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On Generalization of δ-Primary Elements in Multiplicative Lattices

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Abstract: In this paper, we introduce ϕ-δ-primary elements in a compactly generated multiplicative lattice L and obtain its characterizations. We prove many of its properties and investigate the relations between these structures. By a counter example, it is shown that a ϕ-δ-primary element of L need not be δ-primary and found conditions under which a ϕ-δ-primary element of L is δ-primary.

MSC: 06B99.

Keywords: expansion function, δ-primary element, ϕ-δ-primary element, 2-potent δ-primary element, n-potent δ-primary element, global property.

1. Introduction

Prime ideals play a central role in commutative ring theory. In the literature, we find that there are several ways to generalize the notions of a prime ideal and a primary ideal of a commutative ring R with unity. A prime ideal P of R is an ideal with the property that for all $a,\ b \in R$, $ab \in P$ implies either $a \in P$ or $b \in P$. We can either restrict or enlarge where a and/or b lie or restrict or enlarge where ab lies. Same can be thought for primary ideals too. As a generalization of prime ideals of R, ϕ-prime ideals were introduced in [2] and [6] while as a generalization of primary ideals of R, ϕ-primary ideals were introduced in [4]. In an attempt to unify the prime and primary ideals of R under one frame, δ-primary ideals of R were introduced in [12]. Further, the concept of δ-primary ideals of R was generalized by introducing the notion of ϕ-δ-primary ideals of R in [7].

As an extension of these concepts of a commutative ring R to a multiplicative lattice L, C. S. Manjarekar and A. V. Bingi introduced δ-primary elements of L in [8] and introduced ϕ-prime, ϕ-primary elements of L in [9]. In this paper, we introduce and study, ϕ-δ-primary elements of L as a generalization of δ-primary elements of L and unify ϕ-prime and ϕ-primary elements of L under one frame.

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $e \in L$ is called meet principal if $a \wedge be = ((a : e) \wedge b)e$ for all $a,\ b \in L$. An element $e \in L$ is called join principal if $(ae \vee b) : e = (b : e) \vee a$ for all $a,\ b \in L$. An element $e \in L$ is called principal if e is both meet principal and join principal. A multiplicative lattice L is said to be principally generated(PG) if every element of L is a join of principal elements of L. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leqslant \vee X$ implies

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the existence of a finite number of elements $a_1, a_2, \cdots, a_n$ in X such that $a \leqslant a_1 \vee a_2 \vee \cdots \vee a_n$. The set of compact elements of L will be denoted by L_*. If each element of L is a join of compact elements of L, then L is called a compactly generated lattice or simply a CG-lattice.

An element $a \in L$ is said to be proper if $a < 1$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a$, for some $n \in Z_+\}$. A proper element $m \in L$ is said to be maximal if for every element $x \in L$ such that $m < x \leqslant 1$ implies $x = 1$. A proper element $p \in L$ is called a prime element if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ where $a, b \in L$ and is called a primary element if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant \sqrt{p}$ where $a, b \in L_*$. For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$. A multiplicative lattice is called as a Noether lattice if it is modular, principally generated and satisfies ascending chain condition. An element $a \in L$ is called a zero divisor if $ab = 0$ for some $0 \neq b \in L$ and is called idempotent if $a = a^2$. A multiplicative lattice is said to be a domain if it is without zero divisors and is said to be quasi-local if it contains a unique maximal element. A quasi-local multiplicative lattice L with maximal element m is denoted by (L, m). A Noether lattice L is local if it contains precisely one maximal prime. In a Noether lattice L, an element $a \in L$ is said to satisfy restricted cancellation law if for all $b, c \in L$, $ab = ac \neq 0$ implies $b = c$ (see [11]). According to [8], an expansion function on L is a function $\delta : L \longrightarrow L$ which satisfies the following two conditions: ①. $a \leqslant \delta(a)$ for all $a \in L$, ②. $a \leqslant b$ implies $\delta(a) \leqslant \delta(b)$ for all $a, b \in L$ and a proper element $p \in L$ is called δ-primary if for all $a, b \in L$, $ab \leqslant p$ implies either $a \leqslant p$ or $b \leqslant \delta(p)$. According to [9], a proper element $p \in L$ is said to be ϕ-prime if for all $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant \phi(p)$ implies either $a \leqslant p$ or $b \leqslant p$ and a proper element $p \in L$ is said to be ϕ-primary if for all $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant \phi(p)$ implies either $a \leqslant p$ or $b \leqslant \sqrt{p}$ where $\phi : L \longrightarrow L$ is a function on L. The reader is referred to [1] and [5] for general background and terminology in multiplicative lattices.

This paper is motivated by [7]. In this paper, we define a ϕ-δ-primary element in L and obtain their characterizations. Various ϕ_α-δ-primary elements of L are introduced and relations among them are obtained. By counter examples, it is shown that a ϕ-δ-primary element of L need not be ϕ-prime, a ϕ-δ-primary element of L need not be prime and a ϕ-δ-primary element of L need not be δ-primary. In 7 different ways, we have proved that a ϕ-δ-primary element of L is δ-primary under certain conditions. We define a 2-potent δ-primary element of L and a n-potent δ-primary element of L. We investigate some properties of ϕ-δ-primary elements of L with respect to lattice homomorphism and global property. Finally, we show that every idempotent element of L is ϕ_2-δ-primary but converse need not be true. Throughout this paper, ①. L denotes a compactly generated multiplicative lattice with greatest compact element 1 in which every finite product of compact elements is compact, ②. δ denotes an expansion function on L and ③. ϕ denotes a function defined on L.

2. ϕ-δ-primary Elements of L

We begin with introducing the notion of ϕ-δ-primary elements of L which is the generalization of the concept of δ-primary elements of L.

Definition 2.1. *Given an expansion function $\delta : L \longrightarrow L$ and a function $\phi : L \longrightarrow L$, a proper element $p \in L$ is said to be ϕ-δ-primary if for all $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant \phi(p)$ implies either $a \leqslant p$ or $b \leqslant \delta(p)$.*

If $\phi_\alpha : L \longrightarrow L$ is a function on L, then ϕ_α-δ-primary elements of L are defined by following settings in the Definition 2.1 of a ϕ-δ-primary element of L.

- $\phi_0(p) = 0$. Then $p \in L$ is called a **weakly δ-primary** element.
- $\phi_2(p) = p^2$. Then $p \in L$ is called a **2-almost δ-primary** element or a **ϕ_2-δ-primary** element or simply an **almost δ-primary** element.

