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Mathematical Expressions of Heat Transfer to MHD Oscillatory Flow and Homotopy Analysis Method

Research Article

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Abstract: In this research paper we determine the combined effect of a transverse magnetic field and radiative heat transfer to unsteady flow of a conducting optically thin fluid through a channel filled with saturated porous medium and non-uniform walls temperature. The governing dimensionless equations are solved analytically and graphically. The approximate analytical expressions for the dimensionless axial velocity and temperature are obtained using the Homotopy analysis method. Also the analytical expressions for the shear stress function and the rate of heat transfer are also derived. The Homotopy analysis method has proven to be the best for solving complicated non-linear differential equations in MHD flow problems in engineering and applied sciences.

Keywords: Oscillatory flow; Porous medium; Heat transfer; Magnetic field; Homotopy analysis method.

1. Introduction

In recent years, the flow of fluids through porous media has evolved to be of great importance due to its applications in geophysics, engineering, thermal insulation, heat storage and for the recovery of crude oil from the pores of the reservoir rocks; in this case, Darcy's law represents the gross effect. The study of the flow of an electrically conducting fluid named as magnetohydrodynamics (MHD) finds many applications in engineering problems such as MHD power generators, plasma studies, nuclear reactors, geothermal energy extraction and the boundary layer control in the field of aerodynamics [1]. Moreau [2] contains a survey of MHD studies in the technological fields. Motion of Newtonian fluids in the presence of magnetic field has varied applications in many areas including the handling of biological fluids and flow of nuclear fuel slurries, liquid metals and alloys, plasma, mercury amalgams, and blood ([3], [4] and [5]). Another important field of application can be seen in electromagnetic propulsion. Basically, an electro-magnetic propulsion system consists of a power source (such as nuclear reactor), plasma and tube through which the plasma is accelerated by electro-magnetic forces. The study of such systems, which is closely associated with magneto-chemistry, requires a complete understanding of the equation of state and transport properties such as diffusion, the shear stress- shear rate relationship, thermal conductivity, electrical conductivity and radiation. Some of these properties will undoubtedly be influenced by the presence of an external magnetic field that sets plasma in hydrodynamic motion ([6] and [7]).

Meanwhile, there has been many tremendous applications in studying about the magnetohydrodynamic (MHD) flow and heat transfer in porous media because of the effect of magnetic fields on the performance of many systems [8]. Due to the

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increase in applications of free convective and heat transfer flows through porous medium under the influence of magnetic field many researchers have studied about the magnetohydrodynamic free convective heat transfer flow in a porous medium. In Raptis et al. [9] we can study about the unsteady free convective motion through a porous medium bounded by an infinite vertical plate. Ram and Mishra [10] analyzed unsteady flow through MHD porous media. Mansutti [11] have discussed about the steady flow of non- Newtonian fluids past a porous plate with suction or injection.

MHD unsteady free convection Walter's memory flow with constant suction and heat sink was investigated in [12]. In [13] the unsteady MHD memory flow with oscillatory suction, variable free stream and heat sources have been investigated. El Hakiem [14] analyzed about the thermal radiation effects on transient, two-dimensional hydromagnetic free convection along a vertical surface in a highly porous medium using the Roseland diffusion approximation for the radiative heat flux in the energy equation, for the case where free- stream velocity of the fluid vibrates about mean constant value and the surface absorbs the fluid with constant velocity. Aldoss et al. [15] have carried out a study about mixed convection flow from a vertical plate embedded in a porous medium in the presence of a magnetic field. Chamkha [16] has considered MHD free convectional flow from a vertical plate embedded in a thermally stratified porous medium with Hall effects.

In the present paper, we are going to discuss about the method of Homotopy analysis for studying the adverse effects of transverse magnetic field and radiative heat transfer on unsteady flow of a conducting optically thin fluid through a channel filled with saturated porous medium and non-uniform walls temperature.

2. Mathematical Formulation of the Problem

Under the influence of an externally applied homogeneous magnetic field and radioactive heat transfer the flow of a conducting optically thin fluid in a channel filled with saturated porous medium is considered as shown in Fig.1.

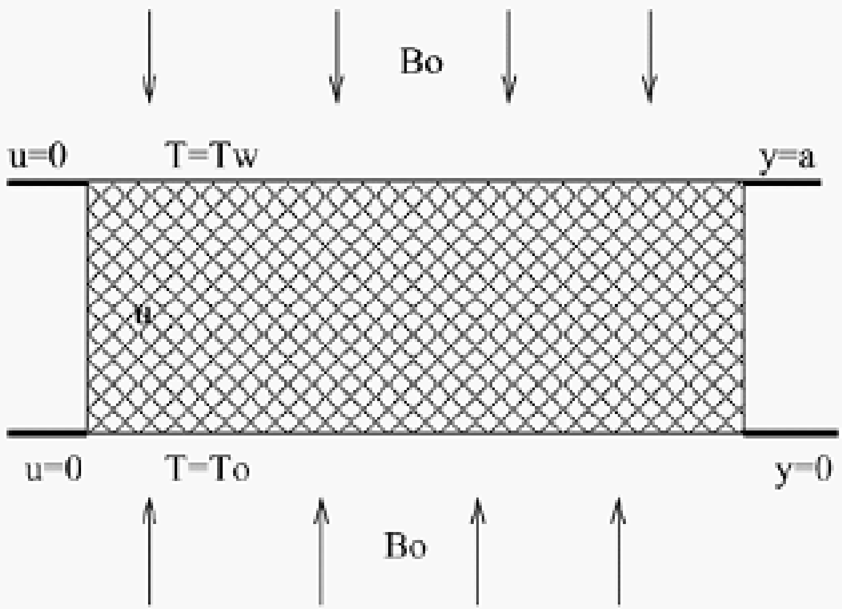

Fig. 1. – Geometry of the problem.

