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On Some Fixed Point Theorem in Ordered $\mathcal{G}$ - Cone Metric Spaces Over Banach Algebra

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Abstract: In this work, concept of ordered $\mathcal{G}$ - cone metric space over Banach algebras and the generalized contractive map are introduced, and convergence properties of sequences are proved. With this modification, we will prove fixed point results for maps that satisfy contraction conditions without assuming normality. Our results are generalized results of Yan [1] and Altun [17].

MSC: 47H10, 54H25.

Keywords: Ordered $\mathcal{G}$ - cone metric spaces, Banach algebras, generalized Lipschitz conditions.

1. Introduction

The concept of b-metric space was initiated by Bakhtin [2], who generalized the metric space. The idea of D-metric space is a generalization of metric spaces and is introduced by Dhage [8]. Mustafa and Sims ([9, 10]) showed that most results on Dhage D-metric spaces introduced a more appropriate generalization of the structure of metric space, which they called $\mathcal{G}$-metric spaces or generalized metric space.

The concept of Cone metric spaces were introduced by Huang and Zhang [4]. In 2013, Liu and Xu [13] introduced the concept of cone metric spaces over Banach algebras by replacing a Banach space E with a Banach algebra. Ismat Beg [15] introduced the concept of generalized cone metric space or $\mathcal{G}$ - cone metric space by replacing the set of real numbers by an ordered Banach space and by proving the convergence properties of the sequence and some fixed point theorems in this space. $\mathcal{G}$ - cone metric space is more general than that $\mathcal{G}$ - metric space and cone metric space.

The main purpose of this paper is to provide some fixed point theorems of map in setting of ordered $\mathcal{G}$ - cone metric spaces with Banach algebra.

2. Preliminaries

Throughout this paper, we assume that $\mathcal{P}$ is a cone in $\mathcal{A}$ with int $\mathcal{P} \neq \phi$ (θ, the additive identity element of $\mathcal{A}$) and $\preceq$ is the partial ordering with respect to $\mathcal{P}$ where $\mathcal{A}$ is a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, satisfying the following properties [5] (for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{R}$) :

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(1). $(xy)z = x(yz)$;

(2). $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(3). $\alpha(xy) = (\alpha x)y; = x(\alpha y)$;

(4). there exists $e \in E$ such that $xe = ex = x$;

(5). $||e|| = 1$;

(6). $||xy|| \leq ||x||.||y||$;

An element $x \in \mathcal{A}$ is called invertible if there exists $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = x^{-1}x = e$.

Proposition 2.1 ([5]). *Let $x \in \mathcal{A}$ be a Banach algebra with a unit e, then the spectral radius $\rho(u)$ of $u \in \mathcal{A}$ holds*

$$\rho(u) = \lim_{n\to\infty} ||u_n||^{\frac{1}{n}} = inf||u^n||^{\frac{1}{n}} < 1$$

Further, $e - u$ is invertible and $(e-u)^{-1} = \sum_{i=0}^{\infty} u^i$.

Consider a Banach algebra $\mathcal{A}, \theta$ be the null vector, e be the identity element of $\mathcal{A}$ and a subset $\mathcal{P}$ of $\mathcal{A}$ is called a cone if it satisfies the following:

(1). $\{\theta, e\} \subset \mathcal{P}$ and $\mathcal{P}$ is closed;

(2). $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subset \mathcal{P}$;

(3). $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$, for all non-negative real numbers α and β;

(4). $\mathcal{P} \cap (-C\mathcal{P}) = \{\theta\}$;

With respect to cone $\mathcal{P}$, a partial ordering $\preceq$ is defined as $u \preceq w$ if and only if $w - u \in \mathcal{P}$ and $u \prec w$ if $u \preceq w$ and $u \neq w$ whereas $u \ll w$ means $w - u \in int\, \mathcal{P}$. If $\mathcal{A}$ is a Banach space and $\mathcal{P} \subset \mathcal{A}$, satisfies the conditions (1), (3) and (4) then $\mathcal{P}$ is called a cone of $\mathcal{A}$.