- $\phi_n(p) = p^n$ $(n \geq 2)$. Then $p \in L$ is called an n-**almost** δ-**primary** element or a ϕ_n-δ-**primary** element $(n \geq 2)$.
- $\phi_\omega(p) = \bigwedge_{i=1}^{\infty} p^n$. Then $p \in L$ is called a ω-δ-**primary** element or ϕ_ω-δ-**primary** element.

Since for an element $a \in L$ with $a \leqslant q$ but $a \nleqslant \phi(q)$ implies that $a \nleqslant q \wedge \phi(q)$, there is no loss generality in assuming that $\phi(q) \leqslant q$. We henceforth make this assumption.

Definition 2.2. *Given any two functions $\gamma_1, \gamma_2 : L \longrightarrow L$, we define $\gamma_1 \leqslant \gamma_2$ if $\gamma_1(a) \leqslant \gamma_2(a)$ for each $a \in L$.*

Clearly, we have the following order:

$$\phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2$$

Further as $\phi(p) \leqslant p$ and $p \leqslant \delta(p)$ for each $p \in L$, the relation between the functions δ and ϕ is $\phi \leqslant \delta$.
According to [8], δ_0 is an expansion function on L defined as $\delta_0(p) = p$ for each $p \in L$ and δ_1 is an expansion function on L defined as $\delta_1(p) = \sqrt{p}$ for each $p \in L$. Further, note that by Theorem 2.2 in [8], a proper element $p \in L$ is δ_0-primary if and only if it is prime and by Theorem 2.3 in [8], a proper element $p \in L$ is δ_1-primary if and only if it is primary.
The following 2 results relate ϕ-prime and ϕ-primary elements of L with some ϕ-δ-primary elements of L.

Theorem 2.3. *A proper element $p \in L$ is ϕ-δ_0-primary if and only if p is ϕ-prime.*

Proof. The proof is obvious. □

Theorem 2.4. *A proper element $p \in L$ is ϕ-δ_1-primary if and only if p is ϕ-primary.*

Proof. The proof is obvious. □

Theorem 2.5. *Let $\delta,\ \gamma : L \longrightarrow L$ be expansion functions on L such that $\delta \leqslant \gamma$. Then every ϕ-δ-primary element of L is ϕ-γ-primary. In particular, a ϕ-prime element of L is ϕ-δ-primary for every expansion function δ on L.*

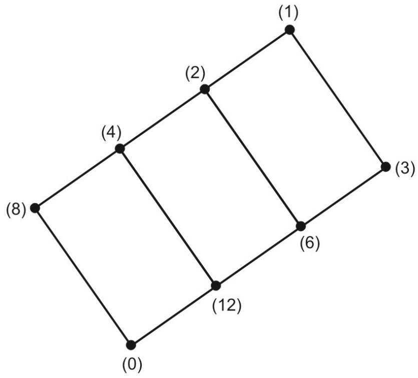
Proof. Let a proper element $p \in L$ be ϕ-δ-primary. Suppose $ab \leqslant p$ and $ab \nleqslant \phi(p)$ for $a,\ b \in L$. Then either $a \leqslant p$ or $b \leqslant \delta(p) \leqslant \gamma(p)$ and so p is ϕ-γ-primary. Next, for any expansion function δ on L, we have $\delta_0 \leqslant \delta$. So a ϕ-δ_0-primary element of L is ϕ-δ-primary and we are done since a ϕ-prime element of L is ϕ-δ_0-primary. □

Corollary 2.6. *A prime element of L is ϕ-δ-primary for every expansion function δ on L.*

Proof. The proof follows by using Theorem 2.5 to the fact that every prime element of L is ϕ-prime. □

The following example shows that (by taking ϕ as ϕ_2 and δ as δ_1 for convenience)
①. a ϕ-δ-primary element of L need not be ϕ-prime,
②. a ϕ-δ-primary element of L need not be prime.

Example 2.7. *Consider the lattice L of ideals of the ring $R = < Z_{24}\ ,\ +\ ,\ \cdot >$. Then the only ideals of R are the principal ideals (0),(2),(3),(4),(6),(8),(12),(1). Clearly, $L = \{$(0),(2),(3),(4),(6),(8),(12),(1)$\}$ is a compactly generated multiplicative lattice. Its lattice structure and multiplication table is as shown in Figure 1. It is easy to see that the element $(4) \in L$ is ϕ_2-δ_1-primary while (4) is not ϕ_2-prime because though $(2) \cdot (6) \subseteq (4)$, $(2) \cdot (6) \nsubseteq (4)^2$ but $(2) \nsubseteq (4)$ and $(6) \nsubseteq (4)$. Also, (4) is not prime.*



$\cdot$	(0)	(2)	(3)	(4)	(6)	(8)	(12)	(1)
(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
(2)	(0)	(4)	(6)	(8)	(12)	(8)	(0)	(2)
(3)	(0)	(6)	(3)	(12)	(6)	(0)	(12)	(3)
(4)	(0)	(8)	(12)	(8)	(0)	(8)	(0)	(4)
(6)	(0)	(12)	(6)	(0)	(12)	(0)	(0)	(6)
(8)	(0)	(8)	(0)	(8)	(0)	(8)	(0)	(8)
(12)	(0)	(0)	(12)	(0)	(0)	(0)	(0)	(12)
(1)	(0)	(2)	(3)	(4)	(6)	(8)	(12)	(1)

Figure 1.

Now before obtaining the characterizations of a ϕ-δ-primary element of L, we state the following essential lemma which is outcome of Lemma 2.3.13 from [3].

Lemma 2.8. *Let $a_1,\ a_2 \in L$. Suppose $b \in L$ satisfies the following property:*

$(*)$. If $h \in L_*$ with $h \leqslant b$, then either $h \leqslant a_1$ or $h \leqslant a_2$.

Then either $b \leqslant a_1$ or $b \leqslant a_2$.

Theorem 2.9. *Let q be a proper element of L. Then the following statements are equivalent:*

①. q is ϕ-δ-primary.

②. for every $a \in L$ such that $a \nleqslant \delta(q)$, either $(q : a) = q$ or $(q : a) = (\phi(q) : a)$.

③. for every $r,\ s \in L_*$, $rs \leqslant q$ and $rs \nleqslant \phi(q)$ implies either $s \leqslant q$ or $r \leqslant \delta(q)$.

