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Growth of Generalised K-Iterated Entire Functions

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Abstract: Introducing the idea of generalised k-iterations of k entire functions we study some growth properties of such functions to extend some earlier results.

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1. Introduction

Let $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ be an entire function. The maximum term of $f(z)$ on $|z|=r$ is denoted by $\mu(r,f)$ and is defined as $\mu(r,f)=\max|a_n|r^n$. Also $M(r,f)=\max\limits_{|z|=r}|f(z)|$ is called the maximum modulus of $f(z)$ on $|z|=r$. Juneja, Kapoor and Bajpai [4] introduced the concept of (p,q)-order and lower (p,q)-order denoted by $\rho_{(p,q)}(f)$ and $\lambda_{(p,q)}(f)$ respectively and defined as

$$\rho_{(p,q)}(f)=\lim_{r\to\infty}\sup\frac{\log^{[p]}M(r,f)}{\log^{[q]}r}$$

and

$$\lambda_{(p,q)}(f)=\lim_{r\to\infty}\inf\frac{\log^{[p]}M(r,f)}{\log^{[q]}r}, b\leq\lambda_{(p,q)}(f)\leq\rho_{(p,q)}(f)\leq\infty$$

where $b=1$ if $p=q$ and zero otherwise. Also we have $\log^{[0]}x=x, \exp^{[0]}x=x$ and for positive integer m, $\log^{[m]}x=\log\left(\log^{[m-1]}x\right), \exp^{[m]}x=\exp\left(\exp^{[m-1]}x\right)$.

Again for $0\leq r<R$, we get $\mu(r,f)\leq M(r,f)\leq\frac{R}{R-r}\mu(R,f)$. Therefore

$$\rho_{(p,q)}(f)=\lim_{r\to\infty}\sup\frac{\log^{[p]}\mu(r,f)}{\log^{[q]}r}$$
$$and\quad\lambda_{(p,q)}(f)=\lim_{r\to\infty}\inf\frac{\log^{[p]}\mu(r,f)}{\log^{[q]}r}.$$

A real valued function $\phi(r)$ is said to have the property P [2] if

(i). $\phi(r)$ is non-negative and continuous for $r\geq r_0$, say;

(ii). $\phi(r)$ is strictly increasing and $\phi(r)\to\infty$ as $r\to\infty$; and

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(iii). $\log \phi(r) < \delta\phi(r/2)$, for all $\delta > 0$ and for all sufficiently large values of r.

Therefore a function satisfying the property P also satisfies the following relation:

$$\log^{[p]} \phi(r) < \delta \log^{[q]} \phi(r/2) \text{ for all } \delta > 0,\ p > q \text{ and for all sufficiently large values of } r.$$

In [1], Banerjee and Mondal introduced the generalised iteration of two entire functions and made close investigation on growth properties of maximum modulus and maximum term of generalised iterated entire functions.

In the present note we consider k non-constant entire functions $f_1,\ f_2, ..., f_k$ and a constant α with $0 < \alpha \leq 1$ and form the iteration as below:

$$\begin{aligned}
F_1^1(z) &= (1-\alpha)z + \alpha f_1(z) \\
F_2^1(z) &= (1-\alpha)F_1^2(z) + \alpha f_1\left(F_1^2(z)\right) \\
F_3^1(z) &= (1-\alpha)F_2^2(z) + \alpha f_1\left(F_2^2(z)\right) \\
&\vdots \\
F_n^1(z) &= (1-\alpha)F_{n-1}^2(z) + \alpha f_1\left(F_{n-1}^2(z)\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
F_1^2(z) &= (1-\alpha)z + \alpha f_2(z) \\
F_2^2(z) &= (1-\alpha)F_1^3(z) + \alpha f_2\left(F_1^3(z)\right) \\
F_3^2(z) &= (1-\alpha)F_2^3(z) + \alpha f_2\left(F_2^3(z)\right) \\
&\vdots \\
F_n^2(z) &= (1-\alpha)F_{n-1}^3(z) + \alpha f_2\left(F_{n-1}^3(z)\right)
\end{aligned}$$

and

$$\begin{aligned}
F_1^k(z) &= (1-\alpha)z + \alpha f_k(z) \\
F_2^k(z) &= (1-\alpha)F_1^1(z) + \alpha f_k\left(F_1^1(z)\right) \\
F_3^k(z) &= (1-\alpha)F_2^1(z) + \alpha f_k\left(F_2^1(z)\right) \\
&\vdots \\
F_n^k(z) &= (1-\alpha)F_{n-1}^1(z) + \alpha f_k\left(F_{n-1}^1(z)\right).
\end{aligned}$$

Clearly all $F_n^1(z),\ F_n^2(z), ...,\ F_n^k(z)$ are entire functions. With this definition of generalised k-iterated entire functions we extend the results of Banerjee and Mondal [2] in this direction. Throughout this paper we assume that maximum modulus functions of $f_1, f_2, ..., f_k$ and all their generalised iterated functions satisfy the property P.

2. Lemmas

In this section we present two lemmas which will be needed in the sequel.