It is assumed that the fluid has a small electrical conductivity and thus the electromagnetic force produced is also very small. Consider a Cartesian coordinate system (x, y) where x lies along the centre of the channel, y is the distance measured in the normal section. Now using the Boussinesq incompressible fluid model, the governing equations of the fluid motion is obtained as:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\frac{\partial^2 u}{\partial y^2} - \frac{v}{K}u - \frac{\sigma_e B_0^2}{\rho}u + g\beta(T - T_0), \tag{1}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p}\frac{\partial q}{\partial y} \tag{2}$$

with the corresponding boundary conditions are as follows

$$u = 0, \quad T = T_w, \quad on \quad y = 1 \tag{3}$$

$$u = 0, \quad T = T_0, \quad on \quad y = 0 \tag{4}$$

where u denotes the axial velocity, t denotes time, T is the fluid temperature, P is the pressure, g the gravitational force, q the radiative heat flux, β the coefficient of volume expansion due to temperature, c_p the specific heat at constant pressure, k the thermal conductivity, K the porous medium permeability coefficient, $B_0 = (\mu_e H_0)$ the electromagnetic induction, μ_e the magnetic permeability, H_0 the intensity of magnetic field, σ_e the conductivity of the fluid, ρ the fluid density and v is the kinematic viscosity coefficient. It is assumed that both walls temperatureT_0, T_w are high enough to induce radiative heat transfer. It is assumed that the fluid is optically thin with a relatively low density and the radiative heat flux following Cogley et al is given as follows:

$$\frac{\partial q}{\partial y} = 4\alpha^2 (T_0 - T_w) \tag{5}$$

where α is the mean radiation absorption coefficient. We introduce the following dimensionless variables and parameters:

$$\begin{array}{l} Re = \frac{Ua}{v}\,,\ \bar{x} = \frac{x}{a}\,,\ \bar{y} = \frac{y}{a}\,,\ \bar{u} = \frac{u}{U}\,,\ \theta = \frac{T-T_0}{T_w-T_0}\,,\ H^2 = \frac{a^2\sigma_e B_0^2}{\rho v}\,,\ \bar{t} = \frac{tU}{a}, \\ \bar{P} = \frac{aP}{\rho v U}\,,\ Da = \frac{K}{a^2}\,,\ Gr = \frac{g\beta(T_w-T_0)a^2}{vU}\,,\ Pe = \frac{Ua\rho c_p}{k}\,,\ N^2 = \frac{4\alpha^2 a^2}{k} \end{array} \tag{6}$$

Where U denotes the flow mean velocity. The dimensionless governing equations together with the appropriate boundary conditions can be written as follows:

$$Re\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \left(s^2 + H^2\right)u + Gr\theta \tag{7}$$

$$Pe\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} + N^2\theta \tag{8}$$

$$u = 0\,,\quad \theta = 1\,,\quad on \quad y = 1, \tag{9}$$

$$u = 0\,,\quad \theta = 0\,,\quad on \quad y = 0 \tag{10}$$

where $Gr\,,\ H\,,\ N\,,\ Pe\,,\ Re\,,\ Da\,,\ s = \left(\frac{1}{Da}\right)$ are Grashoff number, Hartmann number, Radiation parameter, Peclet number, Reynolds number, Darcy number and porous medium shape factor parameter respectively.

Here we first treat the energy equation separately from the momentum equation so that the Equations (7)-(10) can be written as follows:

$$0 = \lambda + \frac{d^2 u}{dy^2} - \left(s^2 + H^2\right)u + Gr\,\theta \tag{11}$$

$$0 = \frac{d^2\theta}{dy^2} + N^2\theta \tag{12}$$

With the following boundary conditions

$$u = 0\,,\quad \theta = 1\,,\quad on \quad y = 1, \tag{13}$$

$$u = 0\,,\quad \theta = 0\,,\quad on \quad y = 0 \tag{14}$$

3. Analytical Expression for the non-linear Differential Equations (11) - (14) using Homotopy Analysis Method

This paper deals with a basic strong analytic tool for nonlinear problems, namely the Homotopy analysis method (HAM) which was generated by Liao [18], is employed to solve the nonlinear differential eqns. (11) - (14). The Homotopy analysis method is based on a basic concept in topology, i.e. Homotopy by Hilton [19] which is widely applied in numerical techniques as in [20–23]. Unlike perturbation techniques like [24], the Homotopy analysis method is independent of the small/large parameters. Unlike all other reported perturbation and non-perturbation techniques such as the artificial small parameter method [25], the δ-expansion method [26] and Adomian's decomposition method [27], the Homotopy analysis method provides us a simple way to adjust and control the convergence region and rate of approximation series. The Homotopy analysis method has been successfully applied to many nonlinear problems such as heat transfer [28], viscous flows [29], nonlinear oscillations [30], Thomas-Fermi's atom model [31], nonlinear water waves [32], etc. Such varied successful applications of the Homotopy analysis method confirm its validity for nonlinear problems in science and engineering. The Homotopy analysis method is a good technique when compared to other perturbation methods. The existence of the auxiliary parameter h in the Homotopy analysis method provides us with a simple way to adjust and control the convergence region of the solution series.