Proof. ①$\Longrightarrow$②. Suppose ① holds. Let $h \in L_*$ be such that $h \leqslant (q : a)$ and $a \nleqslant \delta(q)$. Then $ah \leqslant q$. If $ah \leqslant \phi(q)$, then $h \leqslant (\phi(q) : a)$. If $ah \nleqslant \phi(q)$, then since q is ϕ-δ-primary and $a \nleqslant \delta(q)$, it follows that $h \leqslant q$. Hence by Lemma 2.8, either $(q : a) \leqslant (\phi(q) : a)$ or $(q : a) \leqslant q$. Consequently, either $(q : a) = (\phi(q) : a)$ or $(q : a) = q$.

②$\Longrightarrow$③. Suppose ② holds. Let $rs \leqslant q$, $rs \nleqslant \phi(q)$ and $r \nleqslant \delta(q)$ for $r,\ s \in L_*$. Then by ②, either $(q : r) = (\phi(q) : r)$ or $(q : r) = q$. If $(q : r) = (\phi(q) : r)$, then as $s \leqslant (q : r)$, it follows that $s \leqslant (\phi(q) : r)$ which contradicts $rs \nleqslant \phi(q)$ and so we must have $(q : r) = q$. Therefore $s \leqslant (q : r)$ gives $s \leqslant q$.

③$\Longrightarrow$①. Suppose ③ holds. Let $ab \leqslant q$, $ab \nleqslant \phi(q)$ and $a \nleqslant \delta(q)$ for $a,\ b \in L$. Then as L is compactly generated, there exist $x,\ x',\ y' \in L_*$ such that $x \leqslant a,\ x' \leqslant a,\ y' \leqslant b,\ x \nleqslant \delta(q)$ and $x'y' \nleqslant \phi(q)$. Let $y \leqslant b$ be any compact element of L. Then $(x \vee x'),\ (y \vee y') \in L_*$ such that $(x \vee x')(y \vee y') \leqslant q$, $(x \vee x')(y \vee y') \nleqslant \phi(q)$ and $(x \vee x') \nleqslant \delta(q)$. So by ③, it follows that $(y \vee y') \leqslant q$ which implies $b \leqslant q$ and therefore q is ϕ-δ-primary. □

Theorem 2.10. *A proper element $q \in L$ is ϕ-δ-primary if and only if for every $a \in L$ such that $a \nleqslant q$ either $(q : a) \leqslant \delta(q)$ or $(q : a) = (\phi(q) : a)$.*

Proof. Assume that a proper element $q \in L$ is ϕ-δ-primary. Let $h \in L_*$ be such that $h \leqslant (q : a)$ and $a \nleqslant q$. Then $ah \leqslant q$. If $ah \leqslant \phi(q)$, then $h \leqslant (\phi(q) : a)$. If $ah \nleqslant \phi(q)$, then since q is ϕ-δ-primary and $a \nleqslant q$, it follows that $h \leqslant \delta(q)$. Hence by Lemma 2.8, either $(q : a) \leqslant (\phi(q) : a)$ or $(q : a) \leqslant \delta(q)$. But as $(\phi(q) : a) \leqslant (q : a)$ we have either $(q : a) \leqslant \delta(q)$ or $(q : a) = (\phi(q) : a)$. Conversely, assume that for every $a \in L$ such that $a \nleqslant q$, either $(q : a) \leqslant \delta(q)$ or $(q : a) = (\phi(q) : a)$. Let $rs \leqslant q$, $rs \nleqslant \phi(q)$ and $r \nleqslant q$ for $r,\ s \in L$. Then either $(q : r) = (\phi(q) : r)$ or $(q : r) \leqslant \delta(q)$. If $(q : r) = (\phi(q) : r)$, then as $s \leqslant (q : r)$, it follows that $s \leqslant (\phi(q) : r)$ which contradicts $rs \nleqslant \phi(q)$ and so we must have $(q : r) \leqslant \delta(q)$. Therefore $s \leqslant (q : r)$ gives $s \leqslant \delta(q)$. Hence q is ϕ-δ-primary. □

Theorem 2.11. *Let $(L,\ m)$ be a quasi-local Noether lattice. If a proper element $p \in L$ is such that $p^2 = m^2 \leqslant p \leqslant m$, then p is ϕ_2-δ_1-primary.*

Proof. Let $xy \leqslant p$ and $xy \nleqslant \phi_2(p)$ for $x,\ y \in L$. If $x \nleqslant m$, then $x = 1$. So $xy \leqslant p$ gives $y \leqslant p$. Similarly, $y \nleqslant m$ gives $x \leqslant p$. Now if $x \leqslant m$, then $x^2 \leqslant m^2 = p^2 \leqslant p$ and hence $x \leqslant \delta_1(p)$. Similarly, $y \leqslant m$ gives $y \leqslant \delta_1(p)$. Hence in any case, p is ϕ_2-δ_1 primary. □

To obtain the relation among ϕ_α-δ-primary elements of L, we prove the following lemma.

Lemma 2.12. *Let $\gamma_1,\ \gamma_2 : L \longrightarrow L$ be functions such that $\gamma_1 \leqslant \gamma_2$ and δ be an expansion function on L. Then every proper γ_1-δ-primary element of L is γ_2-δ-primary.*

Proof. Let a proper element $p \in L$ be γ_1-δ-primary. Suppose $ab \leqslant p$ and $ab \nleqslant \gamma_2(p)$ for $a,\ b \in L$. Then as $\gamma_1 \leqslant \gamma_2$, we have $ab \leqslant p$ and $ab \nleqslant \gamma_1(p)$. Since p is γ_1-δ-primary, it follows that either $a \leqslant p$ or $b \leqslant \delta(p)$ and hence p is γ_2-δ-primary. □

Theorem 2.13. *For a proper element p of L, consider the following statements:*

(a). p is a δ-primary element of L.

(b). p is a ϕ_0-δ-primary element of L.

(c). p is a ϕ_ω-δ-primary element of L.

(d). p is a $\phi_{(n+1)}$-δ-primary element of L.

(e). p is a ϕ_n-δ-primary element of L where $n \geqslant 2$.

(f). p is a ϕ_2-δ-primary element of L.

Then $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \Longrightarrow (f)$.