Remark 2.2 ([5]). *If $\rho(x) < 1$, then $\|x_n\| \to 0$ as $n \to \infty$.*

Definition 2.3 ([5]). *Consider X is a non-empty set, $\mathcal{A}$ be a Banach algebra and $\mathcal{P} \subseteq \mathcal{A}$ be a cone. Suppose the mapping $d : X \times X \to \mathcal{A}$ satisfies the following for all $x, y, z \in X$,*

(1). $d(x,z) = \theta$ if and only if $x = z$, and $\theta \preceq d(x,z)$;

(2). $d(x,z) = d(z,x)$;

(3). $d(x,z) \preceq d(x,y) + d(y,z)$ for every $x, y, z \in X$.

Here d is called a cone metric and (X, d) is called Cone metric space over a Banach algebra $\mathcal{A}$ (In Short CMSBA). Note that $d(x,z) \in \mathcal{P}$ for all $x, y \in X$.

Definition 2.4 ([7]). *Let X be a non-empty set, $\mathcal{A}$, a Banach algebra and $\mathcal{G} : X^3 \to \mathcal{A}$ be a function satisfying the following properties:*

(1). $\mathcal{G}(x,y,z) = \theta$ if and only if $x = y = z$

(2). $\theta \prec \mathcal{G}(x,y,z)$,*for all* $x, y \in X$, *with* $x \neq y$

(3). $\mathcal{G}(x,x,y) \preceq \mathcal{G}(x,y,z)$ *for all* $x,y,z \in X$, *with* $z \neq y$

(4). $\mathcal{G}(x,y,z) = \mathcal{G}(y,z,x) = \mathcal{G}(x,z,y) = \ldots$ *(symmetry)*

(5). $\mathcal{G}(x,y,z) \preceq \mathcal{G}(x,a,a) + \mathcal{G}(a,y,z)$ *for all* $a,x,y,z \in X$ *(rectangle inequality)*

Then $\mathcal{G}$ *is called a* $\mathcal{G}$ *- cone metric over Banach algebra A and the pair (X,* $\mathcal{G}$*) denotes a* $\mathcal{G}$ *- cone metric space over Banach algebra.*

Remark 2.5.

(1). If $\mathcal{A}$ *is a Banach space in Definition* 2.3, *then* $(X, \mathcal{G})$ *becomes a* $\mathcal{G}$ *-cone metric space and if in addition* $z = y$, *then it becomes a cone metric space as in Huang and Zhang [4].*

(2). If $\mathcal{A} = R$ *in Definition 2.3, we obtain a G-metric space as in Mustafa and Sims [10] and if in addition,*$z = y$ *in G(x, y, z), then it becomes a metric space.*

Definition 2.6 ([5])**.** *Let (X, d) be a cone metric space over Banach algebra* $\mathcal{A}$ *and* $\{x_n\}$ *a sequence in X. We say that*

(1). $\{x_n\}$ *is a convergent sequence if, for every* $c \in B$ *with* $\theta \ll c$, *there is an* N *such that* $d(x_n, x) \ll c$ *for all* $n \geq N$. *Ones write it by* $x_n \to x(n \to \infty)$;

(2). $\{x_n\}$ *is a Cauchy sequence if, for every* $c \in B$ *with* $\theta \ll c$, *there is an* N *such that* $d(x_n, x_m) \ll c$ *for all* $n, m \geq N$;

(3). (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 2.7 ([5])**.** *Let* $\mathcal{A}$ *be a Banach algebra and k, a vector in A. If* $0 \leq r(k) < 1$, *then we have*

$$r((e-k)^{-1}) < (1 - r(k))^{-1}.$$

Lemma 2.8 ([3])**.** *Let* $\mathcal{A}$ *be a Banach algebra and x, y be vectors in* $\mathcal{A}$. *If x and y commute, then the following holds:*

(1). $r(xy) \leq r(x)r(y)$;

(2). $r(x+y) \leq r(x) + r(y)$;

(3). $|r(x) - r(y)| \leq r(x-y)$.

Lemma 2.9 ([3])**.** *If* $\mathcal{A}$ *is real Banach algebra with a solid cone* $\mathcal{P}$ *and* $\{x_n\}$ *is a sequence in* $\mathcal{A}$. *Suppose* $||x_n|| \to 0(n \to \infty)$ *for any* $\theta \ll c$. *Then* $x_n \ll c$ *for all* $n > N^1, N^1 \in N$.