Lemma 2.1 ([3]). *Let $f(z)$ and $g(z)$ be two entire functions with $g(0)=0$. Let β satisfies $0<\beta<1$ and $C(\beta)=\frac{(1-\beta)^2}{4\beta}$. Then for $r>0$*

$$M(r,f\circ g)\geq M(C(\beta)M(\beta r,g),f).$$

Further if $g(z)$ is any entire function, then with $\beta=\frac{1}{2}$, for sufficiently large values of r

$$M(r,f\circ g)\geq M\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f\right).$$

Clearly

$$M(r,f\circ g)\geq M\left(\frac{1}{16}M\left(\frac{r}{2},g\right),f\right). \tag{1}$$

On the other hand the opposite inequality

$$M(r,f\circ g)\leq M(M(r,g),f) \tag{2}$$

is an immediate consequence of the definition.

Lemma 2.2 ([6]). *Let $f(z)$ and $g(z)$ be entire functions with $g(0)=0$. Let β satisfies $0<\beta<1$ and $C(\beta)=\frac{(1-\beta)^2}{4\beta}$. Also let $0<\delta<1$. Then*

$$\mu(r,f\circ g)\geq(1-\delta)\mu(C(\beta)\mu(\beta\delta r,g),f).$$

Again if $g(z)$ is any entire function, then with $\beta=\delta=\frac{1}{2}$, for sufficiently large values of r

$$\mu(r,f\circ g)\geq\frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|,f\right).$$

Clearly

$$\mu(r,f\circ g)\geq\frac{1}{2}\mu\left(\frac{1}{16}\mu\left(\frac{r}{4},g\right),f\right). \tag{3}$$

3. Main Results

Theorem 3.1. *Let $f_1,f_2,...,f_k$ be k entire functions with positive lower (p,q)-order and of finite (p,q)-order. Then for every positive constant γ, $p>q$ and every real number x*

$$\lim_{r\to\infty}\frac{\log^{[p]}M\left(r,F_n^1\right)}{\left\{\log^{[p]}M\left(r^{\gamma},f_1\right)\right\}^{1+x}}=\infty. \tag{4}$$

Proof. Suppose $1+x>0$. Otherwise the result is obvious. Let $0<\alpha<1$. Choose $0<\varepsilon<\min\left\{\lambda_{(p,q)}(f_1),\lambda_{(p,q)}(f_2),...,\lambda_{(p,q)}(f_k)\right\}$. For sufficiently large values of r, using (1), we get

$$\begin{aligned}M\left(r,F_n^1\right)&=M\left(r,(1-\alpha)F_{n-1}^2+\alpha f_1\left(F_{n-1}^2\right)\right)\\&\geq M\left(r,\alpha f_1\left(F_{n-1}^2\right)\right)-M\left(r,(1-\alpha)F_{n-1}^2\right)\\&\geq\alpha M\left(\frac{1}{16}M\left(\frac{r}{2},F_{n-1}^2\right),f_1\right)-(1-\alpha)M\left(r,F_{n-1}^2\right).\end{aligned}$$

Therefore for all sufficiently large values of r, we get

$$\log M\left(r,F_n^1\right)\geq\log M\left(\frac{1}{16}M\left(\frac{r}{2},F_{n-1}^2\right),f_1\right)-\log M\left(r,F_{n-1}^2\right)+O(1).$$

Therefore

$$\begin{aligned}
\log^{[p]} M\left(r, F_n^1\right) &\geq \log^{[p]} M\left(\frac{1}{16} M\left(\frac{r}{2}, F_{n-1}^2\right), f_1\right) - \log^{[p]} M\left(r, F_{n-1}^2\right) + O(1) \\
&> \left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]}\left(\frac{1}{16} M\left(\frac{r}{2}, F_{n-1}^2\right)\right) - \log^{[p]} M\left(r, F_{n-1}^2\right) + O(1) \\
&> \left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]} M\left(\frac{r}{2}, F_{n-1}^2\right) - \frac{1}{2}\left(\left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]} M\left(\frac{r}{2}, F_{n-1}^2\right)\right) + O(1), \quad \text{by property P} \\
&= \frac{1}{2}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]} M\left(\frac{r}{2}, F_{n-1}^2\right) + O(1) \\
&\geq \frac{1}{2}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[p]} M\left(\frac{r}{2}, F_{n-1}^2\right) + O(1). \qquad (5)
\end{aligned}$$

Therefore using (5) we get

$$\log^{[p]} M\left(r, F_n^1\right) > \frac{1}{2^2}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)\left(\lambda_{(p,q)}(f_2) - \varepsilon\right) \log^{[p]} M\left(\frac{r}{2^2}, F_{n-2}^3\right) + O(1).$$