Proof. Obviously, every δ-primary element of L is weakly δ-primary and hence $(a) \Longrightarrow (b)$. The remaining implications follow by using Lemma 2.12 to the fact that $\phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2$ □

Corollary 2.14. *Let $p \in L$ be a proper element. Then p is ϕ_ω-δ-primary if and only if p is ϕ_n-δ-primary for every $n \geqslant 2$.*

Proof. Assume that $p \in L$ is ϕ_n-δ-primary for every $n \geqslant 2$. Let $ab \leqslant p$ and $ab \nleqslant \bigwedge_{n=1}^{\infty} p^n$ for $a,\ b \in L$. Then $ab \leqslant p$ and $ab \nleqslant p^n$ for some $n \geqslant 2$. Since p is ϕ_n-δ-primary, we have either $a \leqslant p$ or $b \leqslant \delta(p)$ and hence p is ϕ_ω-δ-primary. The converse follows from Theorem 2.13. □

Now we show that under a certain condition, a ϕ_n-δ-primary element of L $(n \geqslant 2)$ is δ-primary.

Theorem 2.15. *Let L be a local Noetherian domain. A proper element $p \in L$ is ϕ_n-δ-primary for every $n \geqslant 2$ if and only if p is δ-primary.*

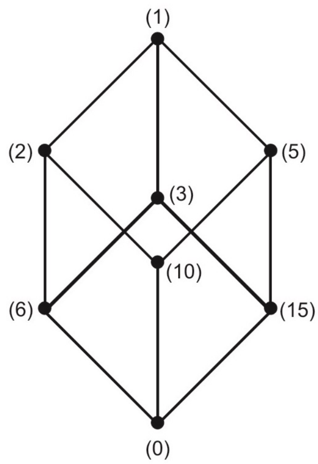
Proof. Assume that a proper element $p \in L$ is ϕ_n-δ-primary for every $n \geqslant 2$. Let $ab \leqslant p$ for $a,\ b \in L$. If $ab \nleqslant \phi_n(p)$ for $n \geqslant 2$, then as $p \in L$ is ϕ_n-δ-primary, we have $a \leqslant p$ or $b \leqslant \delta(q)$. If $ab \leqslant \phi_n(p) = p^n$ for all $n \geqslant 1$, then as L is local Noetherian, by Corollary 3.3 of [5], it follows that $ab \leqslant \bigwedge_{n=1}^{\infty} p^n = 0$ and so $ab = 0$. Since L is domain, we have either $a = 0$ or $b = 0$ which implies either $a \leqslant p$ or $b \leqslant \delta(q)$ and hence p is δ-primary. Converse follows from Theorem 2.13. □

Corollary 2.16. *Let L be a local Noetherian domain. A proper element $p \in L$ is ϕ_ω-δ-primary if and only if p is δ-primary.*

Proof. The proof follows from Theorem 2.15 and Corollary 2.14. $\square$

Clearly, every δ-primary element of L is ϕ-δ-primary. The following example shows that its converse need not be true (by taking ϕ as ϕ_2 and δ as δ_1 for convenience).

Example 2.17. *Consider the lattice L of ideals of the ring $R = < Z_{30}\ ,\ +\ ,\ \cdot >$. Then the only ideals of R are the principal ideals (0),(2),(3),(5),(6),(10),(15),(1). Clearly $L = \{(0),(2),(3),(5),(6),(10),(15),(1)\}$ is a compactly generated multiplicative lattice. Its lattice structure and multiplication table is as shown in Figure 2. It is easy to see that the element $(6) \in L$ is ϕ_2-δ_1-primary but not δ_1-primary.*



·	(0)	(2)	(3)	(5)	(6)	(10)	(15)	(1)
(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
(2)	(0)	(2)	(6)	(10)	(6)	(10)	(0)	(2)
(3)	(0)	(6)	(3)	(15)	(6)	(0)	(15)	(3)
(5)	(0)	(10)	(15)	(5)	(0)	(10)	(15)	(5)
(6)	(0)	(6)	(6)	(0)	(6)	(0)	(0)	(6)
(10)	(0)	(10)	(0)	(10)	(0)	(10)	(0)	(10)
(15)	(0)	(0)	(15)	(15)	(0)	(0)	(15)	(15)
(1)	(0)	(2)	(3)	(5)	(6)	(10)	(15)	(1)

Figure 2.

In the following successive seven theorems, we show conditions under which a ϕ-δ-primary element of L is δ-primary.

Theorem 2.18. *Let L be a Noether lattice. Let $0 \neq q \in L$ be a non-nilpotent proper element satisfying the restricted cancellation law. Then q is ϕ-δ-primary for some $\phi \leqslant \phi_2$ if and only if q is δ-primary.*

Proof. Assume that $q \in L$ is a δ-primary element. Then obviously, q is ϕ-δ-primary for every ϕ and hence for some $\phi \leqslant \phi_2$. Conversely, let $q \in L$ be ϕ-δ-primary for some $\phi \leqslant \phi_2$. Then by Lemma 2.12, $q \in L$ is ϕ_2-δ-primary (almost δ-primary). Let $xy \leqslant q$ for $x,\ y \in L$. If $xy \nleqslant \phi_2(q)$, then as q is ϕ_2-δ-primary, we have either $x \leqslant q$ or $y \leqslant \delta(q)$. If $xy \leqslant \phi_2(q) = q^2$, consider $(x \vee q)y = xy \vee qy \leqslant q$. If $(x \vee q)y \nleqslant \phi_2(q)$, then as q is ϕ_2-δ-primary, we have either $x \leqslant (x \vee q) \leqslant q$ or $y \leqslant \delta(q)$. So assume that $(x \vee q)y \leqslant \phi_2(q)$. Then $qy \leqslant q^2 \neq 0$ which implies $y \leqslant q \leqslant \delta(q)$ by Lemma 1.11 of [11]. Hence q is δ-primary. $\square$

Corollary 2.19. *Every non-zero and non-nilpotent ϕ_2-δ-primary element of a Noether lattice L satisfying the restricted cancellation law is δ-primary.*

Proof. The proof follows from proof of the Theorem 2.18. $\square$

The following result is general form of Theorem 2.18.

