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Equivalent Condition to a Frame Sequence

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Abstract

Let G be a second countable locally compact abelian group and H a countable closed discrete subgroup of G such that the quotient group G/H is compact. A simple proof of characterization of frame generated by a function $\phi \in L^2(G)$ and its translates by elements of H in terms of the boundedness of the periodization of its Fourier transform $\hat{\phi}$ is given.

Keywords: Abelian group; frame Sequence; Fourier transform; subgroup.

1. Introduction

If we consider a vector space then each element in the space has an easy unique representation with the help of a basis of a vector space. Frames are nothing but, a generalization of a basis in vector space. So, the first natural question arises in the mind : Why do we need frames if we already have a representation in terms of the basis? Is there something special or extra that we get from frames? The answer for above question is affirmative. Yes, frames are something different. The uniqueness of the representation of elements no more bother us. In other words, we are not bound to choose coefficients uniquely and if some of the coefficients are lost, still we can recover the signals or functions in the space.

John J. Bendetto and Shidong Li in [2] have given necessary and sufficient conditions for integer translates of a function in $L^2(R)$ to generate a frame. M. Bownik in [10] extended this result in $L^2(R^n)$ using Helson's [11] ideas on range functions. Using similar techniques this has been extended in [3] for locally compact abelian groups. Using the techniques of John J. Benedetto and Shidong Li [2], we give a simple proof of this result for locally compact abelian groups. Let G denotes a second countable locally compact abelian group and H a discrete countable subgroup of G such that the quotient group G/H is compact. $H^1 = \{\gamma \in \hat{G} : \langle h, \gamma \rangle = 1 \;\; for \;\; all \;\; h \in H\}$. Then $H^1 \cong \overline{(G/H)}$. Hence, H^1 is discrete

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and We consider the shift-invariant subspaces of $L^2(G)$ of the following form:

$$V = \left\{ \sum_{h \in H} c_h \phi(. - h) : \{c_h\} \in \ell^2(H) \right\}.$$

2. Fourier Analysis on the Space $L^2(\hat{H})$

For $f \in L^2\left(\hat{H}\right)$, the Fourier coefficients of f are defined by

$$F_\lambda(f) := \int_{\hat{H}} f(\gamma)\, \langle \lambda, -\gamma \rangle \, d\gamma;$$

where $\lambda \in H$. The Fourier series of f is defined as $\sum_{\lambda \in H} F_\lambda(f) \langle \lambda, \gamma \rangle$.
For a finite subset E of H and $f \in L^2\left(\hat{H}\right)$, $S_E f$ denotes the partial sum of Fourier series of f defined by $S_E f := \sum_{k \in E} F_k(f) \langle k \rangle$. By a trigonometric polynomial we mean the functions of the form L_S defined by $L_S(\gamma) = \sum_{\lambda \in S} c_\lambda \langle \lambda, \gamma \rangle$, where S is a finite subset of H and $c_\lambda \in C$. $P_E\left(\hat{H}\right)$ denotes the set of all trigonometric polynomials of the form L_A such that $A \subseteq E \subseteq H$.

Lemma 2.1. *Given $f \in L^2\left(\hat{H}\right)$ and finite subsets A and E of H with $A \subseteq E$, $(f - S_E F)$ is orthogonal to all trigonometric polynomials L_A.*

Proof. It suffices to show that $(f - S_E F)$ is orthogonal to the characters of the form $g(\gamma) = \langle k, \gamma \rangle$ for every $k \in A$. Now,

$$\begin{aligned} \langle f - S_E f, g \rangle &= \langle f, g \rangle - \sum_{h \in E} F_h(f) \langle \langle h, . \rangle, \langle k, . \rangle \rangle \\ &= F_k(f) - F_k(f) = 0. \end{aligned}$$

□

Lemma 2.2. $\|S_E f\|^2_{L^2(\hat{H})} = \sum_{k \in E} |(F_k(f))|^2$ *for every finite subset E of H and $f \in L^2\left(\hat{H}\right)$.*

Proof. It can be easily proved. For $f \in L^2\left(\hat{H}\right)$ and $A \subseteq E \subseteq H$, the following hold:

(1) For every trigonometric polynomial $L_A \in P_E\left(\hat{H}\right)$,

$$\|f - S_E f\|_{L^2(\hat{H})} \leq \|f - L_A\|_{L^2(\hat{H})}.$$

(2) Equality holds in the above inequality if and only if $L_A = S_E f$.