Lemma 2.10 ([7])**.** *If E is a real Banach space with a solid cone* $\mathcal{P}$ *and if* $||x_n|| \to 0$ *as* $n \to \infty$, *then for any* $\theta \ll c$, *there exists* $N \in \mathbb{N}$ *such that, for any* $n > N$, *we have* $x_n \ll c$.

Example 2.11 ([3])**.** *Let* $\mathcal{A}$ *be the Banach space of all continuous real-valued functions* $C(K)$ *on a compact Hausdorff topological space K, with multiplication defined pointwise. Then* $\mathcal{A}$ *is a Banach algebra, and the constant function* $f(t) = 1$ *is the unit of* $\mathcal{A}$. *Let* $P = \{f \in \mathcal{A} : f(t) \geq 0$ *for all* $t \in K\}$. *Then* $\mathcal{P} \subset \mathcal{A}$ *is a normal cone with a normal constant* $M = 1$. *Let* $X = C(K)$ *with the metric* $d : X \times X \to A$ *defined by* $d(f,g) = |f(t) - g(t)|$, *where* $t \in K$. *Then* (X, d) *is a cone metric space over a Banach algebra* $\mathcal{A}$.

Definition 2.12 ([7]). *Let $(X, \mathcal{G})$ be a $\mathcal{G}$ - cone metric space over Banach algebra. $\mathcal{G}$ is said to be symmetric if:*

$$\mathcal{G}(x, y, y) = \mathcal{G}(x, x, y)$$

for all $x, y, z \in X$.

Definition 2.13 ([7]). *A $\mathcal{G}$ - cone metric space over Banach algebra $\mathcal{A}$ is said to be $\mathcal{G}$-bounded if for any $x, y, z \in X$, there exists $K \succ \theta$ such that $||\mathcal{G}(x, y, z)|| \preceq K$.*

Definition 2.14 ([7]). *Let $(X, \mathcal{G})$ be a $\mathcal{G}$ - cone metric space over Banach algebra and $\{x_n\}$ a sequence in X, $c \gg \theta$ with $c \in \mathcal{A}$. Then*

(1). $\{x_n\}$ converges to $x \in X$ if and only if $\mathcal{G}(x_m, x_n, x) \ll c$ for all $n, m > N^1, N^1 \in N$.

(2). $\{x_n\}$ is Cauchy sequence if and only if $\mathcal{G}(x_n, x_m, x_p) \ll c$ for all $n, m > p > N^1, N^1 \in N$.

(3). $(X, \mathcal{G})$ is complete $\mathcal{G}$ -cone metric space over Banach algebra if every Cauchy sequence converges.

Definition 2.15 ([19]). *Let X be a nonempty set. Then $(X, G, \sqsubseteq)$ is called an ordered G - cone metric space if:*

(1). (X, G) is a G-cone metric space,

(2). (X, $\sqsubseteq$) is a partially ordered set

Let $(X, \sqsubseteq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \sqsubseteq y$ or $y \sqsubseteq x$ holds.

Definition 2.16 ([1]). *Let $(X, \sqsubseteq)$ be a partially ordered set. We say that $x, y \in X$ are comparable if $x \sqsubseteq y$ or $y \sqsubseteq x$ holds. Similarly, $f : X \to X$ is said to be comparable if for any comparable pair $x, y \in X, f(x), f(y)$ are comparable.*

Definition 2.17 ([1]). *Let $(X, \sqsubseteq)$ be a partially ordered set. Two maps $f, g : X \to X$ are said to be weakly comparable if both $f(x)$, $gf(x)$ and $g(x)$, $fg(x)$ are comparable for all $x \in X$.*

Lemma 2.18 ([17]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space $(X, \mathcal{G})$ is complete. Let $f : X \to X$ be a continuous and nondecreasing mapping with $\sqsubseteq$. Suppose that the following two assertions hold:*

(1). there exists $k \in (0, 1)$ such that $\mathcal{G}(Tx, Ty, Tz) \preceq h\mathcal{G}(x, y, z)$ for each $x, y, z \in X$ with $y \sqsubseteq x$;

(2). there exists $x_0 \in X$ such that $x_0 \sqsubseteq f(x_0)$.

Then f has a fixed point $x^* \in X$.