In this paper we have used the Homotopy analysis method for the non-linear boundary value problem which is expressed in the Equations (11) and (12) with the boundary conditions (13) and (14). And we have obtained the approximate analytical expression for the dimensionless axial velocity $u\left(y\right)$ and dimensionless Temperature $\theta\left(y\right)$ (see Appendix B) as follows:

$$\begin{aligned} u(y) = {} & A\,\exp\left(-\sqrt{s^2+H^2}\,y\right) + B\,\exp\left(\sqrt{s^2+H^2}\,y\right) + \tfrac{\lambda}{s^2+H^2} \\ & -h\,\left(C_1\,\exp\left(-\sqrt{s^2+H^2}\,y\right) + C_2\,\exp\left(\sqrt{s^2+H^2}\,y\right) + \tfrac{G\,\sin(N\,y)}{\sin(N)(N^2+s^2+H^2)}\right) \end{aligned} \tag{15}$$

where

$$A = -\frac{\lambda}{s^2+H^2} - \frac{\lambda\left(1-\exp\left(-\sqrt{s^2+H^2}\,\right)\right)}{(s^2+H^2)\left(\exp\left(-\sqrt{s^2+H^2}\,\right)-\exp\left(\sqrt{s^2+H^2}\,\right)\right)} \tag{16}$$

$$B = \frac{\lambda\left(1-\exp\left(-\sqrt{s^2+H^2}\,\right)\right)}{(s^2+H^2)\left(\exp\left(-\sqrt{s^2+H^2}\,\right)-\exp\left(\sqrt{s^2+H^2}\,\right)\right)} \tag{17}$$

$$C_1 = -\frac{G}{(N^2+s^2+H^2)\left(\exp\left(-\sqrt{s^2+H^2}\right)-\exp\left(\sqrt{s^2+H^2}\right)\right)} \tag{18}$$

$$C_2 = \frac{G}{(N^2+s^2+H^2)\left(\exp\left(-\sqrt{s^2+H^2}\right)-\exp\left(\sqrt{s^2+H^2}\right)\right)} \quad \text{and} \tag{19}$$

$$\theta(y) = \frac{\sin(N\,y)}{\sin\left(N\right)} \tag{20}$$

The analytical expression for the shear stress function is given by

$$\left(\frac{du}{dy}\right)_{y=0} = \frac{-\begin{pmatrix} 2s^2hG\sin\left(N\right)\exp\left(-\sqrt{s^2+H^2}\right)+s^2\lambda\sin\left(N\right)-2s^2\lambda\sin\left(N\right)\exp\left(-\sqrt{s^2+H^2}\right) \\ +s^2\lambda\sin\left(N\right)\exp\left(-2\sqrt{s^2+H^2}\right)+2hG\sin\left(N\right)H^2\exp\left(-\sqrt{s^2+H^2}\right) \\ -2\lambda\sin\left(N\right)H^2\exp\left(-\sqrt{s^2+H^2}\right)+\lambda\sin\left(N\right)H^2\exp\left(-2\sqrt{s^2+H^2}\right) \\ -2\lambda\sin\left(N\right)N^2\exp\left(-\sqrt{s^2+H^2}\right)-hG\sqrt{s^2+H^2}N+\lambda\sin\left(N\right)N^2 \\ +\lambda\sin\left(N\right)N^2\exp\left(-2\sqrt{s^2+H^2}\right) \\ +\lambda\sin\left(N\right)H+hGN\sqrt{s^2+H^2}\exp\left(-2\sqrt{s^2+H^2}\right) \end{pmatrix}}{\begin{pmatrix} \exp\left(-2\sqrt{s^2+H^2}\right)N^2+\exp\left(-2\sqrt{s^2+H^2}\right)s^2+ \\ \exp\left(-2\sqrt{s^2+H^2}\right)H^2-N^2-H^2-s^2 \end{pmatrix}\sin\left(N\right)\sqrt{s^2+H^2}} \tag{21}$$

The rate of heat transfer across the channel's wall is given by

$$\left(\frac{d\theta}{dy}\right)_{y=0} = \frac{N}{\sin(N)} \tag{22}$$

4. Results and Discussion

The analytical expression for the non-linear MHD oscillatory flow in a channel filled with porous medium has been obtained using Homotopy analysis method. Figure 1 show the geometry of the problem which can be seen in section 2. In HAM the selection of suitable values of h is very crucial to make our solution accurate. After choosing the appropriate values of the auxiliary parameter h we have plotted the velocity profiles for various values of the radiation parameter and also for various values of the Hartmann number i.e. the magnetic field intensity which are depicted in Figure 2 and Figure 3. We have made use of the following parameters,$s = 1$, $H = 1$, $\lambda = 1$, $G = 1$, $N = 1$ as fixed parameters wherever necessary.

From the plotted velocity profiles Figure 2 and Figure 3 it can be noticed that the fluid velocity profile curves are parabolic with maximum magnitude along the centerline channel and minimum at the walls. The influence of radiation parameter and the Hartman number on the velocity profiles are studied as, the magnitude of fluid velocity increases with an increase in radiation parameter and decreases with an increase in Hartmann number i.e. magnetic field intensity. Likewise we have also studied the varied influence of the Grashoff number G, constant λ and the porous medium shape factor s on the fluid axial velocity$u(y)$. Figure 4 and Figure 5 depicts that the fluid axial velocity increases with an increase in the Grashoff number$G$ and the constant λ. Figure 6 reveals the decreasing nature of the fluid velocity with increasing porous medium shape factor values s.