Let $L_A \in P_E\left(\hat{H}\right)$. Then there exist complex numbers c_k such that $L_A(\gamma) = \sum_{k \in A} c_k \langle k, \gamma \rangle$. Then by taking $c_k = 0$ for $k \in E/A$, we can write

$$f - L_A = f - \sum_{k \in E} c_k \langle k, . \rangle - S_E f + S_E f,$$

where $S_E f(\gamma) = \sum\limits_{k\in E} F_k(f)\langle k,\gamma\rangle$. We can also see that

$$f - L_A = (f - S_E f) + \left(\sum_{k\in E} (F_k(f) - c_k)\langle k,.\rangle\right).$$

As a consequence of Lemma 2.1,

$$\|f - L_A\|^2_{L^2(\hat{H})} = \|f - S_E f\|^2_{L^2(\hat{H})} + \left\|\sum_{k\in E} (F_k(f) - c_k)\langle k,.\rangle\right\|^2_{L^2(\hat{H})} \tag{1}$$

Therefore,

$$\|f - L_A\|^2_{L^2(\hat{H})} \geq \|f - S_E f\|^2_{L^2(\hat{H})}.$$

Also equality holds if and only if the second summand on the right hand side of equation (1) is equal to 0 and hence if and only if $F_K(f) = c_k$ for all $k \in E$ and so $L_A = S_E f$ as required. □

Lemma 2.3. *There exists a sequence $\{K_i\}_0^\infty$ of finite subsets of H with distinct elements and H^1 periodic functions F_{K_i} satisfying the following:*

(1). $K_i \subseteq K_{i+1}$ for every i and $\bigcup\limits_{i=0}^{\infty} K_i = H$.

(2). $\int\limits_{\hat{H}} |F_{K_i}(\gamma)|\, d\gamma = 1$ and there exists $M > 0$ such that $\int\limits_{\hat{H}} |F_{K_i}(\gamma)|\, d\gamma \leq M$ for all i.

(3). $F_{K_i} * f = \sigma_{K_i} f$, where $\sigma_{K_i} f = \frac{1}{|K_i|}\sum\limits_{j=0}^{i=1} S_{K_j}(f)$.

(4). For every neighbourhood N_0 of identity, $\int\limits_{\hat{H}/N_0} |F_{K_i}(\gamma)|\, d\gamma \to 0$ as $i \to \infty$.

We may use the fact that $H \cong Z^d \times F$ for some finite abelian group F. When H is isomorphic to Z, the Dirichlet's kernel and Fejer kernel take the following forms:

$$D_{K_i}(\gamma) = \sum_{|j|\leq i} \langle x_j, \gamma\rangle = \frac{[\langle x_{i+1},\gamma\rangle - \langle x_{-i},\gamma\rangle]}{[\langle x_1,\gamma\rangle - \langle x_0,\gamma\rangle]}$$

and

$$\begin{aligned} F_{K_i}(\gamma) &= \frac{D_{K_0}(\gamma) + D_{K_1}(\gamma) + ... + D_{K_{i-1}}(\gamma)}{|K_i|} \\ &= -\frac{1}{|K_i|\,|\langle x_1,\gamma\rangle - 1|^2} 2\ Re\left[\langle x_1,\gamma\rangle^i - 1\right], \end{aligned}$$

where, $j \mapsto x_j$ is an isomorphism between Z and H. Moreover, $\int\limits_{\hat{H}} |F_{K_i}(\gamma)|\, d\gamma = 1$ and $\int\limits_{\hat{H}} |F_{K_i}(\gamma)|\, d\gamma \leq M$ for some $M > 0$. Also for every neighbourhood N_0 of the identity in $\hat{H}$, $\int\limits_{\hat{H}/N_0} |F_{K_i}(\gamma)|\, d\gamma \to 0$ as $i \to \infty$, since on $\hat{H}/N_0$,