Proceeding in this way we get for $n = km$, $m \in \mathbb{N}$

$$\begin{aligned}
\log^{[p]} M\left(r, F_n^1\right) &> \frac{1}{2^{n-k}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n-k}{k}}\left(\lambda_{(p,q)}(f_2) - \varepsilon\right)^{\frac{n-k}{k}} \ldots \left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n-k}{k}} \times \log^{[p]} M\left(\frac{r}{2^{n-k}}, F_k^1\right) + O(1) \\
&> \frac{1}{2^{n-(k-1)}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_2) - \varepsilon\right)^{\frac{n}{k}-1} \ldots \left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n}{k}-1} \\
&\quad \times \log^{[q]} M\left(\frac{r}{2^{n-(k-1)}}, F_{k-1}^2\right) + O(1) \\
&\quad \vdots \\
&> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n}{k}} \ldots \left(\lambda_{(p,q)}(f_{k-1}) - \varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n}{k}-1} \times \log^{[q]} M\left(\frac{r}{2^{n-1}}, F_1^k\right) + O(1) \\
&= \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n}{k}} \ldots \left(\lambda_{(p,q)}(f_{k-1}) - \varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n}{k}-1} \\
&\quad \times \log^{[q]} M\left[\frac{r}{2^{n-1}}, (1-\alpha) z + \alpha f_k(z)\right] + O(1) \\
&\geq \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n}{k}} \ldots \left(\lambda_{(p,q)}(f_{k-1}) - \varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n}{k}-1} \\
&\quad \times \left[\log^{[q]} M\left(\frac{r}{2^{n-1}}, \alpha f_k(z)\right) - \log^{[q]} M\left(\frac{r}{2^{n-1}}, (1-\alpha) z\right)\right] + O(1) \\
&= \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n}{k}} \ldots \left(\lambda_{(p,q)}(f_{k-1}) - \varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n}{k}-1} \\
&\quad \times \left[\exp^{[p-q-1]}\left(\log^{[p-1]} M\left(\frac{r}{2^{n-1}}, f_k(z)\right)\right) - \log^{[q]} M\left(\frac{r}{2^{n-1}}, z\right)\right] + O(1) \\
&\geq \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n}{k}} \ldots \left(\lambda_{(p,q)}(f_{k-1}) - \varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n}{k}-1} \\
&\quad \times \left[\exp^{[p-q-1]}\left\{\log^{[q-1]} \frac{r}{2^{n-1}}\right\}^{\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)} - \log^{[q]} \frac{r}{2^{n-1}}\right] + O(1). \qquad (6)
\end{aligned}$$

Now we choose r sufficiently large so that for every $\gamma > 0$

$$\left\{\log^{[p]} M\left(r^\gamma, f_1\right)\right\}^{1+x} < \left(\rho_{(p,q)}(f_1) + \varepsilon\right)^{1+x}\left\{\log^{[q]} r^\gamma\right\}^{1+x}. \qquad (7)$$

From (6) and (7) for sufficiently large r we get

$$\begin{aligned}
\frac{\log^{[p]} M\left(r, F_n^1\right)}{\left\{\log^{[p]} M\left(r^\gamma, f_1\right)\right\}^{1+x}} &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n}{k}} \ldots \left(\lambda_{(p,q)}(f_{k-1}) - \varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n}{k}-1} \\
&\quad \times \frac{\left[\exp^{[p-q-1]}\left\{\log^{[q-1]} \frac{r}{2^{n-1}}\right\}^{\left(\lambda_{(p,q)}(f_k) - \varepsilon\right)} - \log^{[q]} \frac{r}{2^{n-1}}\right] + O(1)}{\left(\rho_{(p,q)}(f_1) + \varepsilon\right)^{1+x}\left\{\log^{[q]} r^\gamma\right\}^{1+x}} \\
&\to \infty \quad \text{as} \quad r \to \infty.
\end{aligned}$$

Now let $n = km - 1$, $m \in \mathbb{N}$. Then

$$\begin{aligned}
\log^{[p]} M\left(r, F_n^1\right) &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n+1}{k}} .. \left(\lambda_{(p,q)}\left(f_{k-2}\right)-\varepsilon\right)^{\frac{n+1}{k}}\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}} \\
&\quad \times\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}}[\exp^{[p-q-1]}\left(\log^{[p-1]} M\left(\frac{r}{2^{n-1}}, f_{k-1}(z)\right)\right)-\log^{[q]} M\left(\frac{r}{2^{n-1}}, z\right)]+O(1) \\
&\geq \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n+1}{k}} \ldots\left(\lambda_{(p,q)}\left(f_{k-2}\right)-\varepsilon\right)^{\frac{n+1}{k}}\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}} \\
&\quad \left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}}\left[\exp^{[p-q-1]}\left\{\log^{[q-1]} \frac{r}{2^{n-1}}\right\}^{\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)}-\log^{[q]} \frac{r}{2^{n-1}}\right]+O(1) .
\end{aligned}$$

Therefore the statement (4) follows.

When $n = km - (k-1)$, $m \in \mathbb{N}$. Then

$$\begin{aligned}
\log^{[p]} M\left(r, F_n^1\right) &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n-1}{k}}\left(\lambda_{(p,q)}\left(f_2\right)-\varepsilon\right)^{\frac{n-1}{k}} \ldots\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n-1}{k}} \\
&\quad \times\left[\exp^{[p-q-1]}\left\{\log^{[q-1]} \frac{r}{2^{n-1}}\right\}^{\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)}-\log^{[q]} \frac{r}{2^{n-1}}\right]+O(1) . \qquad (8)
\end{aligned}$$

Consequently the statement (4) follows.