3. Main Result

Theorem 3.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that $(X, \mathcal{G})$ be a complete $\mathcal{G}$-cone metric space over a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a non normal cone. Suppose that the mapping $T : X \to X$ is continuous and comparable and the following two assertions holds:*

(1). there exists $h \in \mathcal{P}$ with $\rho(h) \in (0, 1)$ such that $\mathcal{G}(Tx, Ty, Tz) \preceq h\mathcal{G}(x, y, z)$ for any comparable pair $x, y, z \in X$;

(2). there exists $x_0 \in X$ such that $x_0, f(x_0)$ are comparable.

Then f has a fixed point $x^* \in X$.

Proof. If $T(y_0) = y_0$ then the proof is completed. Let $T(y_0) \neq y_0$ from condition second and T is comparable, we deduce that $T^i(y_0)$ and T^{i+1} are comparable for any $i \geq 0$. Replacing $y_n = T^n(y_0)$, we recover x_i, x_{i+1} are comparable by condition first, we have

$$\mathcal{G}(y_n, y_{n+1}, y_{n+1}) = \mathcal{G}(Ty_{n-1}, Ty_n, Ty_n) \preceq h\mathcal{G}(y_{n-1}, y_n, y_n) \preceq \cdots \preceq h^n\mathcal{G}(y_0, y_1, y_1) \tag{1}$$

Thus, for $m > n$,

$$\mathcal{G}(y_n, y_m, y_m) \preceq \mathcal{G}(y_n, y_{n+1}, y_{n+1}) + \mathcal{G}(y_{n+1}, y_{n+2}, y_{n+2}) \cdots + \mathcal{G}(y_{m-1}, y_m, y_m) \tag{2}$$

From Equations (1) and (2), we have

$$\begin{aligned}
\mathcal{G}(y_n, y_m, y_m) &\preceq \mathcal{G}(y_n, y_{n+1}, y_{n+1}) + \mathcal{G}(y_{n+1}, y_{n+2}, y_{n+2}) \cdots + \mathcal{G}(y_{m-1}, x_m, y_m) \\
&\preceq [h^n + h^{n+1} + h^{n+2} \cdots + h^{m-1}]\mathcal{G}(y_0, y_1, y_1) \\
&= (e + h + ... + h^{m-n-1})h^n\mathcal{G}(y_0, y_1, y_1) \\
&\preceq \left(\sum_{i=0}^{\infty} h^i\right) h^n\mathcal{G}(y_0, y_1, y_1) \\
&\preceq \left[\frac{h^n}{e-h}\right]\mathcal{G}(y_0, y_1, y_1)
\end{aligned}$$

Using Lemmas 2.6 and 2.7, we have

$$\begin{aligned}
\rho\left[\frac{h^n}{e-h}\right] &\leq \rho(h^n).\rho[(e-h)^{-1}] \\
&\leq \frac{(\rho(h))^n}{(e-\rho(h))}
\end{aligned}$$

By Remark 2.2 and Lemma 2.10

$$\left\|\frac{h^n}{e-h}\right\| \to 0 \text{ as } n \to \infty$$

It follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exist $N \in \mathbb{N}$ such that $m > n > N$, we have that

$$\mathcal{G}(y_n, y_m, y_m) \preceq \left[\frac{h^n}{e-h}\right]\mathcal{G}(y_0, y_1, y_1) \ll c$$

which implies that $\{y_n\}$ is Cauchy. Since X is complete, there exists $y^* \in X$ such that $y_n \to y^*$ as $n \to \infty$ and $Ty^* \in X$ By continuity, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{G}(y_{n+1}, y_n, Ty^*) &= \mathcal{G}\left(\lim_{n\to\infty} y_{n+1}, \lim_{n\to\infty} y_n, Ty^*\right) \\
&= \mathcal{G}(y^*, y^*, Ty)
\end{aligned}$$

Thus $Ty^* = y^*$. To prove the uniqueness of y^*, suppose z^* is another fixed point of T such that $y^* \neq z^*$. Then

$$\mathcal{G}(y^*, y^*, z^*) = \mathcal{G}(y^*, y^*, z^*) \preceq h\mathcal{G}(y^*, y^*, z^*) \ \text{ (a contradiction)}$$

So, $y^* = z^*$. Hence, the fixed point is unique. □

Theorem 3.2. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that $(X, \mathcal{G})$ be a complete $\mathcal{G}$ - cone metric space over a Banach algebra $\mathcal{A}$. Suppose that the mapping $T : X \to X$ is comparable and the following assertions holds:*

(1). There exists $h \in \mathcal{P}$ with $\rho(h) \in (0, 1)$ such that $\mathcal{G}(Tx, Ty, Tz) \preceq h\mathcal{G}(x, y, z)$ for any comparable pair $x, y, z \in X$.