$$|F_{K_i}(\gamma)| = \left|\frac{1}{|K_i|}\frac{Re\left[1 - \langle x_1,\gamma\rangle^i\right]}{Re\left[1 - \langle x_1,\gamma\rangle\right]}\right| \leq \frac{2}{|K_i|\, C},$$

where $C = Re[1 - \langle x_1, \gamma \rangle]$. When H is isomorphic to Z^d we define the Dirichlet kernel and Fejer kernel to be

$$D_K(\gamma) = \prod_{j=1}^{d} D_{K_j}(\gamma_j) \quad and \quad F_K(\gamma) = \prod_{j=1}^{d} F_{K_j}(\gamma_j),$$

where $K = (K_1, K_2, \ldots, K_d)$. When H is isomorphic to $Z^d \times F$, we define $F_N(\gamma) = \prod_{j=1}^{d} F_{N_j}(\gamma_j) D_{n_0}(\gamma_0)$, $n_0 = |F|$ and $\gamma = (\gamma_0, \gamma_1, ..., \gamma_d)$. Then $F_N * f = \sigma_{Nf}$ and $\int_{\hat{H}} |F_N(\gamma)| d\gamma = 1$ and $\int_{\hat{H}} |F_N(\gamma)| d\gamma \leq M$ and $\int_{\hat{H}|N_0} |F_N(\gamma)| d\gamma \to 0$. If $f : \hat{G} \to C$ is continuous and periodic with period H^1, then $\sigma_{Nf} \to f$ uniformly on $\hat{H}$. Let $\varepsilon > 0$. As $\hat{H}$ is compact, f is uniformly continuous on $\hat{H}$. Hence, there is exists a neighbourehood N_0 of 0 (identity in $\hat{H}$) such that $|f(\gamma_1) - f(\gamma_2)| < \frac{\varepsilon}{2}$ whenever $\gamma_1, \gamma_2 \in \hat{H}$ and $\gamma_1 - \gamma_2 \in N_0$. In view of Lemma 2.3 we can find i_0 such that $\int_{\hat{H}/N_0} \left|F_{k_i}(\gamma)\right| d\gamma < \frac{\varepsilon}{2}$ for all $i \geq i_0$ and also we obtain

$$\begin{aligned}
|\sigma_{K_i}(f, \gamma) - f(\gamma)| &= \left| \int_{\hat{H}} [f(\gamma - u) - f(\gamma)] F_{K_i}(u) du \right| \\
&\leq \int_{N_0} |f(\gamma - u) - f(\gamma)| |F_{K_i}(u)| du + \int_{\hat{H}/N_0} |f(\gamma - u) - f(\gamma)| |F_{K_i}(u)| du \\
&\leq \int_{N_0} \frac{\varepsilon}{2} |F_{K_i}(u)| du + M_1 \frac{\varepsilon}{2} \\
&< M\frac{\varepsilon}{2} + M_1 \frac{\varepsilon}{2}
\end{aligned}$$

for sufficiently large i. For $f \in L^2(\hat{H})$, $\lim_{i \to \infty} \|f - S_{K_i}(f)\|_{L^2}(\hat{H}) = 0$. Let $\varepsilon > 0$. As continuous functions on $\hat{H}$ are dense in $L^2(\hat{H})$, there exist a continuous function g on $\hat{H}$ such that $\|f - g\|_{L^2(\hat{H})} \leq \frac{\varepsilon}{2}$. By Theorem 2, $\lim_{i \to \infty} \|g - S_{K_i} g\|_{L^2(\hat{H})} = 0$. Now,

$$\|f - S_{K_i} f\|_{L^2(\hat{H})} \leq \|f - S_{K_i} g\|_{L^2(\hat{H})} \leq \|f - g\|_{L^2(\hat{H})} + \|g - S_{K_i} g\|_{L^2(\hat{H})}.$$

The proof follows from the above inequality.

3. Frames Formed by Translates

We give a characterization for translates of ϕ by elements of H to generate a frame in terms of boundedness condition of the H^1-periodic function $G_\phi(\gamma)$. This extends the result of [2] for locally compact abelian groups.