This completes the proof. $\square$

Corollary 3.2. *Let $f_1, f_2, ..., f_k$ be k entire functions with positive lower (p,q)-order and of finite (p,q)-order. Then for every positive constant $\gamma, p > q$ and every real number x,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M\left(r, F_n^1\right)}{\left\{\log^{[p]} M\left(r^\gamma, f_k\right)\right\}^{1+x}} = \infty.$$

Proof. Using $\left\{\log^{[p]} M\left(r^\gamma, f_k\right)\right\}^{1+x} < \left(\rho_{(p,q)}\left(f_k\right)+\varepsilon\right)^{1+x}\left\{\log^{[q]} r^\gamma\right\}^{1+x}$ in place of (7) we get the result. $\square$

Note 3.3. *Theorem 3.1 does not hold when $\lambda_{(p,q)}(f_1) = 0$ or $\lambda_{(p,q)}(f_2) = 0$ or... or $\lambda_{(p,q)}(f_k) = 0$, which follows from the following example.*

Example 3.4. *Let $f_1 = f_2 = ... = f_k = z; x = 0$ and $\gamma = 1$. Then $\lambda_{(p,q)}(f_k) = 0$ and $F_n^1(z) = f_1(z)$ for every n and*

$$\lim_{r\to\infty} \frac{\log^{[p]} M\left(r, F_n^1\right)}{\left\{\log^{[p]} M\left(r^\gamma, f_1\right)\right\}^{1+x}} = 1.$$

Theorem 3.5. *Let $f_1, f_2, ..., f_k$ be k entire functions of finite (p,q) order and $\lambda_{(p,q)}(f_1) > 0$. Then for $h > 0$, $p > q$ and for every real number x*

$$\lim_{r\to\infty} \frac{\left\{\log^{[p+(n-2)(p-q)]} M\left(r, F_n^1\right)\right\}^{1+x}}{\log^{[p]} M\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^h, f_1\right)} = 0$$

where, $h > (1+x)\rho_{(p,q)}(f_k)$ if $p = 2, q = 1$ and $h > \rho_{(p,q)}(f_k)$ otherwise for $n = km, m \in \mathbb{N}$.

Proof. Here $\lambda_{(p,q)}(f_1) > 0$. So f_1 is not a polynomial. Therefore $M(r, f_1) \geq r$ for all large values of r. Again $M(r, f_k) \geq \mu r$ for some $\mu > 0$ and for all large values of r. Thus for all large values of r

$$M(r, f_1) \geq cr \quad \text{and} \quad M(r, f_k) \geq cr \qquad (9)$$

where, $c = \min\{1, \mu\}$. So $\frac{1}{c} \geq 1$. Now for all large values of r, using (2) and (9), we get

$$M\left(r, F_n^1\right) \leq (1-\alpha) M\left(r, F_{n-1}^2\right)+\alpha M\left(r, f_1\left(F_{n-1}^2\right)\right)$$

$$\begin{aligned} &\leq (1-\alpha)\frac{1}{c}M\left(M\left(r,F_{n-1}^{2}\right),f_{1}\right)+\frac{1}{c}\alpha M\left(M\left(r,F_{n-1}^{2}\right),f_{1}\right) \\ &= \frac{1}{c}M\left(M\left(r,F_{n-1}^{2}\right),f_{1}\right). \end{aligned}$$

Therefore

$$\log^{[p]} M\left(r,F_{n}^{1}\right) < \left(\rho_{(p,q)}\left(f_{1}\right)+\varepsilon\right)\log^{[q]} M\left(r,F_{n-1}^{2}\right)+O\left(1\right). \tag{10}$$

Now from (10), we get

$$\log^{[p+(p-q)]} M(r,F_{n}^{1}) < (\rho_{(p,q)}\left(f_{2}\right)+\varepsilon)\log^{[q]} M\left(r,F_{n-2}^{3}\right)+O\left(1\right).$$

Therefore

$$\log^{[p+2(p-q)]} M(r,F_{n}^{1}) < (\rho_{(p,q)}\left(f_{3}\right)+\varepsilon)\log^{[q]} M\left(r,F_{n-3}^{4}\right)+O\left(1\right).$$

Proceeding similarly we get for $n=km$, $m\in\mathbb{N}$

$$\begin{aligned} \log^{[p+(n-2)(p-q)]} M(r,F_{n}^{1}) &< (\rho_{(p,q)}(f_{k-1})+\varepsilon)\log^{[q]} M\left(r,F_{1}^{k}\right)+O\left(1\right) \\ &= \left(\rho_{(p,q)}\left(f_{k-1}\right)+\varepsilon\right)\log^{[q]} M\left(r,(1-\alpha)z+\alpha f_{k}\left(z\right)\right)+O\left(1\right) \\ &= \left(\rho_{(p,q)}\left(f_{k-1}\right)+\varepsilon\right)\left[\log^{[q]} M\left(r,\alpha f_{k}\left(z\right)\right)+\log^{[q]} M(r,(1-\alpha)z)\right]+O\left(1\right) \\ &\leq \left(\rho_{(p,q)}\left(f_{k-1}\right)+\varepsilon\right)\left[\log^{[q]} M\left(r,f_{k}\left(z\right)\right)+\log^{[q]} M(r,f_{k}(z))\right]+O\left(1\right) \\ &= 2\left(\rho_{(p,q)}\left(f_{k-1}\right)+\varepsilon\right)\log^{[q]} M\left(r,f_{k}\left(z\right)\right)+O\left(1\right) \\ &< 2\left(\rho_{(p,q)}\left(f_{k-1}\right)+\varepsilon\right)\left\{\exp^{[p-q-1]}\left(\log^{[q-1]} r\right)^{\left(\rho_{(p,q)}(f_{k})+\varepsilon\right)}\right\}+O\left(1\right). \end{aligned} \tag{11}$$

Again for sufficiently large vales of r, we get

$$\log^{[p]} M\left(r,f_{1}\right) > \left(\lambda_{(p,q)}\left(f_{1}\right)-\varepsilon\right)\log^{[q]} r.$$