(2). There exists $y_0 \in X$ such that $y_0, f(y_0)$ are comparable.

(3). If a sequence $\{y_n\}$ converges to y in X and y_i, y_{i+1} are comparable for all $i \geq 0$ theny_i,y are comparable.

Then T has a fixed point $y^* \in X$.

Proof. Assume $y_n = T(y_0)$, we get that y_n, y_{n+1} are comparable for all $n \geq 0$ and $\{y_n\}$ converges to y^* as in the proof of Theorem 3.1. Now the Condition (3) implies y_n, y^* are comparable. Therefore, the Condition (1) gives that

$$\mathcal{G}(Ty^*, y^*, y^*) \preceq \mathcal{G}(Ty^*, Ty_n, Ty) + \mathcal{G}(Ty_n, y^*, Ty) \preceq h\mathcal{G}(y^*, y_n, y_n) + \mathcal{G}(y_{n+1}, y^*, y^*)$$

Hence for each $c \gg \theta$ we have $\mathcal{G}(Ty^*, y^*, y^*) \ll c$, so $\mathcal{G}(Ty^*, y^*, y^*) = \theta$, which implies that y^* is a fixed point of T. □

Corollary 3.3. *Let $(X, \sqsubseteq)$ be a partially ordered set, $\mathcal{P}$ be an ordered cone and suppose there is a metric G on X such that (X, G) is complete G-cone metric space over a Banach space $\mathcal{A}$. Let $T : X \to X$ be a mapping such that*

$$\mathcal{G}(Tx, Ty, Tz) \preceq \mathcal{G}(x, y, z), \ \ \forall \ \ x, y, z \in X$$

where, $0 < \|h\| < 1, h \in \mathcal{P}, x, y, z \in X$. Then T has a unique fixed point in X.

Proof. The proof holds from the proof of Theorem 3.1 above as $\rho(h) \leq \|h\|$. □

Remark 3.4. *In Theorem 3.1, we only suppose that the spectral radius of $h < 1$, neither $h \prec e$ nor $\|h\| < 1$ is assumed. This is vital. In fact, the condition $\rho(h) < 1$ is weaker than that $\|h\| < 1$, as is given by (Hao and Shaoyuan [5]). Thus if Theorem 3.1 holds then Corollary 3.3 holds. The reverse is not true.*

Theorem 3.5. *Let $(X, \sqsubseteq)$ be a partially ordered set and $(X, \mathcal{G})$ be a complete $\mathcal{G}$ - cone metric space over a Banach algebra $\mathcal{A}$. Suppose that the mapping $T : X \to X$ be comparable and the following two assertions hold:*

(1). There exist $a, b, c, d \in \mathcal{P}$ with $\rho(a) + 2\rho(b) + 2\rho(c) + 2\rho(d) < 1$ such that

$$\mathcal{G}(Tx, Ty, z) \preceq a\mathcal{G}(x, y, z) + 2b\mathcal{G}(x, Tx, z) + 2c\mathcal{G}(y, Ty, z) + 2d\mathcal{G}(Tx, Ty, z)$$

for all $x, y, z \in X$,

(2). There exists $y_0 \in X$ such that $y_0, T(y_0)$ are comparable.

Then T has a unique fixed point in X.

Proof. Let $y_0 \in X$. If $T(y_0) = y_0$ then the proof is completed. Let $T(y_0) \neq y_0$ from condition second and T is comparable, we deduce that $T^i(y_0)$ and T^{i+1} are comparable for any $i \geq 0$. Replacing $y_n = T^n(y_0)$, we recover x_i, x_{i+1} are comparable by condition first, we have a sequence $y_1 = Ty_0, y_2 = T^2y_0, \cdots, y_n = T^ny_0, \ n \geq 1$. We have