Lemma 3.1. *Let $\phi \in L^2(G)$ and that $V = \overline{span}\{T_K \phi : k \in H\}$. Then the frame condition*

$$A\|f\|^2 \leq \sum_{k \in H} |\langle f, T_K \phi \rangle|^2 \leq B\|f\|^2 \tag{2}$$

is valid for each $f \in V$ *if and only if* (2) *is valid for* $f \in span\,\{T_K\phi\}$.

If (2) is valid for each $f \in V$, then (2) is trivially true for $f \in span\,\{T_K\phi\}$. Suppose that (2) is valid for $f \in span\,\{T_K\phi\}$ and let $f \in V$. Choose $f_n \in span\,\{T_K\phi\}$ such that $\lim\limits_{n\to\infty} \|f - f_n\|_2 = 0$. Since

$$\left| |\langle f_n, T_K\phi\rangle|^2 - |\langle f, T_K\phi\rangle|^2 \right| \leq \|\phi\|_2^2 \left(\|f_n\|_2 + \|f\|_2\right) \|f_n - f\|_2,$$

we get

$$\lim_{n\to\infty} |\langle f_n, T_K\phi\rangle|^2 = |\langle f, T_K\phi\rangle|^2 \tag{3}$$

Now, as a consequence of equation (3) and Fatou's Lemma applied to sums, we have by the right side of (2) for $span\,\{T_K\phi\}$ that

$$\sum_{k\in H} |\langle f, T_K\phi\rangle|^2 \leq \liminf_{n\to\infty} \sum_{k\in H} |\langle f_n, T_K\phi\rangle|^2 \leq B \liminf_{n\to\infty} \|f_n\|_2^2 = B\,\|f\|_2^2 \tag{4}$$

Thus, the right side of (2) is valid for V. By using the triangle inequality,

$$\left(\sum_{k\in H} |\langle f, T_K\phi\rangle|^2\right)^{\frac{1}{2}} \geq \left(\sum_{k\in H} |\langle f_n, T_K\phi\rangle|^2\right)^{\frac{1}{2}} - \left(\sum_{k\in H} |\langle f_n - f, T_K\phi\rangle|^2\right)^{\frac{1}{2}}$$

is true for any n. First inequality of (2) for f can be obtained easily since the lower bound in (2) holds in $span\,\{T_K\phi\}$ and hence holds for f_n. Also by inequality (4) the upper bound holds for all $f \in V$. Hence, we have

$$\left(\sum_{k\in H} |\langle f, T_K\phi\rangle|^2\right)^{\frac{1}{2}} \geq A^{\frac{1}{2}}\,\|f_n\|_2 - B^{\frac{1}{2}}\,\|f_n - f\|_2\,.$$

We obtain the lower bound of inequality (2) by taking the limit as n tending to ∞ on the right hand side of above inequality.

Lemma 3.2. *For* $\phi \in L^2(G)$ *and a finite set* $S \subset H$*, define* $f : \sum\limits_{k\in S} c_k, T_K\phi$ *and* $L_S(\gamma) := \sum\limits_{k\in S} c_k \langle -k, \gamma\rangle$*. Then* $f \in L^2(G)$*,* $L_S \in L^\infty\left(\hat{H}\right)$ *and*

$$\|f\|_2^2 = \int_{\hat{H}} |L_S(\gamma)|^2\, G_\phi(\gamma)\, d\gamma < +\infty \tag{5}$$

Proof. Using the Parseval's identity and H^1-periodicity of L_S we obtain,

$$\begin{aligned}
\|f\|_2^2 &= \left\langle \sum_{m\in S} c_m T_m\phi, \sum_{n\in S} c_n T_n\phi \right\rangle \\
&= \left\langle \sum_{m\in S} c_m \overline{T_m\phi}, \sum_{n\in S} c_n \overline{T_n\phi} \right\rangle \\
&= \left\langle L_S\hat{\phi}, L_S\hat{\phi} \right\rangle \\
&= \int_{\hat{G}} |L_S(\gamma)|^2\, |\hat{\phi}(\gamma)|^2\, d\gamma
\end{aligned}$$

$$= \int_{\hat{H}} \sum_{k \in H^1} |L_S(\gamma + k)|^2 \left|\hat{\phi}(\gamma + k)\right|^2 d\gamma$$
$$= \int_{\hat{H}} |L_S(\gamma)|^2 \sum_{k \in H^1} \left|\hat{\phi}(\gamma + k)\right|^2 d\gamma$$
$$= \int_{\hat{H}} |L_S(\gamma)|^2 G_\phi d\gamma$$