Hence for all sufficiently large values of r we get from (11)

$$\frac{\left\{\log^{[p+(n-2)(p-q)]} M\left(r,F_{n}^{1}\right)\right\}^{1+x}}{\log^{[p]} M\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^{h},f_{1}\right)} < \frac{\left\{2\left(\rho_{(p,q)}\left(f_{k-1}\right)+\varepsilon\right)\exp^{[p-q-1]}\left(\log^{[q-1]} r\right)^{\left(\rho_{(p,q)}(f_{k})+\varepsilon\right)}\right\}^{1+x}+O\left(1\right)}{\left(\lambda_{(p,q)}\left(f_{1}\right)-\varepsilon\right)\exp^{[p-q-1]}\left(\log^{[q-1]} r\right)^{h}}$$

Now we choose $0<\varepsilon<\min\left\{\lambda_{(p,q)}\left(f_{1}\right),\frac{h}{1+x}-\rho_{(p,q)}\left(f_{k}\right)\right\}$ if $p=2$, $q=1$ and $0<\varepsilon<\min\left\{\lambda_{(p,q)}\left(f_{1}\right),h-\rho_{(p,q)}\left(f_{k}\right)\right\}$ otherwise. Therefore the statement follows. □

Corollary 3.6. *Let $f_1, f_2, ..., f_k$ be k entire functions of finite (p,q)-order and $\lambda_{(p,q)}\left(f_{1}\right)>0$. Then for $h>0$, $p>q$ and for every real number x*

$$\lim_{r\to\infty}\frac{\left\{\log^{[p+(n-2)(p-q)]} M\left(r,F_{n}^{1}\right)\right\}^{1+x}}{\log^{[p]} M\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^{h},f_{1}\right)}=0$$

where, $h>(1+x)\rho_{(p,q)}\left(f_{1}\right)$ if $p=2$, $q=1$ and $h>\rho_{(p,q)}\left(f_{1}\right)$ otherwise for $n=km-(k-1)$, $m\in\mathbb{N}$.

Proof. When $n=km-(k-1)$, $m\in\mathbb{N}$ proceeding similarly we get

$$\frac{\left\{\log^{[p+(n-2)(p-q)]} M\left(r,F_{n}^{1}\right)\right\}^{1+x}}{\log^{[p]} M\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^{h},f_{1}\right)} < \frac{\left\{2\left(\rho_{(p,q)}\left(f_{k}\right)+\varepsilon\right)\exp^{[p-q-1]}\left(\log^{[q-1]} r\right)^{\left(\rho_{(p,q)}(f_{1})+\varepsilon\right)}\right\}^{1+x}+O\left(1\right)}{\left(\lambda_{(p,q)}\left(f_{1}\right)-\varepsilon\right)\exp^{[p-q-1]}\left(\log^{[q-1]} r\right)^{h}}$$

Now we choose $0<\varepsilon<\min\left\{\lambda_{(p,q)}\left(f_{1}\right),\frac{h}{1+x}-\rho_{(p,q)}\left(f_{1}\right)\right\}$ if $p=2$, $q=1$ and $0<\varepsilon<\min\left\{\lambda_{(p,q)}\left(f_{1}\right),h-\rho_{(p,q)}\left(f_{1}\right)\right\}$ otherwise. Therefore the statement follows. □

Note 3.7. *Theorem 3.5 does not hold if* $h = (1+x)\,\rho_{(p,q)}(f_k)$ *where* $p=2$, $q=1$, $\alpha=1$ *and* $n=km$, $m\in\mathbb{N}$, *which follows from the following example.*

Example 3.8. *Let* $f_k(z)=z$, $p=2$, $q=1$, $\alpha=1$ *and* $n=km$, $m\in\mathbb{N}$. *Then* $\rho_{(p,q)}(f_k)=0$ *and therefore* $h=0$. *Here*

$$\lim_{r\to\infty}\frac{\left\{\log^{[p+(n-2)(p-q)]}M\left(r,F_n^1\right)\right\}^{1+x}}{\log^{[p]}M\left(\exp^{[p-1]}\left(\log^{[q-1]}r\right)^{h},f_1\right)}=\infty.$$

Theorem 3.9. *Let* $f_1,f_2,...,f_k$ *be* k *entire functions of finite* (p,q) *order and* $\lambda_{(p,q)}(f_k)>\rho_{(p,q)}(f_1)\geq\lambda_{(p,q)}(f_1)>0$. *Then for* $p>q$ *and* $n=km$, $m\in\mathbb{N}$

$$\lim_{r\to\infty}\frac{\log^{[p]}M\left(r,F_n^1\right)}{\log^{[p]}M\left(\exp^{[p-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_1)},f_k\right)}=\infty.$$

Proof. We choose ε such that $0<\varepsilon<\lambda_{(p,q)}(f_k)-\rho_{(p,q)}(f_1)$. We get from (6) for all $r\geq r_0$

$$\begin{aligned}\log^{[p]}M\left(r,F_n^1\right) &\geq \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}(f_{k-1})-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k)-\varepsilon\right)^{\frac{n}{k}-1}\\ &\quad\times\left[\exp^{[p-q-1]}\left\{\log^{[q-1]}\frac{r}{2^{n-1}}\right\}^{\left(\lambda_{(p,q)}(f_k)-\varepsilon\right)}-\log^{[q]}\frac{r}{2^{n-1}}\right]+O(1).\end{aligned}$$