$$\begin{aligned}\mathcal{G}(y_{n+1}, y_n, z) &= \mathcal{G}(Ty_n, Ty_{n-1}, z) \preceq a\mathcal{G}(y_n, y_{n-1}, z) \\ &\qquad + 2b\mathcal{G}(y_n, Ty_n, z) + 2c\mathcal{G}(y_{n-1}, Ty_{n-1}, z) + 2d\mathcal{G}(Ty_n, Ty_{n-1}, z) \\ \mathcal{G}(y_{n+1}, y_n, z) &= \mathcal{G}(Ty_n, Ty_{n-1}, z) \preceq a\mathcal{G}(y_n, y_{n-1}, z)\end{aligned}$$

$$
\begin{aligned}
&+ 2b\mathcal{G}(y_n, Ty_n, z) + 2c\mathcal{G}(y_{n-1}, Ty_{n-1}, z) + 2d\mathcal{G}(Ty_n, Ty_{n-1}, z)\\
\mathcal{G}(y_{n+1}, y_n, z) &= \mathcal{G}(Ty_n, Ty_{n-1}, z) \preceq a\mathcal{G}(y_n, y_{n-1}, z)\\
&+ 2b\mathcal{G}(y_{n+1}, y_n, z) + 2c\mathcal{G}(y_n, y_{n-1}, z) + 2d\mathcal{G}(y_{n+1}, y_n, z)\\
\mathcal{G}(y_{n+1}, y_n, z) &\preceq \left[\frac{a+2c}{e-2b-2d}\right]\mathcal{G}(y_1, y_0, z)
\end{aligned}
$$

Hence

$$
\mathcal{G}(y_{n+1}, y_n, z) \preceq \left[\frac{a+2c}{e-2b-2d}\right]^n \mathcal{G}(y_1, y_0, z)
$$

Set $\beta = \left[\frac{a+2c}{e-2b-2d}\right]$ such that

$$
\mathcal{G}(y_{n+1}, y_n, z) \preceq \beta^n \mathcal{G}(y_1, y_0, z)
$$

Let K be $\mathcal{G}$ bound for X. For $y_\alpha \in X, 0 \leq \alpha \leq n$, we have

$$
\|\mathcal{G}(y_1, y_0, y_\alpha)\| \preceq \mathcal{K}
$$

Use the rectangle inequality with $n > m$, we obtain

$$
\begin{aligned}
\mathcal{G}(y_n, y_m, z) &\preceq \mathcal{G}(y_n, y_{n-1}, y_{n-1}) + \mathcal{G}(y_{n-1}, y_{n-1}, y_{n-1}) + \cdots + \mathcal{G}(y_{m+1}, y_m, z)\\
&\preceq \beta^{n-1}\mathcal{K} + \beta^{n-2}\mathcal{K} + \beta^{n-3}\mathcal{K} + \cdots + \beta^{m+2}\mathcal{K} + \beta^{m+1}\mathcal{G}(y_1, y_0, z)\\
&\preceq (\beta^{n-1} + \beta^{n-2} + \beta^{n-3} + \cdots + \beta^{m+2})\mathcal{K} + \beta^{m+1}\mathcal{G}(y_1, y_0, z)\\
&\preceq (\beta^{m+2} + \beta^{m+3} + \cdots + \beta^{n-1})\mathcal{K} + \beta^{m+1}\mathcal{G}(y_1, y_0, z)\\
&\preceq (1 + \beta + \beta^2 + \cdots + \beta^{n-m-3})\beta^{m+2}\mathcal{K} + \beta^{m+1}\mathcal{G}(y_1, y_0, z)\\
&\preceq \beta^{m+2}\mathcal{K}\sum_{i=0}^{n-m-3}\beta^i + \beta^{m+1}\mathcal{G}(y_1, y_0, z)\\
&\preceq \beta^{m+2}\mathcal{K}(e-\beta)^{-1}\beta^{m+1}\mathcal{G}(y_1, y_0, z)
\end{aligned}
$$