Since $L_S \in L^\infty\left(\hat{H}\right)$ and $G_\phi(\gamma) \in L^1\left(\hat{H}\right)$,

$$\|f\|_2^2 = \int_{\hat{H}} |L_F(\gamma)|^2 G_\phi(\gamma)\, d\gamma < \infty.$$

□

Lemma 3.3. *For $\phi \in L^2(G)$ and a finite subset $S \subset H$, define $f := \sum_{h \in S} c_h, T_H \phi$ and $L_S(\gamma) \equiv \sum_{h \in S} c_h \langle -h, \gamma \rangle$. If $G_\phi \in L^2\left(\hat{H}\right)$, then $f \in L^2(G)$, $L_S \in L^\infty\left(\hat{H}\right)$ and*

$$\sum_{h \in H} |\langle f, T_h \phi \rangle|^2 = \int_{\hat{H}} |L_S(\gamma)|^2 G_\phi(\gamma)^2 d\gamma < \infty \tag{6}$$

Proof. Let K be a finite subset of H. Then

$$\sum_{k \in K} |\langle f, T_k \phi \rangle|^2 = \sum_{k \in K} \langle f, T_k \phi \rangle \overline{\langle f, T_k \phi \rangle}$$
$$= \sum_{k \in K} \left\langle \sum_{h \in S} c_h, T_h \phi, T_k \phi \right\rangle \overline{\left\langle \sum_{u \in S} c_u, T_u \phi, T_k \phi \right\rangle}$$
$$= \sum_{k \in K} \sum_{h \in S} \sum_{u \in S} c_h \overline{c_u} \langle T_h \phi, T_k \phi \rangle \overline{\langle T_u \phi, T_k \phi \rangle}$$
$$= \sum_{k \in K} \sum_{h \in S} \sum_{u \in S} c_h \overline{c_u} \int_{\hat{G}} \left|\hat{\phi}(\gamma)\right|^2 \langle k - h, \gamma \rangle\, d\gamma \int_{\hat{G}} \left|\hat{\phi}(\lambda)\right|^2 \langle u - k, \lambda \rangle\, d\lambda$$
$$= \sum_{k \in K} \sum_{h \in S} \sum_{u \in S} c_h \overline{c_u} \int_{\hat{G}} \left|\hat{\phi}(\gamma)\right|^2 \langle -h, \gamma \rangle \int_{\hat{G}} \left|\hat{\phi}(\lambda)\right|^2 \langle u, \lambda \rangle \langle k, \gamma \bar{\lambda} \rangle\, d\lambda d\gamma$$
$$= \sum_{h \in S} \sum_{u \in S} c_h \overline{c_u} \int_{\hat{G}} \left|\hat{\phi}(\gamma)\right|^2 \langle -h, \gamma \rangle \int_{\hat{G}} \left|\hat{\phi}(\lambda)\right|^2 \langle u, \lambda \rangle \left(\sum_{k \in K} \langle k, \gamma \bar{\lambda} \rangle \right) d\lambda d\gamma$$
$$= \sum_{h \in S} \sum_{u \in S} c_h \overline{c_u} \int_{\hat{G}} \left|\hat{\phi}(\gamma)\right|^2 \langle -h, \gamma \rangle \int_{\hat{H}} G_\phi(\lambda) \langle u, \lambda \rangle D_K\left(\gamma \lambda^{-1}\right) d\lambda d\gamma$$

Now,

$$\int_{\hat{H}} G_\phi(\lambda) \langle u, \lambda \rangle D_K\left(\gamma \lambda^{-1}\right) d\lambda = \sum_{k \in K} \left(\overline{G_\phi, u}\right)(k) \langle k, \gamma \rangle$$

where

$$\left(\overline{G_\phi, u}\right)(k) = \int_{\hat{H}} G_\phi(\lambda) \langle u, \lambda \rangle (-k, \lambda)\, d\lambda$$