Theorem 2.20. *Let L be a Noether lattice. Let $0 \neq q \in L$ be a non-nilpotent proper element satisfying the restricted cancellation law. Then q is ϕ-δ-primary for some $\phi \leqslant \phi_n$ and for all $n \geqslant 2$ if and only if q is δ-primary.*

Proof. Assume that $q \in L$ is a δ-primary element. Then obviously, q is ϕ-δ-primary for every ϕ and hence for some $\phi \leqslant \phi_n$, for all $n \geqslant 2$. Conversely, let $q \in L$ be ϕ-δ-primary for some $\phi \leqslant \phi_n$ and for all $n \geqslant 2$. Then by Lemma 2.12,

$q \in L$ is ϕ_n-δ-primary (n-almost δ-primary) and for all $n \geqslant 2$. Let $xy \leqslant q$ for $x,\ y \in L$. If $xy \nleqslant \phi_n(q)$ for some $n \geqslant 2$, then as q is ϕ_n-δ-primary, we have either $x \leqslant q$ or $y \leqslant \delta(q)$ and we are done. So let $xy \leqslant \phi_n(q) = q^n$ for all $n \geqslant 2$. Consider $(x \vee q)y = xy \vee qy \leqslant q$. If $(x \vee q)y \nleqslant \phi_n(q)$, then as q is ϕ_n-δ-primary, we have either $x \leqslant (x \vee q) \leqslant q$ or $y \leqslant \delta(q)$. So assume that $(x \vee q)y \leqslant \phi_n(q)$. Then $qy \leqslant q^n \leqslant q^2 \neq 0$ as $n \geqslant 2$. This implies $y \leqslant q \leqslant \delta(q)$ by Lemma 1.11 of [11]. Hence q is δ-primary. $\square$

Corollary 2.21. *Every non-zero and non-nilpotent ϕ_n-δ-primary element ($\forall\ n \geqslant 2$) of a Noether lattice L satisfying the restricted cancellation law is δ-primary.*

Proof. The proof follows from proof of the Theorem 2.20. $\square$

Definition 2.22. *A proper element $p \in L$ is said to be* **2-potent δ-primary** *if for all $a,\ b \in L$, $ab \leqslant p^2$ implies either $a \leqslant p$ or $b \leqslant \delta(p)$.*

Obviously, every 2-potent δ_0-primary element of L is 2-potent prime and vice versa. Also, every 2-potent δ_0-primary element of L is 2-potent δ-primary.

Theorem 2.23. *Let a proper element $q \in L$ be 2-potent δ-primary. Then q is ϕ-δ-primary for some $\phi \leqslant \phi_2$ if and only if q is δ-primary.*

Proof. Assume that $q \in L$ is a δ-primary element. Then obviously, q is ϕ-δ-primary for every ϕ and hence for some $\phi \leqslant \phi_2$. Conversely, let $q \in L$ be ϕ-δ-primary for some $\phi \leqslant \phi_2$. Then by Lemma 2.12, $q \in L$ is ϕ_2-δ-primary (almost δ-primary). Let $xy \leqslant q$ for $x,\ y \in L$. If $xy \nleqslant \phi_2(q)$, then as q is ϕ_2-δ-primary, we have either $x \leqslant q$ or $y \leqslant \delta(q)$. If $xy \leqslant \phi_2(q) = q^2$, then as q is 2-potent δ-primary, we have either $x \leqslant q$ or $y \leqslant \delta(q)$. Hence q is δ-primary. $\square$

Corollary 2.24. *Every ϕ_2-δ-primary element of L which is 2-potent δ-primary is δ-primary.*

Proof. The proof follows from proof of the Theorem 2.23. $\square$

Theorem 2.25. *Let a proper element $q \in L$ be 2-potent δ_0-primary. Then q is ϕ-δ-primary for some $\phi \leqslant \phi_2$ if and only if q is δ-primary.*

Proof. The proof follows by using Theorem 2.23 to the fact that every 2-potent δ_0-primary element of L is 2-potent δ-primary. $\square$

Corollary 2.26. *Every ϕ_2-δ-primary element of L which is 2-potent δ_0-primary is δ-primary.*

Definition 2.27. *Let $n \geqslant 2$. A proper element $p \in L$ is said to be* **n-potent δ-primary** *if for all $a,\ b \in L$, $ab \leqslant p^n$ implies either $a \leqslant p$ or $b \leqslant \delta(p)$.*

Obviously, every n-potent δ_0-primary element of L is n-potent δ-primary.

The following result is general form of Theorem 2.23.

Theorem 2.28. *A proper element $q \in L$ is ϕ-δ-primary for some $\phi \leqslant \phi_n$ where $n \geqslant 2$ if and only if q is δ-primary, provided q is k-potent δ-primary for some $k \leqslant n$.*

Proof. Assume that $q \in L$ is a δ-primary element. Then obviously, q is ϕ-δ-primary for every ϕ and hence for some $\phi \leqslant \phi_n$ where $n \geqslant 2$. Conversely, let $q \in L$ be ϕ-δ-primary for some $\phi \leqslant \phi_n$ where $n \geqslant 2$. Then by Lemma 2.12, $q \in L$ is ϕ_n-δ-primary (n-almost δ-primary). Let $xy \leqslant q$ for $x,\ y \in L$. If $xy \nleqslant \phi_k(q) = q^k$, then $xy \nleqslant \phi_n(q) = q^n$ as $k \leqslant n$. Since q is ϕ_n-δ-primary, we have either $x \leqslant q$ or $y \leqslant \delta(q)$. If $xy \leqslant \phi_k(q) = q^k$, then as q is k-potent δ-primary, we have either $x \leqslant q$ or $y \leqslant \delta(q)$. Hence q is δ-primary. $\square$

Corollary 2.29. *Every ϕ_n-δ-primary element of L which is k-potent δ-primary is δ-primary where $k \leqslant n$.*

Theorem 2.30. *Let a proper element $q \in L$ be ϕ-δ-primary. If $q^2 \nleqslant \phi(q)$, then q is δ-primary.*

Proof. Let $ab \leqslant q$ for $a,\ b \in L$. If $ab \nleqslant \phi(q)$, then as q is ϕ-δ-primary, we have either $a \leqslant q$ or $b \leqslant \delta(q)$. So assume that $ab \leqslant \phi(q)$. First suppose $aq \nleqslant \phi(q)$. Then $ad \nleqslant \phi(q)$ for some $d \leqslant q$ in L. Also $a(b \vee d) = ab \vee ad \leqslant q$ and $a(b \vee d) \nleqslant \phi(q)$. As q is ϕ-δ-primary, either $a \leqslant q$ or $(b \vee d) \leqslant \delta(q)$. Hence either $a \leqslant q$ or $b \leqslant \delta(q)$. Similarly, if $bq \nleqslant \phi(q)$, we can show that either $a \leqslant q$ or $b \leqslant \delta(q)$. So we can assume that $aq \leqslant \phi(q)$ and $bq \leqslant \phi(q)$. Since $q^2 \nleqslant \phi(q)$, there exist $r,\ s \leqslant q$ in L such that $rs \nleqslant \phi(q)$. Then $(a \vee r)(b \vee s) \leqslant q$ but $(a \vee r)(b \vee s) \nleqslant \phi(q)$. As q is ϕ-δ-primary, we have either $(a \vee r) \leqslant q$ or $(b \vee s) \leqslant \delta(q)$. Therefore either $a \leqslant q$ or $b \leqslant \delta(q)$ and hence q is δ-primary. □