In Figure 7, we have plotted the temperature profile, from which we can infer that a general increase in the fluid temperature occurs with an increase in radiation parameter through the absorption of heat. Figure 8 shows the distribution of wall shear stress. The effect of increasing values of the magnetic field intensity H is to decrease the magnitude of the wall shear stress while an increase in the radiation parameter through heat absorption causes a further increase in the magnitude of wall shear stress.

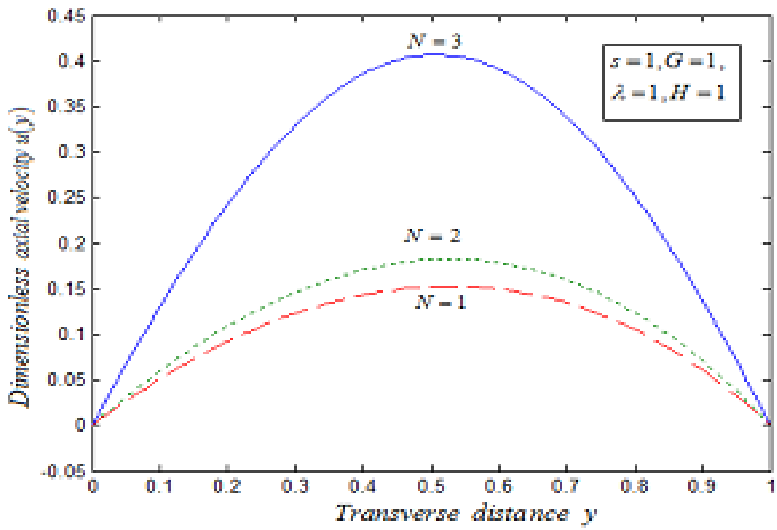

Fig.2 Dimensionless axial velocity $u(y)$ versus the dimensionless transverse distance$y$. The curves are plotted for various values of the radiation parameter N with some fixed values of$s$, H , λ and G in the Equation (15).

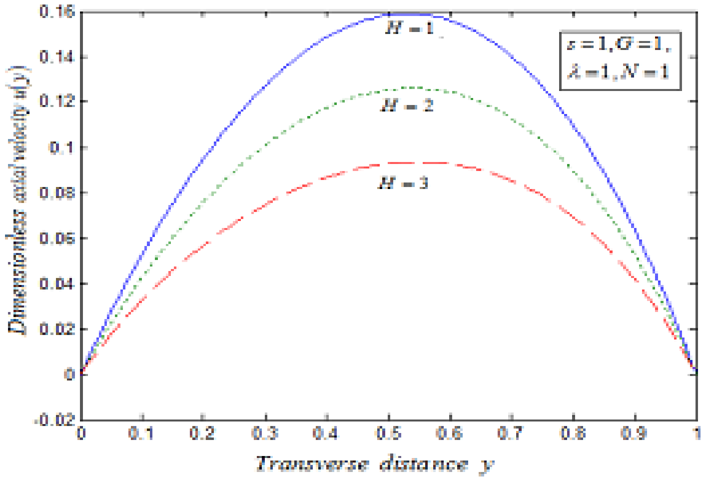

Fig.3 Dimensionless axial velocity $u(y)$ versus the dimensionless transverse distance y. The curves are plotted for various values of the radiation parameter H with some fixed values for s, N, λ and G in the Equation (15).

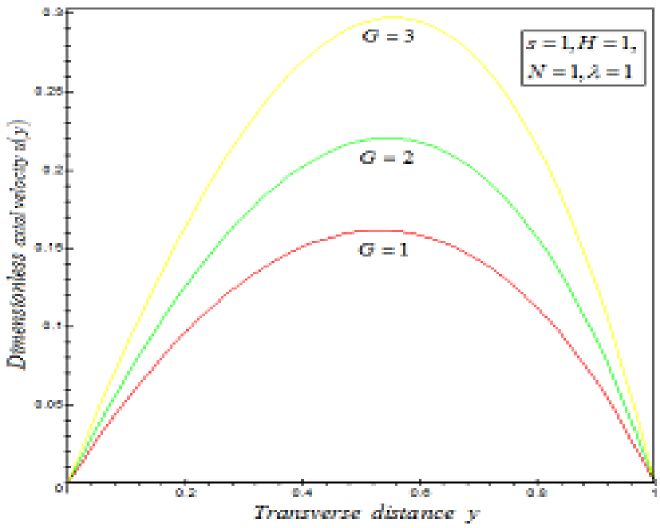

Fig.4. Dimensionless axial velocity $u(y)$ versus the dimensionless transverse distance y. The curves are plotted for various values of the Grashoff number G with some fixed values for s, H, λ and N in the Equation (15).