By Lemma 2.7 and 2.8, we have

$$
\begin{aligned}
\rho(\beta^{m+2}(e-\beta)^{-1})\mathcal{K} &\leq (\rho(e-\beta)^{-1})(\rho(\beta))^{m+2}\\
&\leq \frac{(\rho(\beta))^{m+2}}{1-\rho(\beta)}\\
&< 1
\end{aligned}
$$

and

$$
\rho(\beta)^{m+1} \leq (\rho(\beta))^{m+1}
$$

Using Remark 2.2 and Lemma 2.10

$$
\|\beta^{m+2}(e-\beta)^{-1}\| \to 0, \|\beta^{m+2}\| \to 0 \quad \text{as} \quad m \to \infty
$$

for any $c \in \mathcal{A}$ with $\theta \ll c$, there exist $N \in \mathbb{N}$ such that $m > n > N$, we have that

$$
\mathcal{G}(y_n, y_m, y_m) \preceq \mathcal{K}^n(e-\mathcal{K})^{-1}\mathcal{G}(y_0, y_1, y_1) \ll c
$$

which implies that $\{y_n\}$ is Cauchy. Since X is complete, there exists $y^* \in X$ such that $y_n \to y^*$ as $n \to \infty$ and $Ty^* \in X$. Then we get

$$\|\beta^{m+2}(e-\beta)^{-1}\mathcal{K}\| \to 0, \|\beta^{m+1}\mathcal{G}(x_1, x_0, z)\| \to 0, \text{as} \quad m \to 0$$

Use Lemma 2.9 and for any $c \in \mathcal{A}$ with $\frac{c}{2} \gg \theta, n > m > N^1, N^1 \in N$, we have

$$\begin{aligned}\mathcal{G}(x_n, x_m, z) &\preceq [\beta^{m+2}(e-\beta)^{-1}]\mathcal{K} + \beta^{m+1}\mathcal{G}(x_0, x_1, z)\\ &\ll \frac{c}{2} + \frac{c}{2} = c\end{aligned}$$

which implies that $\{y_n\}$ is Cauchy. Since X is complete, there exists $y^* \in X$ such that $y_n \to y^*$ as $n \to \infty$ and $Ty^* \in X$ By continuity, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{G}(y_{n+1}, y_n, Ty^*) &= \mathcal{G}\left(\lim_{n\to\infty} y_{n+1}, \lim_{n\to\infty} y_n, Ty^*\right)\\ &= \mathcal{G}(y^*, y^*, Ty^*)\end{aligned}$$

Thus

$$Ty^* = y^*$$

To prove the uniqueness of y^* suppose z^* is another fixed point of T such that $y^* \neq z^*$. Then

$$\begin{aligned}\mathcal{G}(y^*, y^*, z^*) &= \mathcal{G}(Ty^*, Ty^*, Tz^*)\\ \mathcal{G}(Ty^*, Ty^*, Tz^*) &\preceq a\mathcal{G}(y^*, y^*, z^*) + 2b\mathcal{G}(y^*, Ty^*, z^*) + 2d\mathcal{G}(Ty^*, Tz^*, z^*)\\ \mathcal{G}(y^*, y^*, z^*) &\leq a\mathcal{G}(y^*, y^*, z^*) + 2b\mathcal{G}(y^*, y^*, z^*) + 2d\mathcal{G}(y^*, z^*, z^*)\\ (e-a-2b-2d) &\preceq \theta \;\; \text{(contradiction)}\end{aligned}$$

So, $y^* = z^*$. Hence, the fixed point is unique. □

Corollary 3.6. *Let $(X, \sqsubseteq)$ be a ordered $\mathcal{G}$ - cone metric space. Suppose that the mapping $T : X \to X$ satisfies the generalized condition*

$$\mathcal{G}(Tx, Ty, z) \leq a_1\mathcal{G}(x; y; z) + a_2\mathcal{G}(x; Tx; z) + a_3\mathcal{G}(y; Ty; z) + a_4\mathcal{G}(Tx; Ty; z)$$

$\forall\ x, y, z \in X$, where $0 < \rho(a_1) + \rho(a_2) + \rho(a_3) + \rho(a_4) < 1, a_1, a_1, a_1, a_1 \in \mathbb{R}$, $x, y, z \in X$. Then T has a unique fixed point in X. and for any $x \in X$.