Therefore,

$$\begin{aligned}\sum_{k\in K}|\langle f,T_K\phi\rangle|^2 &= \sum_{h\in S}\sum_{u\in S}c_h\overline{c_u}\int_{\hat{G}}\left|\hat{\phi}(\gamma)\right|^2\langle -h,\gamma\rangle\, S_K\left(G_\phi,u\right)(\lambda)\,d\gamma\\ &= \sum_{h\in S}\sum_{u\in S}c_h\overline{c_u}\int_{\hat{H}}G_\phi(\gamma)\,(-h,\gamma)\,S_K\left(G_\phi u\right)(\gamma)\,d\gamma\end{aligned}$$

Now,

$$\left|\int_{\hat{H}}G_\phi(\gamma)\,(-h,\gamma)\,S_K\left(G_\phi u\right)(\gamma)-G_\phi(\gamma)\,\langle u,\gamma\rangle\,d\gamma\right|\leq\left\|G_\phi\right\|_{L^2(\hat{H})}\left\|S_K\left(G_\phi u\right)-G_\phi u\right\|_{L^2(\hat{H})}.$$

Therefore, as a consequence of Theorem 2, we get

$$\lim_{|K|\to\infty}\int_{\hat{H}}G_\phi(\gamma)\,\langle -h,\gamma\rangle\,S_K\left(G_\phi u\right)(\gamma)\,d\gamma\to\int_{\hat{H}}\left|G_\phi(\gamma)\right|^2\langle u-h,\gamma\rangle\,d\gamma,$$

where $|K|\to\infty$ we mean there exist natural numbers $n_1<n_2<\ldots$ and finite subsets $K_1\subseteq K_2\subseteq\ldots$ such that $|K_i|=n_i\to\infty$ as $i\to\infty$ and $U_iK_i=H$. Hence,

$$\begin{aligned}\sum_{k\in K}|\langle f,T_K\phi\rangle|^2 &= \sum_{h\in S}\sum_{u\in S}c_h\overline{c_u}\int_{\hat{H}}\left|G_\phi(\gamma)\right|^2\langle u-h,\gamma\rangle\,d\gamma\\ &= \int_{\hat{H}}\left|G_\phi(\gamma)\right|^2\sum_{h\in S}\sum_{u\in S}c_h\overline{c_u}\,\langle u-h,\gamma\rangle\,d\gamma\\ &= \int_{\hat{H}}\left|G_\phi(\gamma)\right|^2|L_S(\gamma)|^2\,d\gamma.\end{aligned}$$

□

Lemma 3.4. *Let $\phi\in L^2(G)$ be such that $G_\phi\in L^2\left(\hat{H}\right)$ and let $V\equiv\overline{span}\{T_K\phi:k\in H\}$ be a closed subspace of $L^2(G)$. Then the sequence $\{T_K\phi\}$ is a frame for V with frame bounds A and B if and only if for all trigonometric polynomials $L(\gamma)\equiv L_S(\gamma)\equiv\sum_{k\in S}c_k\langle -k,\gamma\rangle$ on $\hat{H}$,*

$$A\int_{\hat{H}}|L(\gamma)|^2G_\phi(\gamma)\,d\gamma\leq\int_{\hat{H}}|L(\gamma)|^2G_\phi(\gamma)^2\,d\gamma\leq B\int_{\hat{H}}|L(\gamma)|^2G_\phi(\gamma)\,d\gamma<+\infty \tag{7}$$

Proof. Suppose that $\{T_K(\phi)\}$ is frame for V. For a given L_S, define

$$f(\cdot):=\sum_{k\in S}c_kT_K\phi(\cdot)$$

In view of Lemma 3.2 and 3.3, we obtain

$$\|f\|_2^2=\int_{\hat{H}}|L(\gamma)|^2G_\phi(\gamma)\,d\gamma\ \text{ and} \tag{8}$$

$$\sum \left|\langle f, T_K\phi\rangle\right|^2 = \int\limits_{\hat{H}} \left|L\left(\gamma\right)\right|^2 G_\phi\left(\gamma\right)^2 d\gamma \tag{9}$$

Since $\{T_K(\phi)\}$ is a frame for V, there exist constants $A, B > 0$ such that for $g \in V$,

$$A \parallel g \parallel^2 \leq \sum \left|\langle g, T_K\phi\rangle\right|^2 \leq B \parallel g \parallel^2 < \infty \tag{10}$$

Replacing $g = f$ in (10) and using (8) and (9), we get (7).