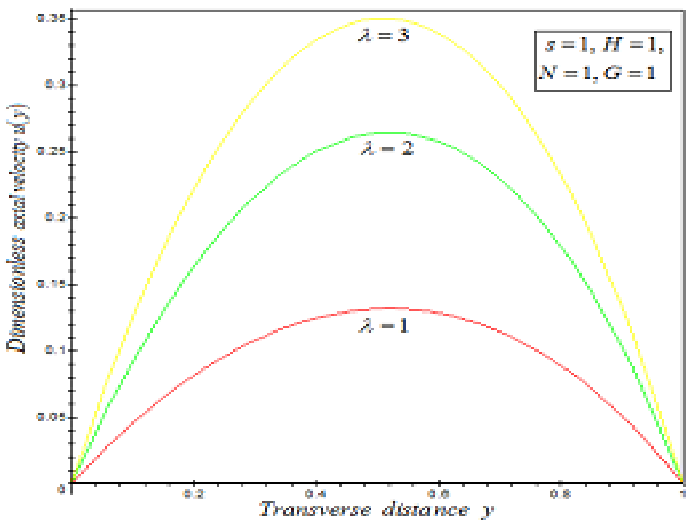

Fig.5. Dimensionless axial velocity $u(y)$ versus the dimensionless transverse distance y. The curves are plotted for various values of the constant $\lambda$with some fixed values for s, H, G and N in the Equation (15).

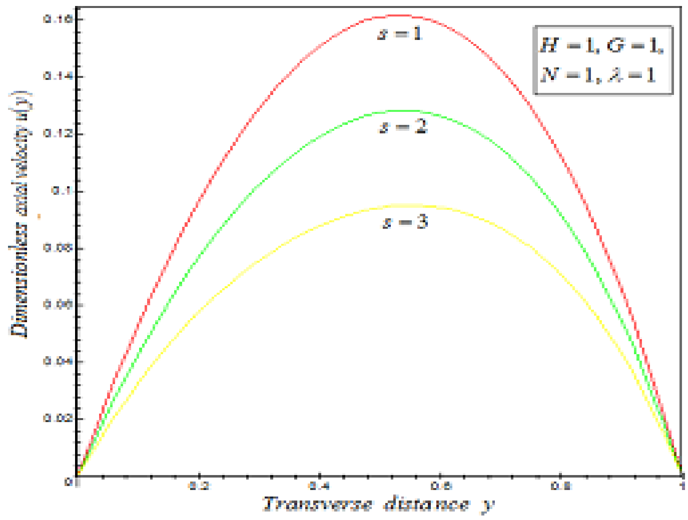

Fig.6. Dimensionless axial velocity $u(y)$ versus the dimensionless transverse distance y. The curves are plotted for various values of the porous medium shape factor s with some fixed values for H, G, λ *and* N in the Equation (15).

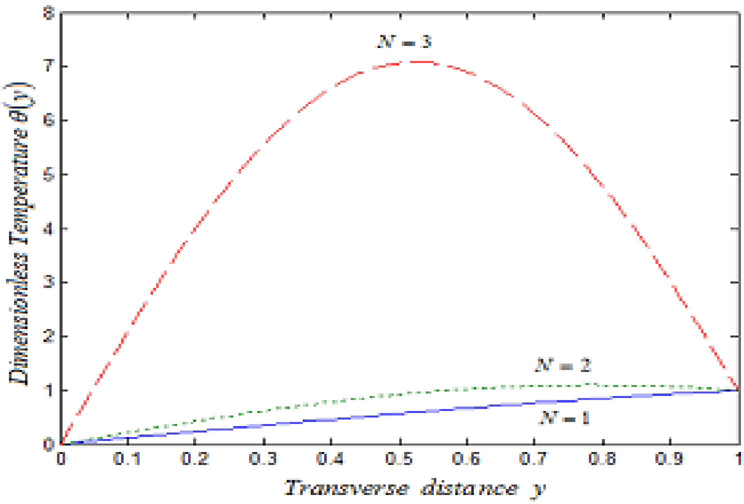

Fig.7. Dimensionless temperature $\theta(y)$ versus the dimensionless transverse distance y. The curves are plotted for various values of the radiation parameter N for the Equation (20)

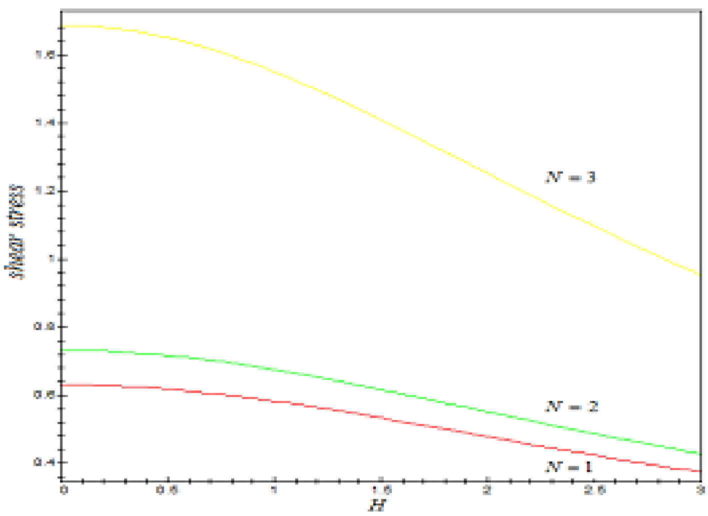

Fig.8. Shear stresses versus the Hartmann number i.e. the magnetic field intensity for various values of the radiation parameter are plotted using Equation (21).

5. Conclusion

This research paper investigates the combined effects of transverse magnetic field and radiative heat transfer to MHD oscillatory flow in a channel filled with porous medium. Using Homotopy analysis method we have obtained the analytical solutions for the dimensionless non-linear problem. The velocity and temperature profiles are obtained analytically depicting the adverse effects of radiation parameter, Hartmann number, constantλ, porous medium shape factor and the Grashoff number.