From the Theorem 2.30, it follows that,

- if a proper element $q \in L$ is ϕ-δ-primary but not δ-primary, then $q^2 \leqslant \phi(q)$,
- a ϕ-δ-primary element $q < 1$ of L with $q^2 \nleqslant \phi(q)$ is δ-primary.

Clearly, given an expansion function δ on L, $\delta(p) \leqslant \delta(\delta(p))$ for each $p \in L$. Moreover, for each $p \in L$, $\delta_1(\delta_1(p)) = \delta_1(p)$, by property $(p3)$ of radicals in [10]. Also, obviously $\delta_0(\delta_0(p)) = \delta_0(p)$ for each $p \in L$.
Now we present the consequences of the Theorem 2.30 in the form of following corollaries.

Corollary 2.31. *If a proper element $q \in L$ is ϕ-δ-primary but not δ-primary, then $\delta_1(q) = \delta_1(\phi(q))$.*

Proof. By Theorem 2.30, we have $q^2 \leqslant \phi(q)$. So $q \leqslant \delta_1(\phi(q))$ which gives $\delta_1(q) \leqslant \delta_1(\delta_1(\phi(q))) = \delta_1(\phi(q))$. Since $\phi(q) \leqslant q$, we have $\delta_1(\phi(q)) \leqslant \delta_1(q)$. Hence $\delta_1(q) = \delta_1(\phi(q))$. □

Corollary 2.32. *If a proper element $q \in L$ is ϕ-δ-primary where $\phi \leqslant \phi_3$, then q is ϕ_n-δ-primary for every $n \geqslant 2$.*

Proof. If q is δ-primary, then by Theorem 2.13, q is ϕ_ω-δ-primary. So assume that q is not δ-primary. Then by Theorem 2.30 and by hypothesis, we get $q^2 \leqslant \phi(q) \leqslant q^3$. Hence $\phi(q) = q^n$ for every $n \geqslant 2$. Consequently, q is ϕ_n-δ-primary for every $n \geqslant 2$. □

Corollary 2.33. *If a proper element $q \in L$ is ϕ-δ-primary where $\phi \leqslant \phi_3$, then q is ϕ_ω-δ-primary.*

Proof. The proof follows from Corollary 2.32 and Corollary 2.14. □

Corollary 2.34. *If a proper element $q \in L$ is ϕ_0-δ-primary but not δ-primary, then $q^2 = 0$.*

Proof. The proof is obvious. □

Theorem 2.35. *Let q be a ϕ-δ-primary element of L. If $\phi(q)$ is a δ-primary element of L, then q is δ-primary.*

Proof. Let $ab \leqslant q$ for $a,\ b \in L$. If $ab \nleqslant \phi(q)$, then as q is ϕ-δ-primary, we have either $a \leqslant q$ or $b \leqslant \delta(q)$ and we are done. Now if $ab \leqslant \phi(q)$, then as $\phi(q)$ is δ-primary, we have either $a \leqslant \phi(q)$ or $b \leqslant \delta(\phi(q))$. This implies that either $a \leqslant q$ or $b \leqslant \delta(q)$ because $\phi(q) \leqslant q$ and $\delta(\phi(q)) \leqslant \delta(q)$. □

The next result shows that the join of a family of ascending chain of ϕ-δ-primary elements of L is again ϕ-δ-primary.

Theorem 2.36. *Let $\{p_i \mid i \in \triangle\}$ be a chain of ϕ-δ-primary elements of L and let the function ϕ be such that $x \leqslant y$ imply $\phi(x) \leqslant \phi(y)$ for all $x, y \in L$. Then the element $p = \bigvee_{i \in \triangle} p_i$ is also ϕ-δ-primary.*

Proof. Since $1 \in L$ is compact, $\bigvee_{i\in\triangle} p_i = p \neq 1$. Let $ab \leqslant p$, $ab \nleqslant \phi(p)$ and $a \nleqslant p$ for $a,\ b \in L$. Then as $\{p_i \mid i \in \triangle\}$ is a chain, we have $ab \leqslant p_i$ for some $i \in \triangle$ but $a \nleqslant p_i$ and $ab \nleqslant \phi(p_i)$ because for each $k \in \triangle$, we have $p_k \leqslant p$ and this implies $\phi(p_k) \leqslant \phi(p)$. As each p_i is ϕ-δ-primary, it follows that $b \leqslant \delta(p_i)$. Since $p_i \leqslant p$, we have $\delta(p_i) \leqslant \delta(p)$ and so $b \leqslant \delta(p)$. Hence p is ϕ-δ-primary. □

The following theorem shows that a under certain condition, $(p : q) \in L$ is ϕ-δ-primary if $p \in L$ is ϕ-δ-primary element where $q \in L$.

Theorem 2.37. *Let a proper element $p \in L$ be ϕ-δ-primary. Then $(p : q)$ is ϕ-δ-primary for all $q \in L$ if $(\phi(p) : q) \leqslant \phi(p : q)$.*

Proof. Clearly, $pq \leqslant p$ implies $p \leqslant (p : q)$ and so $\delta(p) \leqslant \delta(p : q)$. Now let $ab \leqslant (p : q)$, $ab \nleqslant \phi(p : q)$ and $a \nleqslant (p : q)$ for $a,\ b \in L$. Then $abq \leqslant p$, $abq \nleqslant \phi(p)$ and $aq \nleqslant p$ since $ab \nleqslant (\phi(p) : q)$. Now as p is ϕ-δ-primary, we have $b \leqslant \delta(p) \leqslant \delta(p : q)$ and hence $(p : q)$ is ϕ-δ-primary. □

In the next result, we show that under a certain condition $\delta_1(p) \leqslant \delta(p)$, for every ϕ-δ-primary $p \in L$.