Conversely, suppose that (7) is true for all trigonometric polynomials $L = L_S$. By Lemma 3.2 and Lemma 3.3 we get

$$A \parallel f \parallel^2 \leq \sum \left|\langle f, T_K\phi\rangle\right|^2 \leq B \parallel f \parallel^2$$

for all $f \quad span\{T_K\phi\}$ and hence the result follows from Lemma 3.1. Let $\phi \in L^2(G)$ and let $V = \overline{span}\{T_K\phi : k \in H\}$ be a closed subspace of $L^2(G)$. Then the sequence $\{T_K\phi\}$ is a frame for V if and only if there are positive constants A and B such that

$$A \leq G_\phi\left(\gamma\right) \leq B \ \text{ a.e. on } \ \hat{H}\backslash N, \tag{11}$$

where $N = \left\{\gamma \in \hat{H} : G_\phi\left(\gamma\right) = 0\right\}$.

Suppose that (11) holds and that $L\left(\gamma\right) = L_S\left(\gamma\right) = \sum\limits_{k\in S} c_k \left\langle -k, \gamma\right\rangle$ be a trigonometric polynomial on $\hat{H}$. It is easy to see that

$$\begin{aligned} A \int\limits_{\hat{H}} \left|L\left(\gamma\right)\right|^2 G_\phi\left(\gamma\right) d\gamma &= A \int\limits_{\hat{H}\backslash N} \left|L\left(\gamma\right)\right|^2 G_\phi\left(\gamma\right) d\gamma \\ &\leq \int\limits_{\hat{H}\backslash N} \left|L\left(\gamma\right)\right|^2 G_\phi\left(\gamma\right)^2 d\gamma \\ &= \int\limits_{\hat{H}\backslash N} \left|L\left(\gamma\right)\right|^2 G_\phi\left(\gamma\right)^2 d\gamma \\ &= B \int\limits_{\hat{H}\backslash N} \left|L\left(\gamma\right)\right|^2 G_\phi\left(\gamma\right) d\gamma. \end{aligned}$$

Hence, by Lemma 3.4, $\{T_K\phi\}$ forms a frame for V.

Conversely suppose that the sequence $\{T_K\phi\}$ is a frame for V. Then there exist positive constants A and B such that for $f \in V$,

$$A \parallel f \parallel^2 \leq \sum \left|\langle f, T_K\phi\rangle\right|^2 \leq B \parallel f \parallel^2 .$$

Hence, in view of Lemma 3.3 and Lemma 3.4 for $f = \sum\limits_{k\in S} c_k T_K\phi$, we obtain

$$\sum_{h\in H} \left|\langle f, T_h\phi\rangle\right|^2 = \int\limits_{\hat{H}} \left|L_S\left(\gamma\right)\right|^2 G_\phi\left(\gamma\right)^2 d\gamma$$

and

$$\|f\|_2^2 = \int\limits_{\hat{H}} |L_S(\gamma)|^2 G_\phi(\gamma)^2 \, d\gamma.$$

Assuming $G_\phi(\gamma) < A$ on $E \subseteq \hat{H} \backslash N$ for some measurable set E of positive Lebesgue measure, we shall obtain a contradiction. By using our assumption of G_ϕ we can choose $L \in L^\infty\left(\hat{H}\right) \subseteq L^2\left(\hat{H}\right)$, not necessarily a trigonometric polynomial such that $L = 0$ on E^c, $L > 0$ on E and

$$A \int\limits_{\hat{H}} |L(\gamma)|^2 G_\phi(\gamma) \, d\gamma > \int\limits_{\hat{H}} |L(\gamma)|^2 G_\phi(\gamma)^2 \, d\gamma.$$

Thus,

$$c = \int\limits_{\hat{H}} |L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma > 0 \tag{12}$$