From the plotted velocity profiles thus obtained we learnt that the fluid velocity is directly proportional to the Radiation parameter N, constant λ and the Grashoff number G whereas it is inversely proportional to the Magnetic field intensity i.e. Hartmann number$H$ and also to the porous medium shape factor s. Also from our results formed we also infer that the temperature increases with increasing radiation parameter. However, the fluid velocity also increases with increasing radiation parameter but decreases with increasing Hartmann number. Also we have computed the wall shear stress and rate of heat transfer at the channel walls. Generally, our result from the plot of shear stress function shows that increasing magnetic field intensity reduces wall shear stress while increasing radiation parameter through heat absorption causes an increase in the magnitude of wall shear stress.

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References

[1] Y.J.Kim, *Unsteady MHD Convective Heat transfer past a semi-infinite vertical plate*, Int. J. Eng. Sci., 38(2000), 833-845.

[2] R.Moreau, *Magnetohydrodynamics*, Kulwer Academic Publishers, Dordrecht, (1990).

[3] O.D.Makinde, *MHD Steady flow and heat transfer on the sliding plate*, A.M.S.E., Modelling, Measurement and Control, 70(1)(2001), 61-70.

[4] O.D.Makinde, *Magneto-hydrodynamic stability of plane-Poiseuille flow using multideck asymptotic technique*, Mathematical and Computer Modelling, 37(3-4)(2003), 251-259.

[5] A.R.Rao and K.S.Deshikachar, *Asymptotic solution for steady streaming MHD flow in a channel of a variable cross section*, Int. J. Engng. Sci., 24(10)(1986), 1628.

[6] T.Hayat and Y.Wang, *Magnetohydrodynamic flow due to monoaxial rotations of a porous disk and a fourth-grade fluid at infinity*, Math. Probl., 2(2003), 47-64.

[7] O.D.Makinde, *Heat transfer to MHD Oscillatory flow in a permeable channel*, A.M.S.E., Modelling, Measurement and Control, 70(2)(2001), 23-29.

[8] O.D.Makinde and P.Sibanda, *Wall driven steady flow and heat transfer in a porous tube*, Comp. Assist. Mech. Eng. Sc., 5(1998), 389-398.

[9] A.Raptis and Perdikis, *Unsteady free convective through a porous medium bounded by an infinite vertical plate*, International Journal of Engineering Sciences, 23(1985), 99-105.

[10] Ram, Gulab and R.S.Mishra, *Unsteady flow through MHD porous media.* Indian J. Pure Applied Mathematics, 8(6)(1976), 637-647.

[11] D.Mansutti, G.Pontrelli and KR.Rajagopal, *Steady flows of non-Newtonian fluids past a porous plate with suction or injection*, International Journal Num. Method Fluids, 17(1993), 929-941.

[12] Ramana, M.V.Murthy, G.Noushima Humera, Rafi'uddin and M.Chenna Krishna Reddy, *MHD Unsteady free convective Walter's memory flow with constant suction and heat Sink*, ARPN Journal of Engineering and Applied Sciences, 2(5)(2007), 12-16.

[13] S.Mustafa, Rafi'uddin and M.V.Ramana Murthy, *Unsteady MHD Memory Flow with Oscillatory suction, Variable free stream and heat source*, ARPN Journal of Engineering and Applied Sciences, 3(3)(2008), 17-24.

[14] M.A.El-Hakiem, *MHD Oscillatory Flow on Free Convection-radiation through a Porous medium with constant suction velocity*, Journal of Magnetism and Magnet Materials, 220(2-3)(2000), 271-276.

[15] T.K.Aldoss, M.A.Al-Nimr, M.A.Jarrah and B.Al-Shaer, *Mixed convection flow from a vertical plate embedded in porous medium in the presence of magnetic field*, Heat Transfer, 28A(1995), 635-645.

[16] A.J.Chamkha, *Non-similar solutions for heat and mass transfer by hydromagnetic mixed convection flow over a plate in porous media with surface suction or injection*, Int. J. Numer. Meth. Heat Fluid Flow, 10(2000), 142-162.

[17] A.C.L.Cogley, W.G.Vinvent and E.S.Giles, *Free Convection effects and radiative heat transfer in MHD stokes problem for the flow of dusty conducting fluid through porous medium*, American Institue of Aeronautics and Astronautics, (1968).

[18] S.J.Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, PhD thesis, Shanghai Jiao Tong University, (1992).

[19] P.J.Hilton, *An introduction to homotopy theory*, Cambridge University Press, (1953).

[20] J.C.Alexander and J.A.Yorke, *The homotopy continuation method: numerically implementable topological procedures*, Trans. Am. Math. Soc., 242(1978), 271-284.

[21] T.F.C.Chan and H.B.Keller, *Arc-length continuation and multi-grid techniques for non-linear elliptic eigenvalue problems*, SIAM J. Sci. Statist. Comput., 3(1982), 173-193.

[22] N.Dinar and H.B.Keller, *Computations of taylor vortex flows using multigrid continuation methods*, Tech. Rep. California Institute of Technology, (1985).

[23] E.E.Grigolyuk and V.I.Shalashilin, *Problems of Nonlinear Deformation: The Continuation Method Applied to Nonlinear Problems in Solid Mechanics*, Kluwer, (1991).