Theorem 2.38. *If a proper element $p \in L$ is ϕ-δ-primary element such that $\delta_1(\phi(p)) \leqslant \delta(p)$, then $\delta_1(p) \leqslant \delta(p)$.*

Proof. Assume that a proper element $p \in L$ is ϕ-δ-primary. For $a \in L$, let $a \leqslant \delta_1(p) = \sqrt{p}$. Then there exists a least positive integer k such that $a^k \leqslant p$. If $k = 1$, then $a \leqslant p \leqslant \delta(p)$. Now let $k > 1$. If $a^k \leqslant \phi(p)$, then $a \leqslant \delta_1(\phi(p)) \leqslant \delta(p)$. So let $a^k \nleqslant \phi(p)$. Clearly, $a^{k-1}a \leqslant p$ where $a^{k-1} \nleqslant p$. As $p \in L$ is ϕ-δ-primary, it follows that $a \leqslant \delta(p)$. Thus in any case, we have $\delta_1(p) \leqslant \delta(p)$. □

Note that, if $p \in L$ is δ-primary, then by consequence of Theorem 2.5 of [8], we have $\phi(p) \leqslant p$ implies $\delta_1(\phi(p)) \leqslant \delta_1(p) \leqslant \delta(p)$ and hence $\delta_1(\phi(p)) \leqslant \delta(p)$.

Corollary 2.39. *If a proper element $p \in L$ is ϕ-δ-primary element such that $\delta_1(\phi(p)) \leqslant \delta(p)$ with $\delta(p) \leqslant \delta_1(p)$, then $\delta_1(p) = \delta(p)$.*

Proof. The proof follows from Theorem 2.38. □

According to [8], an expansion function δ on L_1 and on L_2 is said to have global property if for any lattice isomorphism $f : L_1 \longrightarrow L_2$, $\delta(f^{-1}(a)) = f^{-1}(\delta(a))$ for all $a \in L_2$ where L_1 and L_2 are multiplicative lattices. Similarly, now we define global property of a function ϕ on multiplicative lattices.

Definition 2.40. *Let L_1 and L_2 be multiplicative lattices. A function ϕ on L_1 and on L_2 is said to have* **global property** *if for any lattice isomorphism $f : L_1 \longrightarrow L_2$, $\phi(f^{-1}(a)) = f^{-1}(\phi(a))$ for all $a \in L_2$.*

Lemma 2.41. *Let the function β on L_1 and on L_2 have the global property where L_1 and L_2 are multiplicative lattices. If the function $g : L_1 \longrightarrow L_2$ is a lattice isomorphism, then $g(\beta(q)) = \beta(g(q))$ for all $q \in L_1$.*

Proof. For $q \in L_1$, the global property of β gives $\beta(q) = \beta(g^{-1}(g(q))) = g^{-1}(\beta(g(q)))$. Then since g is onto, we have $g(\beta(q)) = \beta(g(q))$. □

The next result shows that if $q \in L$ is ϕ-δ-primary with some conditions on δ and ϕ, then $\delta(q) \in L$ is ϕ-prime.

Theorem 2.42. *Let the expansion function δ on L be a lattice isomorphism. Let the function ϕ on L have the global property. If a proper element $q \in L$ is ϕ-δ-primary and satisfies $\delta(\delta(q)) \leqslant \delta(q)$, then $\delta(q)$ is a ϕ-prime element of L.*

Proof. By Lemma 2.41, we have $\delta(\phi(q)) = \phi(\delta(q))$. Let $xy \leqslant \delta(q)$, $xy \nleqslant \phi(\delta(q)) = \delta(\phi(q))$ and $x \nleqslant \delta(q)$ for $x,\ y \in L$. So $\delta^{-1}(x)\cdot\delta^{-1}(y) = \delta^{-1}(xy) \leqslant q$, $\delta^{-1}(x)\cdot\delta^{-1}(y) = \delta^{-1}(xy) \nleqslant \phi(q)$ and $\delta^{-1}(x) \leqslant q$. As q is ϕ-δ-primary, we have $\delta^{-1}(y) \leqslant \delta(q)$ which implies $y \leqslant \delta(\delta(q)) \leqslant \delta(q)$ and hence $\delta(q)$ is a ϕ-prime element of L. □

Note that, in the Theorem 2.42, the idea behind taking the expansion function δ on L as a lattice isomorphism and the function ϕ on L with the global property is to get $\delta(\phi(q)) = \phi(\delta(q))$. The following theorem is a similar version of Theorem 2.42.

Theorem 2.43. *If a proper element $q \in L$ is ϕ-δ_1-primary such that $\delta_1(\phi(q)) = \phi(\delta_1(q))$, then $\delta_1(q)$ is a ϕ-prime element of L.*

Proof. Assume that $ab \leqslant \delta_1(q)$, $ab \nleqslant \phi(\delta_1(q))$ and $a \nleqslant \delta_1(q)$ for $a,\ b \in L$. Then there exists $n \in Z_+$ such that $a^n \cdot b^n = (ab)^n \leqslant q$. If $(ab)^n \leqslant \phi(q)$, then by hypothesis $ab \leqslant \delta_1(\phi(q)) = \phi(\delta_1(q))$, a contradiction. So we must have $a^n \cdot b^n = (ab)^n \nleqslant \phi(q)$. Since q is ϕ-δ_1-primary and $a^n \nleqslant q$ for all $n \in Z_+$, we have $b^n \leqslant \delta_1(q)$ and hence $b \leqslant \delta_1(\delta_1(q)) = \delta_1(q)$. This shows that $\delta_1(q)$ is a ϕ-prime element of L. □

Lemma 2.44. *Let the expansion function δ on L_1 and on L_2 have the global property where L_1 and L_2 are multiplicative lattices. Let the function ϕ on L_1 and on L_2 have the global property. If $f : L_1 \longrightarrow L_2$ is a lattice isomorphism, then for any ϕ-δ-primary element $p \in L_2$, $f^{-1}(p) \in L_1$ is ϕ-δ-primary.*

Proof. Assume that a proper element $p \in L_2$ is ϕ-δ-primary. Let $ab \leqslant f^{-1}(p)$, $ab \nleqslant \phi(f^{-1}(p)) = f^{-1}(\phi(p))$ and $a \nleqslant f^{-1}(p)$ for $a,\ b \in L_1$. Then $f(ab) = f(a)\cdot f(b) \leqslant p$, $f(ab) = f(a)\cdot f(b) \nleqslant \phi(p)$ and $f(a) \nleqslant p$. As p is ϕ-δ-primary, we have $f(b) \leqslant \delta(p)$. Now the global property of δ gives $b \leqslant f^{-1}(\delta(p)) = \delta(f^{-1}(p))$ showing that $f^{-1}(p) \in L_1$ is ϕ-δ-primary. □

The following result gives another characterization of ϕ-δ-primary elements of L.