Example 3.7. *Let $X = [0, \infty), \mathcal{A} = R^3, \mathcal{P} = (u, v, w) \in R^3 : u, v, w \geq 0$. Define $\mathcal{G} : X^3 \to \mathcal{A}$ by*

$$\mathcal{G}(u, v, w) = (|u-v|, |v-w|)$$

Then $(X, \mathcal{G})$ is a ordered $\mathcal{G}$ - cone metric space. Define $T : X \to X$ by

$$Tu = \frac{u}{4}, u \in X$$

then

$$\mathcal{G}(T^2u, T^2v, T^2w) = \mathcal{G}\left(\frac{u}{16}, \frac{v}{16}, \frac{w}{16}\right) = \left(\frac{1}{16}|u-v|, \frac{1}{16}|v-w|\right)$$

and

$$\mathcal{G}(u, v, w) = (|u-v|, |v-w|).$$

Therefore

$$\mathcal{G}(T^2u, T^2v, T^2w) = \frac{1}{16}\mathcal{G}(u, v, w)$$

Thus with

$$\mu = \frac{1}{16} \in [0, 1); \quad n = 2$$
$$u = \mathcal{G}(u, v, w) \in \{\mathcal{G}(u, v, w), \mathcal{G}(u, Tv, Tw), \mathcal{G}(u, T^2v, T^2w), \mathcal{G}(u, T^2u, T^2u)\}$$

All the conditions of the theorem are satisfied. $u = 0$ *is a unique fixed point of* T.

4. Conclusions

The aim of this paper is to introduce the concept of ordered $\mathcal{G}$ - cone metric space with Banach algebra, which generalizes cone metric space with Banach algebra and we explain some properties of such metric spaces. In addition, we provide some fixed point theorems for generalize contraction mappings in such spaces. Also presented example is constructed to support our result. Our results extend and unify many existing results in the recent literature ([1, 5, 15, 19]).

References

[1] Q. Yan, J. Yin and T. Wang, *Fixed point and common fixed point theorems on ordered cone metric spaces over Banach algebras*, J. Nonlinear Sci. Appl., 9(4)(2016), 1581-1589.

[2] I. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Func. An., Gos. Ped. Inst. Unianowsk, 30(1989), 26-37.

[3] A. Ahmed and J. N. Salunke, *Fixed Point Theorem of Expanding Mapping without Continuity in Cone Metric Space over Banach Algebra*, In International Conference on Recent Trends in Engineering and Science, 20(2017), 19-22.

[4] L. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 332(2)(2007), 1468-1476.

[5] L. Hao and X. Shaoyuan, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory and Applications, (2013), 1-10.

[6] S. Xu and S. Radenovic, *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebra without assumption of normality*, Fixed Point Theory Appl., (2014), 1-12.

[7] O. K. Adewale and E. K. Osawaru, *G-cone metric Spaces over Banach Algebras and Some Fixed Point Results*, International Journal of Mathematical Analysis and Optimization: Theory and Applications, 2(2019), 546-557.

[8] B. Dhage, *Generalized metric space and mapping with fixed points*, Bulletin of the Calcutta Mathematical Society, 84(1992), 329-336.

[9] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Analysis, 7(2)(2006), 289-297.

[10] Z. Mustafa and B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory and Applications, (2009), 1-10.

[11] J. O. Olaleru, *Some generalizations of fixed point theorems in cone metric spaces*, Mathematics E-Note, 11(2010), 41-49.

[12] S. Radenovic and B. E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl., 57(2009), 1701-1707.

[13] L. Hao and X. Shaoyuan, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory and Applications,(2013), 1-10.

[14] I. Beg, M. Abbas and T. Nazir, *Generalized cone metric spaces*, J. Nonlinear Sci. Appl., 3(1)(2010), 21-31.

[15] O. K. Adewale and E. K. Osawaru, *G-cone metric Spaces over Banach Algebras and Some Fixed Point Results*, International Journal of Mathematical Sciences and Optimization Theory and Applications, 2(2019), 546-557.

[16] D. Turkoglu and N. Bilgili, *Some fixed point theorem for mapping on complete G-cone metric spaces*, Journal of Applied Functional Analysis, 7(1-2)(2012), 118-137.

[17] I. Altun and G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mat. Palermo, 58(2)(2009), 319–325.

[18] R. Saadati, S. M. Vaezpour, P. Vetro and B. E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Modelling, 52(2010), 797–801.

[19] H. K. Nashine, Z. Kadelburg, R. P. Pathak and S. Radenović, *Coincidence and fixed point results in ordered G-cone metric spaces*, Mathematical and Computer Modelling, 57(3-4)(2013), 701-709.