \times V_1) \cap A = \varphi$. Therefore $(x, y) \notin \overline{A}$, the closure of A with respect $\tau_1^* \times \tau_2^*$. Thus $\overline{A} \subset A$ which shows that A is also closed with respect $\tau_1^* \times \tau_2^*$. This proves that $(\tau_1 \times \tau_2)^* \subseteq \tau_1^* \times \tau_2^*$. □

The following example shows that strict inclusion is possible.

Example 3.3. *Let X be the set of all natural numbers. Let τ_1 be the topology with a basis $\{\{1,2\}, \{3,4\}, \ldots \{2n-1, 2n\}, \ldots\}$.*

If $\Im$ is the class of all finite subsets of X, then τ_1^* is the discrete topology on X. Let Y be the real line with the usual topology and $\vartheta = \{\varphi\}$. Then $\tau_2 = \tau_2^*$, the usual topology. Let $A = \{2, 3\}$ which is closed in τ_1^*. To prove $p_1^{-1}(A)$ is not closed in $(\tau_1 \times \tau_2)^*$. Consider the point $(4, 2)$. Let $U \times V$ be the neighborhood of $(4, 2)$ with respect to $\tau_1 \times \tau_2$, then U contains $\{3, 4\}$ and V contains an interval $(2 - \delta, 2 + \delta)$, for some $\delta > 0$. Then $(U \times V) \cap p_1^{-1}(A) \supset \{3\} \times (2 - \delta, 2 + \delta) \notin \Im \times \vartheta$, as $\vartheta = \{\varphi\}$, and so $(4, 2) \in (p_1^{-1}(A))^*$. Hence $p_1^{-1}(A)$ is closed in $\tau_1^* \times \tau_2^*$, but not closed in $(\tau_1 \times \tau_2)^*$. Thus $(\tau_1 \times \tau_2)^* \subset \tau_1^* \times \tau_2^*$. Now we obtain conditions under which $(\tau_1 \times \tau_2)^* = \tau_1^* \times \tau_2^*$.

Theorem 3.4. *Let (X, τ_1) and (Y, τ_2) be two topological spaces with ideals $\Im$ and ϑ on X and Y respectively. Consider the ideal $\Im \times \vartheta$ on $X \times Y$. Then $(\tau_1 \times \tau_2)^* = \tau_1^* \times \tau_2^*$ if and only if*

(a). $\tau_1 = \tau_1^*$ (or) for every $y \in Y$, there is an $V_y \in \tau_2$ such that $y \in V_y \in \vartheta$ and

(b). $\tau_2 = \tau_2^*$ (or) for every $x \in X$, there is an $U_x \in \tau_1$ such that $x \in U_x \in \Im$.

Proof. Assume that $(\tau_1 \times \tau_2)^* = \tau_1^* \times \tau_2^*$.

Suppose that $\tau_1 \neq \tau_1^*$. Let A be closed set in τ_1^*, but not in τ_1, so there exist $x \in \overline{A}$ but $x \notin A$. As $A \times Y$ is closed in $\tau_1^* \times \tau_2^*$, it is closed in $(\tau_1 \times \tau_2)^*$. Let $y \in Y$, then $(x, y) \notin A \times Y$, so there exist $\tau_1 \times \tau_2$-neighborhood $V_x \times V_y$ of (x, y) such that $(V_x \times V_y) \cap (A \times Y) \in (\Im \times \vartheta)$. Hence $(V_x \cap A) \times (V_y \cap Y) \in (\Im \times \vartheta)$, so $V_y \in \vartheta$. This derives (a).

Suppose $\tau_2 \neq \tau_2^*$. Then, as in the previous case, we can prove that for every $x \in X$, there is an $U_x \in \tau_1$ such that $x \in U_x \in \Im$. This derives (b).

Now let us prove the converse part.

Case (i) : Suppose $\tau_1 = \tau_1^*$ and $\tau_2 = \tau_2^*$. Then $(\tau_1 \times \tau_2)^* \subset \tau_1^* \times \tau_2^* = \tau_1 \times \tau_2 \subset (\tau_1 \times \tau_2)^*$ so $(\tau_1 \times \tau_2)^* = \tau_1^* \times \tau_2^*$.

Case (ii): Suppose $\tau_1 = \tau_1^*$ and $\tau_2 \neq \tau_2^*$, that is $\tau_1 = \tau_1^*$ and to each $x \in X$, there exist τ_1-open set U of x such that $U \in \Im$. Let $A \times B$ be a closed set in $\tau_1^* \times \tau_2^*$. So both A and B are closed in τ_1^* and τ_2^* respectively. Let $(x, y) \notin A \times B$, then either $x \notin A$ or $y \notin B$. If $x \notin A$, as $A = A \cup A^*$ (closed with respect to τ_1^* (equal to τ_1)) and $x \notin A^*$, there exist a neighborhood U of x with respect to τ_1 such that $U \cap A = \varphi$. Then, If we let A be any neighborhood of y in Y with respect to τ_2, then $(U \times V) \cap (A \times B) = \varphi$, and therefore $(x, y) \notin (A \times B)^*$ which proves that $A \times B$ is closed with respect to $(\tau_1 \times \tau_2)^*$. Let $x \in A$ but $y \notin B$, so $y \notin B^*$ (with respect to τ_2), so there exist a neighborhood V of y such that $V \cap B \in \vartheta$. As $x \in A$, by assumption, there exist a neighborhood U of x such that $U \in \Im$. Therefore $U \times V$ is a neighborhood of $\tau_1 \times \tau_2$ such that $(U \times V) \cap (A \times B) \in \Im \times \vartheta$. Hence $A \times B$ is closed in $(\tau_1 \times \tau_2)^*$. This, of course, proves that $\tau_1^* \times \tau_2^* \subseteq (\tau_1 \times \tau_2)^*$.