Again we also have for all $r\geq r_0$

$$\log^{[p]}M(r,f_k)<\left(\rho_{(p,q)}(f_k)+\varepsilon\right)\log^{[q]}r.$$

We choose r so large that

$$\exp^{[p-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_1)}\geq r_0.$$

Then

$$\log^{[p]}M\left(\exp^{[p-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_1)},f_k\right)<\left(\rho_{(p,q)}(f_k)+\varepsilon\right)\left\{\exp^{[p-q-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_1)}\right\}$$

Thus for sufficiently large r

$$\begin{aligned}\frac{\log^{[p]}M\left(r,F_n^1\right)}{\log^{[p]}M\left(\exp^{[p-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_1)},f_k\right)} &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}(f_{k-1})-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}(f_k)-\varepsilon\right)^{\frac{n}{k}-1}\\ &\quad\times\frac{\left[\exp^{[p-q-1]}\left\{\log^{[q-1]}\frac{r}{2^{n-1}}\right\}^{\left(\lambda_{(p,q)}(f_k)-\varepsilon\right)}-\log^{[q]}\frac{r}{2^{n-1}}\right]+O(1)}{\left(\rho_{(p,q)}(f_k)+\varepsilon\right)\left\{\exp^{[p-q-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_1)}\right\}}\\ &\to\infty\quad\text{as}\quad r\to\infty.\end{aligned}$$

Therefore

$$\lim_{r\to\infty}\frac{\log^{[p]}M\left(r,F_n^1\right)}{\log^{[p]}M\left(\exp^{[p-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_1)},f_k\right)}=\infty.$$

This completes the proof. □

Corollary 3.10. *Let* $f_1,f_2,...,f_k$ *be* k *entire functions of finite* (p,q) *order and* $\lambda_{(p,q)}(f_1)>\rho_{(p,q)}(f_k)\geq\lambda_{(p,q)}(f_k)>0$. *Then for* $p>q$ *and* $n=km-(k-1)\,,m\in\mathbb{N}$

$$\lim_{r\to\infty}\frac{\log^{[p]}M\left(r,F_n^1\right)}{\log^{[p]}M\left(\exp^{[p-1]}\left(\log^{[q-1]}r\right)^{\rho_{(p,q)}(f_k)},f_1\right)}=\infty.$$

Proof. We choose ε such that $0 < \varepsilon < \lambda_{(p,q)}(f_1) - \rho_{(p,q)}(f_k)$. Here we choose r so large that

$$\exp^{[p-1]}\left(\log^{[q-1]} r\right)^{\rho_{(p,q)}(f_k)} \geq r_0.$$

From (8) we have for sufficiently large r,

$$\begin{aligned}\frac{\log^{[p]} M\left(r, F_n^1\right)}{\log^{[p]} M\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^{\rho_{(p,q)}(f_k)}, f_1\right)} &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)^{\frac{n-1}{k}}\left(\lambda_{(p,q)}(f_2) - \varepsilon\right)^{\frac{n-1}{k}} \ldots \left(\lambda_{(p,q)}(f_k) - \varepsilon\right)^{\frac{n-1}{k}} \\ &\quad \times \frac{\left[\exp^{[p-q-1]}\left\{\log^{[q-1]} \frac{r}{2^{n-1}}\right\}^{\left(\lambda_{(p,q)}(f_1) - \varepsilon\right)} - \log^{[q]} \frac{r}{2^{n-1}}\right] + O(1)}{\left(\rho_{(p,q)}(f_1) + \varepsilon\right)\left\{\exp^{[p-q-1]}\left(\log^{[q-1]} r\right)^{\rho_{(p,q)}(f_k)}\right\}} \\ &\to \infty \quad \text{as} \quad r \to \infty.\end{aligned}$$

Hence the result follows. □

Theorem 3.11. *Let $f_1, f_2, ..., f_k$ be k entire functions with positive lower (p,q)-order and of finite (p,q)-order and suppose*

$$\log^{[p]} \mu\left(r, F_n^i\right) \leq \delta \log^{[q]} \mu\left(\frac{r}{4}, F_n^i\right), \quad \textit{for} \quad i = 1, 2, ..., k$$

holds for every $\delta > 0$ and for every positive integer n. Then for every positive constant γ, $p > q$ and every real number x

$$\lim_{r \to \infty} \frac{\log^{[p]} \mu\left(r, F_n^1\right)}{\left\{\log^{[p]} \mu\left(r^{\gamma}, f_1\right)\right\}^{1+x}} = \infty. \tag{12}$$

Proof. Suppose $1 + x > 0$. Otherwise the result is obvious. Let $0 < \alpha < 1$. Choose $0 < \varepsilon < \min\left\{\lambda_{(p,q)}(f_1), \lambda_{(p,q)}(f_2), ..., \lambda_{(p,q)}(f_k)\right\}$. For sufficiently large values of r, using (3), we get

$$\begin{aligned}\mu\left(r, F_n^1\right) &= \mu\left(r, (1-\alpha) F_{n-1}^2 + \alpha f_1\left(F_{n-1}^2\right)\right) \\ &\geq \mu\left(r, \alpha f_1\left(F_{n-1}^2\right)\right) - \mu\left(r, (1-\alpha) F_{n-1}^2\right) \\ &\geq \frac{1}{2}\alpha\mu\left(\frac{1}{16}\mu\left(\frac{r}{4}, F_{n-1}^2\right), f_1\right) - (1-\alpha)\mu\left(r, F_{n-1}^2\right).\end{aligned}$$