Theorem 2.45. *Let the expansion function δ on L_1 and on L_2 have the global property where L_1 and L_2 are multiplicative lattices. Let the function ϕ on L_1 and on L_2 have the global property. Let $f : L_1 \longrightarrow L_2$ be a lattice isomorphism. Then a proper element $a \in L_1$ is ϕ-δ-primary if and only if $f(a) \in L_2$ is ϕ-δ-primary.*

Proof. Assume that a proper element $a \in L_1$ is ϕ-δ-primary. Clearly, by Lemma 2.41, the global property of δ gives $f(\delta(a)) = \delta(f(a))$. Also, by Lemma 2.41, the global property of ϕ gives $f(\phi(a)) = \phi(f(a))$. Now, let $xy \leqslant f(a)$, $xy \nleqslant \phi(f(a))$ and $x \nleqslant f(a)$ for $x,\ y \in L_2$. Then there exists $b,\ c \in L_1$ such that $f(b) = x,\ f(c) = y$. So $f(bc) = f(b)\cdot f(c) = xy \leqslant f(a)$, $f(bc) = f(b)\cdot f(c) = xy \nleqslant \phi(f(a)) = f(\phi(a))$ and $f(b) = x \nleqslant f(a)$. As a is ϕ-δ-primary in L_1, $bc \leqslant a$, $bc \nleqslant \phi(a)$ and $b \nleqslant a$, we have $c \leqslant \delta(a)$. So $y = f(c) \leqslant f(\delta(a))$ and hence $y \leqslant \delta(f(a))$ showing that $f(a) \in L_2$ is ϕ-δ-primary. The converse follows from Lemma 2.44. □

Now we relate idempotent element of L with ϕ_n-δ-primary element $(n \geqslant 2)$ of L.

Theorem 2.46. *Every idempotent element of L is ϕ_ω-δ-primary and hence ϕ_n-δ-primary $(n \geqslant 2)$.*

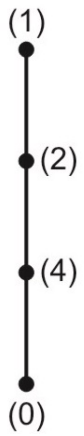
Proof. Let p be an idempotent element of L. Then $p = p^n$ for all $n \in Z_+$. So $\phi_\omega(p) = p$. Therefore p is a ϕ_ω-δ-primary of L. Hence p is a ϕ_n-δ-primary element $(n \geqslant 2)$ of L by Theorem 2.13. □

As a consequence of Theorem 2.46, we have following result whose proof is obvious.

Corollary 2.47. *Every idempotent element of L is ϕ_2-δ-primary.*

However, a ϕ_2-δ-primary element of L need not be idempotent as shown in the following example (by taking δ as δ_1 for convenience).

Example 2.48. *Consider the lattice L of ideals of the ring $R =< Z_8\ ,\ +\ ,\ \cdot >$. Then the only ideals of R are the principal ideals (0),(2),(4),(1). Clearly, $L = \{(0), (2), (4), (1)\}$ is a compactly generated multiplicative lattice. Its lattice structure and multiplication table is as shown in Figure 3. It is easy to see that the element $(4) \in L$ is ϕ_2-δ_1-primary but not idempotent.*



·	(0)	(2)	(4)	(1)
(0)	(0)	(0)	(0)	(0)
(2)	(0)	(4)	(0)	(2)
(4)	(0)	(0)	(0)	(4)
(1)	(0)	(2)	(4)	(1)

Figure 3.

We conclude this paper with the following examples, from which it is clear that,

① a ϕ_2-δ_1-primary element of L need not be 2-potent δ_0-primary,

② a 2-potent δ_0-primary element of L which is ϕ_2-δ_1-primary need not be prime.

Example 2.49. *Consider L as in Example 2.17. Here the element $(6) \in L$ is ϕ_2-δ_1-primary but not 2-potent δ_0-primary.*

Example 2.50. *Consider L as in Example 2.48. Here the element $(4) \in L$ is 2-potent δ_0-primary, ϕ_2-δ_1-primary but not prime.*

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References

[1] F. Alarcon, D. D. Anderson and C. Jayaram, *Some results on abstract commutative ideal theory*, Periodica Mathematica Hungarica, 30(1)(1995), 1-26.

[2] D. D. Anderson and M. Bataineh, *Generalizations of prime ideals*, Communications in Algebra, 36(2)(2008), 686-696.

[3] D. S. Culhan, *Associated primes and primal decomposition in modules and lattice modules and their duals*, Ph. D. Thesis, University of California, Riverside, (2005).

[4] A. Y. Darani, *Generalizations of primary ideals in commutative rings*, Novi Sad J. Math., 42(1)(2012), 27-35.

[5] R. P. Dilworth, *Abstract commutative ideal theory*, Pacific Journal of Mathematics, 12(2)(1962), 481-498.

[6] M. Ebrahimpour and R. Nekooei, *On generalizations of prime ideals*, Communications in Algebra, 40(4)(2012), 1268-1279.

[7] A. Jaber, *Properties of ϕ-δ-primary and 2-absorbing δ-primary ideals of commutative rings*, Asian-European Journal of Mathematics, 13(1)(2020), 1-11.

[8] C. S. Manjarekar and A. V. Bingi, *δ-primary elements in multiplicative lattices*, International Journal of Advance Research, 2(6)(2014), 1-7.

[9] C. S. Manjarekar and A. V. Bingi, *ϕ-Prime and ϕ-Primary Elements in Multiplicative Lattices*, Algebra, (2014), 1-7.

[10] N. K. Thakare and C. S. Manjarekar, *Radicals and uniqueness theorem in multiplicative lattices with chain conditions*, Studia Scientifica Mathematicarum Hungarica, 18(1983), 13-19.

[11] J. Wells, *The restricted cancellation law in a Noether lattice*, Fundamenta Mathematicae, 3(75)(1972), 235-247.

[12] D. Zhao, *δ-primary ideals of commutative rings*, Kyungpook Math. J, 41(2001), 17-22.