Case (iii) : Suppose $\tau_2 = \tau_2^*$ but $\tau_1 \neq \tau_1^*$. Then proof is similar to case (ii).

Case (iv): Suppose, to each $x \in X$, there exist open neighborhood U_x of x in τ_1 such that $U_x \in \Im$ and each $y \in Y$, there exist neighborhood V_y of y in τ_2 such that $V_y \in \vartheta$. Let $A \times B$ is closed in $\tau_1^* \times \tau_2^*$ and $(x, y) \notin (A \times B)$. Consider the neighborhood s U_x and V_y of x and y respectively such that $U_x \in \Im$ and $V_y \in \vartheta$. Then $U_x \times V_y \in \Im \times \vartheta$ and $(V_x \times V_y) \cap (A \times B) \in (\Im \times \vartheta)$, and so $(x, y) \notin (A \times B)^*$. Hence $A \times B$ is closed with respect to $(\tau_1 \times \tau_2)^*$, which also proves that $\tau_1^* \times \tau_2^* \subseteq (\tau_1 \times \tau_2)^*$. □

The Theorems 2.2 and 2.3 can be extended to any finite product space. Now we extend the theorem for finite case.

Theorem 3.5. *If $\wedge$ is finite then* $(\underset{\alpha\in\wedge}{\pi}\tau_\alpha)^* \subset \underset{\alpha\in\wedge}{\pi}\tau_\alpha^*$.

Proof. Let $\Im$ be the ideal for πX_α, induced by $\{\Im_\alpha\}_{\alpha\in\wedge}$. Then $\{V - I :$ V is open with respect to $\pi\tau_\alpha$ and $I \in \Im\}$ is a basis for $(\pi\tau_\alpha)^*$. It is enough to show that I is $\pi\tau_\alpha^*$- closed, for every $I \in \Im$. Let $I \in \Im$ and $x = (x_\alpha) \notin I$. Let, for each $\alpha \in \wedge$, $B_\alpha = \{y \in I : y_\alpha \neq x_\alpha\}$, then $I = \bigcup_{\alpha\in\wedge} B_\alpha$. As $B_\alpha \subset I$, $B_\alpha \in \Im$ and hence $p_\alpha(B_\alpha) \in \Im_\alpha$. Since $x_\alpha \notin p_\alpha(B_\alpha)$ in X_α and $p_\alpha(B_\alpha)$ is closed with respect to τ_α^*, there exists τ_α^*-open sets V_α in X_α such that $x_\alpha \in V_\alpha$ and $V_\alpha \cap p_\alpha(B_\alpha) = \phi$. Let $W = \bigcap_{\alpha\in\wedge} p_\alpha^{-1}(V_\alpha)$. As $\wedge$ is finite, W is a $\pi\tau_\alpha^*$-open neighborhood of x. Note that $W \cap I = \phi$, it follows that I is $\pi\tau_\alpha^*$-closed in πX_α. Hence $(\underset{\alpha\in\wedge}{\pi}\tau_\alpha)^* \subset \underset{\alpha\in\wedge}{\pi}\tau_\alpha^*$, $\wedge$ is finite. □

Remark 3.6. *If $\wedge$ is not finite, then the Theorem 2.2 is not true, as seen from the following example.*

Example 3.7. *For every positive integer n, Let $X_n = R$, τ_n = the usual topology on R and $\Im_n$ be the ideal of all finite subsets of R. Note that $\tau_n = \tau_n^*$, for all n and hence $\underset{n=1}{\overset{\infty}{\pi}}\tau_n = \underset{n=1}{\overset{\infty}{\pi}}\tau_n^*$.*
Let $A_n = \{e_n : e_n = (0, 0, 0, \ldots, 0, 1, 0, \ldots), n = 1, 2, \ldots\}$. Then $A \in \Im$, where $\Im$ be the ideal in $\underset{n=1}{\overset{\infty}{\pi}} X_n$, induced by $\{\Im_n\}_{n=1}^{\infty}$, as $p_n(A) = \{0, 1\} \in \Im_n$, for all n. Hence A is $(\pi\tau_n)^*$-closed. Let $x = (0, 0, \ldots, 0, \ldots)$ i.e. $x_n = 0$, for all n. Any basic open neighborhood G of x in $\underset{n=1}{\overset{\infty}{\pi}}\tau_n = \underset{n=1}{\overset{\infty}{\pi}}\tau_n^*$ is of the form $p_{n_1}^{-1}(U_1) \cap p_{n_2}^{-1}(U_2) \cap \ldots p_{n_k}^{-1}(U_k)$, where $\cup_j$ is open in X_{n_j} with respect to τ_{n_j}. Clearly $G \cap A \neq \phi$ and hence A is not $\pi\tau_\alpha^*$-closed. Hence $\underset{n=1}{\overset{\infty}{\pi}}\tau_n)^* \not\subset \underset{n=1}{\overset{\infty}{\pi}}\tau_n^*$. This is also example for the space with

(a). $\tau_\alpha = \tau_\alpha^*$, for all $\alpha \in \wedge$.

(b). $\pi\tau_\alpha \neq (\pi\,\tau_\alpha)^*$.

If $\wedge$ is not finite, under what conditions $(\pi\,\tau_\alpha)^* \subseteq \pi\tau_\alpha^*$ is true ?.

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