Therefore for all sufficiently large values of r, we get

$$\log \mu\left(r, F_n^1\right) \geq \log \mu\left(\frac{1}{16}\mu\left(\frac{r}{4}, F_{n-1}^2\right), f_1\right) - \log \mu\left(r, F_{n-1}^2\right) + O(1).$$

Therefore

$$\begin{aligned}\log^{[p]} \mu\left(r, F_n^1\right) &\geq \log^{[p]} \mu\left(\frac{1}{16}\mu\left(\frac{r}{4}, F_{n-1}^2\right), f_1\right) - \log^{[p]} \mu\left(r, F_{n-1}^2\right) + O(1) \\ &> \left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]}\left(\frac{1}{16}\mu\left(\frac{r}{4}, F_{n-1}^2\right)\right) - \log^{[p]} \mu\left(r, F_{n-1}^2\right) + O(1) \\ &> \left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]} \mu\left(\frac{r}{4}, F_{n-1}^2\right) - \frac{1}{2}\left(\left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]} \mu\left(\frac{r}{4}, F_{n-1}^2\right)\right) + O(1), \text{ by hypothesis} \\ &= \frac{1}{2}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[q]} \mu\left(\frac{r}{4}, F_{n-1}^2\right) + O(1) \\ &\geq \frac{1}{2}\left(\lambda_{(p,q)}(f_1) - \varepsilon\right) \log^{[p]} \mu\left(\frac{r}{4}, F_{n-1}^2\right) + O(1).\end{aligned} \tag{13}$$

Therefore using (13), we get

$$\log^{[p]}\mu\left(r,F_n^1\right)>\frac{1}{2^2}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)\left(\lambda_{(p,q)}\left(f_2\right)-\varepsilon\right)\log^{[p]}\mu\left(\frac{r}{4^2},F_{n-2}^3\right)+O\left(1\right).$$

Proceeding in this way we get for $n=km$, $m\in\mathbb{N}$

$$\begin{aligned}
\log^{[p]}\mu\left(r,F_n^1\right) &> \frac{1}{2^{n-k}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n-k}{k}}\left(\lambda_{(p,q)}\left(f_2\right)-\varepsilon\right)^{\frac{n-k}{k}}...\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n-k}{k}}\times\log^{[p]}\mu\left(\frac{r}{4^{n-k}},F_k^1\right)+O\left(1\right)\\
&> \frac{1}{2^{n-(k-1)}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}\left(f_2\right)-\varepsilon\right)^{\frac{n}{k}-1}...\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n}{k}-1}\times\log^{[q]}\mu\left(\frac{r}{4^{n-(k-1)}},F_{k-1}^2\right)+O\left(1\right)\\
&\ \vdots\\
&> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n}{k}-1}\times\log^{[q]}\mu\left(\frac{r}{4^{n-1}},F_1^k\right)+O\left(1\right)\\
&= \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n}{k}-1}\\
&\quad\times\log^{[q]}\mu\left[\frac{r}{4^{n-1}},\left(1-\alpha\right)z+\alpha f_k\left(z\right)\right]+O\left(1\right)\\
&\geq \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n}{k}-1}\\
&\quad\times\left[\log^{[q]}\mu\left(\frac{r}{4^{n-1}},\alpha f_k\left(z\right)\right)-\log^{[q]}\mu\left(\frac{r}{4^{n-1}},\left(1-\alpha\right)z\right)\right]+O\left(1\right)\\
&= \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n}{k}-1}\\
&\quad\times\left[\exp^{[p-q-1]}\left(\log^{[p-1]}\mu\left(\frac{r}{4^{n-1}},f_k\left(z\right)\right)\right)-\log^{[q]}\mu\left(\frac{r}{4^{n-1}},z\right)\right]+O\left(1\right)\\
&\geq \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n}{k}-1}\\
&\quad\times\left[\exp^{[p-q-1]}\left\{\log^{[q-1]}\frac{r}{4^{n-1}}\right\}^{\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)}-\log^{[q]}\frac{r}{4^{n-1}}\right]+O\left(1\right). \qquad (14)
\end{aligned}$$

Now we choose r sufficiently large so that for every $\gamma>0$

$$\left\{\log^{[p]}\mu\left(r^\gamma,f_1\right)\right\}^{1+x}<\left(\rho_{(p,q)}\left(f_1\right)+\varepsilon\right)^{1+x}\left\{\log^{[q]}r^\gamma\right\}^{1+x}. \qquad (15)$$

From (14) and (15) for sufficiently large r we get

$$\begin{aligned}
\frac{\log^{[p]}\mu\left(r,F_n^1\right)}{\left\{\log^{[p]}\mu\left(r^\gamma,f_1\right)\right\}^{1+x}} &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n}{k}}...\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n}{k}}\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n}{k}-1}\\
&\quad\times\frac{\left[\exp^{[p-q-1]}\left\{\log^{[q-1]}\frac{r}{4^{n-1}}\right\}^{\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)}-\log^{[q]}\frac{r}{4^{n-1}}\right]+O\left(1\right)}{\left(\rho_{(p,q)}\left(f_1\right)+\varepsilon\right)^{1+x}\left\{\log^{[q]}r^\gamma\right\}^{1+x}}\to\infty\quad\text{as}\quad r\to\infty.
\end{aligned}$$