[24] J.D.Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell, (1958).

[25] A.M.Lyapunov, *General problem on stability of motion*, English Transl. Taylor & Francis, London, (1992).

[26] A.V.Karmishin, A.I.Zhukov and V.G.Kolosov, *Methods of dynamics calculation and testing for thin-walled structures*, Mashinostroyenie, Moscow (in Russian), (1990).

[27] G.Adomian, *Nonlinear stochastic differential equations*, J. Math. Anal. Applic., 55(1996), 441-452.

[28] S.J.Liao and A.Campo, *Analytic solutions of the temperature distribution in Blasius viscous flow problems*. J. Fluid Mech., 453(2002), 411-425.

[29] S.J.Liao, *An explicit, totally analytic approximation of Blasius viscous flow problems*, Intl J. Non-Linear Mech., 34(1999), 759-778.

[30] S.J.Liao and A.T.Chwang, *Application of homotopy analysis method in nonlinear oscillations*, Trans. ASME: J. Appl. Mech., 65(1998) 914-922.

[31] S.J.Liao, *An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude*, Intl J. Non-Linear Mech., 38(2003), 1173-1183.

[32] S.J.Liao and K.F.Cheung, *Homotopy analysis of nonlinear progressive waves in deep water*, J. Engng Maths, 45(2003),

105-116.

[33] S.J.Liao, *An analytic approximation of the drag coefficient for the viscous flow past a sphere*, Intl J. Non-Linear Mech., 37(2002), 1-18.

[34] S.J.Liao, *The Homotopy Analysis method in non-linear differential equations*, Springer and Higher Educattion Press, (2012).

[35] S.J.Liao, *Beyond Perturbation introduction to the Homotopy analysis method*, First Edition, Chapman and Hall, CRC press, Boca Raton, (2003).

[36] M.Subha, V.Ananthaswamy and L.Rajendran, *Analytical solution of non-linear boundary value problem for the electro-hydrodynamic flow equation*, International Jouranal of Automation and Control Engineering, 3(2)(2014), 48-56.

[37] K.Saravanakumar, V.Ananthaswamy, M.Subha and L.Rajendran, *Analytical Solution of nonlinear boundary value problem for in efficiency of convective straight Fins with temperature-dependent thermal conductivity*, ISRN Thermodynamics, Article ID 282481, (2013), 1-18.

[38] V.Ananthaswamy and M.Subha, *Analytical expressions for exothermic explosions in a slab*, International Journal of Research Granthaalayah, 1(2)(2014), 22-37.

[39] V.Ananthaswamy and S.Uma Maheswari, *Analytical expression for the hydrodynamic fluid flow through a porous medium*, International Journal of Automation and Control Engineering, 4(2)(2015), 67-76.

[40] V.Ananthaswamy and L.Sahaya Amalraj, *Thermal stability analysis of reactive hydromagnetic third-grade fluid using Homotopy analysis method*, International Journal of Modern Mathematical Sciences, 14(1)(2016), 25-41.

[41] V.Ananthaswamy and T.Iswarya, *Analytical expressions of mass transfer effects on unsteady flow past an accelerated vertical porous plate with suction*, Nonlinear studies, 23(1)(2016), 73-86.

[42] V.Ananthaswamy and T.Iswarya, *Analytical expressions of the effect of radiation on free convective flow of heat and mass transfer*, Nonlinear studies, 23(1)(2016), 133-147.

Appendix: A

Basic Concept of the Homotopy Analysis Method [18–42]

Consider the following differential equation:

$$N[u(t)] = 0 \tag{A.1}$$

Where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao [33] constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p) - u_0(t)] = phH(t)N[\varphi(t;p)] \tag{A.2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0\ (t)$ is an initial guess of $u(t)$, $\varphi(t:p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p = 0$ and $p = 1$, it holds:

$$\varphi(t;0) = u_0(t) \ \ textand \ \ \varphi(t;1) = u(t) \tag{A.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u\ (t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)p^m \tag{A.4}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \varphi(t;p)}{\partial p^m}|_{p=0} \tag{A.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (A.4) converges at p =1 then we have:

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t). \tag{A.6}$$

Differentiating (A.2) for m times with respect to the embedding parameter p, and then setting $p = 0$ and finally dividing them by $m!$, we will have the so-called m th-order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\Re_m(\underset{m-1}{\vec{u}}) \tag{A.7}$$

where

$$\Re_m(\underset{m-1}{\vec{u}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(t;p)]}{\partial p^{m-1}} \tag{A.8}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{A.9}$$

Applying L^{-1} on both side of equation (A7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\vec{u_{m-1}})] \tag{A10}$$

In this way, it is easily to obtain u_m for $m \geq 1$, at M^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \tag{A.11}$$

When $M \to +\infty$, we get an accurate approximation of the original equation (A.1). For the convergence of the above method we refer the reader to Liao [34]. If equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix: B

Solution of the Non-linear Differential Equations (11) - (14) using the Homotopy Analysis Method