We proceed to find a trigonometric polynomial Ψ so that (12) is true for Ψ. This will provide a contradiction to the inequality (7). Also if $G_\phi \in L^2\left(\hat{H}\right) \backslash L^\infty\left(\hat{H}\right)$, then (12) is still valid for $L \in L^\infty\left(\hat{H}\right)$. So, it is not required that G_ϕ must belong to $L^\infty\left(\hat{H}\right)$ to choose desired Ψ. For any $\Psi \in L^\infty\left(\hat{H}\right)$, we have

$$\begin{aligned} \int\limits_{\hat{H}} |\Psi(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma &= \int\limits_{\hat{H}\backslash E} |\Psi(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma \\ &+ \int\limits_{E} |\Psi(\gamma) - L(\gamma) + L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma \end{aligned} \tag{13}$$

Since, $A - G_\phi > 0$ on E and $G_\phi > 0$ a.e. on $\hat{H}\backslash N$, we have $G_\phi(A - G_\phi) > 0$ a.e. on E and we may consider $G_\phi(A - G_\phi)$ as a weight on a weighted L^2-space on E. Thus, with L, $\Psi \in L^\infty\left(\hat{H}\right)$, we have

$$\begin{aligned} \left(\int\limits_{E} |\Psi(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma\right)^{\frac{1}{2}} &\geq \left(\int\limits_{E} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma\right)^{\frac{1}{2}} \\ &- \left(\int\limits_{E} |L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma\right)^{\frac{1}{2}} \end{aligned} \tag{14}$$

Using equation (13) and (14) we get,

$$\begin{aligned} \int\limits_{\hat{H}} |\Psi(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma &\geq \int\limits_{\hat{H}\backslash E} |\Psi(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma \\ &+ \left[\left(\int\limits_{E} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma\right)^{\frac{1}{2}}\right. \\ &- \left.\left(\int\limits_{E} |L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2\right) d\gamma\right)^{\frac{1}{2}}\right]^2 \end{aligned}$$

$$\geq c - 2c^{\frac{1}{2}} \left(\int_{E} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2 \right) d\gamma \right)^{\frac{1}{2}} + \int_{\hat{H}} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2 \right) d\gamma \tag{15}$$

Assuming $G_\phi \in L^\infty(\hat{H})$ and using a simple estimate we get,

$$\left| 2c^{\frac{1}{2}} \left(\int_{E} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2 \right) d\gamma \right)^{\frac{1}{2}} - \int_{\hat{H}} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2 \right) d\gamma \right| \leq K \left(\|\Psi - L\|_{L^2(\hat{H})} + \|\Psi - L\|^2_{L^2(\hat{H})} \right), \tag{16}$$

where, $K = \max \left(2c^{\frac{1}{2}} \|G_\phi (A - G_\phi)\|^{\frac{1}{2}}_{L^\infty(\hat{H})}, \|G_\phi (A - G_\phi)\|_{L^\infty(\hat{H})} \right)$. Since, the trigonometric polynomials are dense in $L^2(\hat{H})$ we can choose a trigonometric polynomial Ψ such that $\|\Psi - L\|_{L^2(\hat{H})} \leq L$, where $L = \min\left(1, \frac{c}{4K}\right)$. Thus,

$$K \left(\|\Psi - L\|_{L^2(\hat{H})} + \|\Psi - L\|^2_{L^2(\hat{H})} \right) \leq \frac{c}{2} \tag{17}$$

Therefore we can conclude from (15), (16) and (17) that

$$\int_{\hat{H}} |\Psi(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2 \right) d\gamma \geq \frac{c}{2}.$$

Similarly, if $G_\phi \in L^2(\hat{H})$, we can choose a trigonometric polynomial Ψ such that

$$\left| 2c^{\frac{1}{2}} \left(\int_{E} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2 \right) d\gamma \right)^{\frac{1}{2}} - \int_{\hat{H}} |\Psi(\gamma) - L(\gamma)|^2 \left(AG_\phi(\gamma) - G_\phi(\gamma)^2 \right) d\gamma \right| < \frac{c}{2}.$$

□

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