Now let $n=km-1$, $m\in\mathbb{N}$. Then

$$\begin{aligned}
\log^{[p]}\mu\left(r,F_n^1\right) &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n+1}{k}}..\left(\lambda_{(p,q)}\left(f_{k-2}\right)-\varepsilon\right)^{\frac{n+1}{k}}\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}}\\
&\quad\times\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}}[\exp^{[p-q-1]}\left(\log^{[p-1]}\mu\left(\frac{r}{4^{n-1}},f_{k-1}\left(z\right)\right)\right)-\log^{[q]}\mu\left(\frac{r}{4^{n-1}},z\right)]+O\left(1\right)\\
&\geq \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n+1}{k}}...\left(\lambda_{(p,q)}\left(f_{k-2}\right)-\varepsilon\right)^{\frac{n+1}{k}}\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}}\\
&\quad\times\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n-(k-1)}{k}}\left[\exp^{[p-q-1]}\left\{\log^{[q-1]}\frac{r}{4^{n-1}}\right\}^{\left(\lambda_{(p,q)}\left(f_{k-1}\right)-\varepsilon\right)}-\log^{[q]}\frac{r}{4^{n-1}}\right]+O\left(1\right).
\end{aligned}$$

Therefore the statement (12) follows. When $n=km-(k-1)$, $m\in\mathbb{N}$. Then

$$\begin{aligned}
\log^{[p]}\mu\left(r,F_n^1\right) &> \frac{1}{2^{n-1}}\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)^{\frac{n-1}{k}}\left(\lambda_{(p,q)}\left(f_2\right)-\varepsilon\right)^{\frac{n-1}{k}}...\left(\lambda_{(p,q)}\left(f_k\right)-\varepsilon\right)^{\frac{n-1}{k}}\\
&\quad\times\left[\exp^{[p-q-1]}\left\{\log^{[q-1]}\frac{r}{4^{n-1}}\right\}^{\left(\lambda_{(p,q)}\left(f_1\right)-\varepsilon\right)}-\log^{[q]}\frac{r}{4^{n-1}}\right]+O\left(1\right).
\end{aligned}$$

Consequently the statement (12) follows. This completes the proof. $\square$

Corollary 3.12. *Let $f_1, f_2, ..., f_k$ be k entire functions with positive lower (p,q)-order and of finite (p,q)-order. Then for every positive constant $\gamma, p > q$ and every real number x,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu\left(r, F_n^1\right)}{\left\{\log^{[p]} \mu\left(r^{\gamma}, f_k\right)\right\}^{1+x}} = \infty.$$

Theorem 3.13. *Let $f_1, f_2, ..., f_k$ be k entire functions of finite (p,q) order and $\lambda_{(p,q)}\left(f_1\right) > 0$. Then for $h > 0$, $p > q$ and for every real number x*

$$\lim_{r\to\infty} \frac{\left\{\log^{[p+(n-2)(p-q)]} \mu\left(r, F_n^1\right)\right\}^{1+x}}{\log^{[p]} \mu\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^h, f_1\right)} = 0$$

where $h > (1+x)\,\rho_{(p,q)}\left(f_k\right)$ if $p = 2, q = 1$ and $h > \rho_{(p,q)}\left(f_k\right)$ otherwise for $n = km$, $m \in \mathbb{N}$.

Corollary 3.14. *Let $f_1, f_2, ..., f_k$ be k entire functions of finite (p,q) order and $\lambda_{(p,q)}\left(f_1\right) > 0$. Then for $h > 0, p > q$ and for every real number x*

$$\lim_{r\to\infty} \frac{\left\{\log^{[p+(n-2)(p-q)]} \mu\left(r, F_n^1\right)\right\}^{1+x}}{\log^{[p]} \mu\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^h, f_1\right)} = 0$$

where $h > (1+x)\,\rho_{(p,q)}\left(f_1\right)$ if $p = 2, q = 1$ and $h > \rho_{(p,q)}\left(f_1\right)$ otherwise for $n = km - (k-1)$, $m \in \mathbb{N}$.

Theorem 3.15. *Let $f_1, f_2, ..., f_k$ be k entire functions of finite (p,q) order and $\lambda_{(p,q)}\left(f_k\right) > \rho_{(p,q)}\left(f_1\right) \geq \lambda_{(p,q)}\left(f_1\right) > 0$. Then for $p > q$ and $n = km, m \in \mathbb{N}$*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu\left(r, F_n^1\right)}{\log^{[p]} \mu\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^{\rho_{(p,q)}(f_1)}, f_k\right)} = \infty.$$

Corollary 3.16. *Let $f_1, f_2, ..., f_k$ be k entire functions of finite (p,q) order and $\lambda_{(p,q)}\left(f_1\right) > \rho_{(p,q)}\left(f_k\right) \geq \lambda_{(p,q)}\left(f_k\right) > 0$. Then for $p > q$ and $n = km - (k-1)\,, m \in \mathbb{N}$*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu\left(r, F_n^1\right)}{\log^{[p]} \mu\left(\exp^{[p-1]}\left(\log^{[q-1]} r\right)^{\rho_{(p,q)}(f_k)}, f_1\right)} = \infty.$$

The proofs of Theorem 3.13 and Theorem 3.15 are omitted.

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