This appendix contains the derivation of the analytical expressions Equations (15) and (20) for $u(y)$ and $\theta(y)$using the Homotopy analysis method. The Equations (11) - (14) are as follows,

$$0 = \lambda + \frac{d^2u}{dy^2} - \left(s^2 + H^2\right) u + Gr\,\theta \tag{B.1}$$

$$0 = \frac{d^2\theta}{dy^2} + N^2\theta \tag{B.2}$$

With the following boundary conditions

$$u = 0\,, \quad \theta = 1\,, \quad on \quad y = 1, \tag{B.3}$$

$$u = 0\,, \quad \theta = 0\,, \quad on \quad y = 0 \tag{B.4}$$

Solving the Equations (B.2), (B.3) and (B.4) we obtain the exact solution for $\theta(y)$ as

$$\theta(y) = \frac{\sin(N y)}{\sin(N)} \tag{B.5}$$

We construct the Homotopy for the Equation (B.1) as follows,

$$(1-p)\left(\frac{d^2u}{dy^2} - \left(s^2+H^2\right)u + \lambda\right) = hp\left(\frac{d^2u}{dy^2} - \left(s^2+H^2\right)u + \lambda + G\theta\right) \tag{B.6}$$

The approximate solution for the Equation (B.6) is given by,

$$u = u_0 + pu_1 + p^2u_2 + \dots \tag{B.7}$$

By substituting the Equation (B.7) into the Equation (B.6) we get,

$$\begin{aligned}(1-p)&\left(\frac{d^2\left(u_0+pu_1+p^2u_2+\dots\dots.\right)}{dy^2} - \left(s^2+H^2\right)\left(u_0+pu_1+p^2u_2+\dots\dots.\right)+\lambda\right)\\ &= hp\left(\frac{d^2\left(u_0+pu_1+p^2u_2+\dots\dots.\right)}{dy^2} - \left(s^2+H^2\right)\left(u_0+pu_1+p^2u_2+\dots\dots.\right)+\lambda+G\theta\right)\end{aligned} \tag{B.8}$$

Now equating the coefficients of p^0 and p^1 we get the following equations:

$$p^0: \quad \frac{d^2u_0}{dy^2} - \left(s^2+H^2\right)u_0 + \lambda \tag{B.9}$$

$$p^1: \quad \frac{d^2u_1}{dy^2} - \left(s^2+H^2\right)u_1 - \frac{d^2u_0}{dy^2} + \left(s^2+H^2\right)u_0 - \lambda - h\left(\frac{d^2u_0}{dy^2} - \left(s^2+H^2\right)u_0 + \lambda + G\theta\right) \tag{B.10}$$

The initial approximations are as follows:

$$u_0(0) = 0\ ,\ u_0(1) = 0 \tag{B.11}$$

$$u_1(0) = 0\ ,\ u_1(1) = 0 \tag{B.12}$$

Solving (B.9) and (B.11) we obtain the initial solution $u_0$as,

$$u_0 = A\exp\left(-\sqrt{s^2+H^2}\,y\right) + B\exp\left(\sqrt{s^2+H^2}\,y\right) + \frac{\lambda}{s^2+H^2} \tag{B.13}$$

where

$$A = -\frac{\lambda}{s^2+H^2} - \frac{\lambda\left(1-\exp\left(-\sqrt{s^2+H^2}\right)\right)}{\left(s^2+H^2\right)\left(\exp\left(-\sqrt{s^2+H^2}\right)-\exp\left(\sqrt{s^2+H^2}\right)\right)} \tag{B.14}$$

$$B = \frac{\lambda\left(1-\exp\left(-\sqrt{s^2+H^2}\right)\right)}{\left(s^2+H^2\right)\left(\exp\left(-\sqrt{s^2+H^2}\right)-\exp\left(\sqrt{s^2+H^2}\right)\right)} \tag{B.15}$$

Now on solving (B.10) and (B.12) we get

$$u_1 = -h\left(C_1\exp\left(-\sqrt{s^2+H^2}\,y\right) + C_2\exp\left(\sqrt{s^2+H^2}\,y\right) + \frac{G\sin(N y)}{\sin(N)(N^2+s^2+H^2)}\right) \tag{B.16}$$

where

$$C_1 = -\frac{G}{\left(N^2+s^2+H^2\right)\left(\exp\left(-\sqrt{s^2+H^2}\right)-\exp\left(\sqrt{s^2+H^2}\right)\right)} \tag{B.17}$$

$$C_2 = \frac{G}{\left(N^2+s^2+H^2\right)\left(\exp\left(-\sqrt{s^2+H^2}\right)-\exp\left(\sqrt{s^2+H^2}\right)\right)} \tag{B.18}$$

According to Homotopy analysis method we have

$$u = \lim_{p\to 1} u(y) = u_0 + u_1 \tag{B.19}$$

Using the Equations (B.13) and (B.16) in Equation (B.19) we get the solution as given in the text Equation (15).

Appendix C:

Nomenclature

Symbols	Meaning
B_0	Electromagnetic induction
c_p	Specific heat at constant pressure
D_a	Darcy number
g	Gravitational force
H_0	Intensity of magnetic field
k	Thermal conductivity
K	The porous medium permeability
P	Pressure
q	Radiative heat flux
t	Time variable
T	Fluid temperature
T_0	Temperature at $y = 0$
T_w	Temperature at $y = a$
β	Coefficient of volume expansion due to temperature
ω	Frequency of the oscillation
μ_e	Magnetic permeability
v	Kinematic viscosity coefficient
ρ	Fluid density
σ_e	Conductivity of the fluid
G	Grashoff number
H	Hartmann number
N	Radiation parameter
s	Porous medium shape factor
u	The axial velocity
x	Axial distance
y	Transverse distance
λ	A constant
θ	Dimensionless temperature