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Solution and Stability of a,b,c,d Mixed Type Functional Equation in BS (Banach Space) and BA (Banach Algebra) Using Two Different Methods

Research Article

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Abstract: In this article, the authors introduce the general solution and generalized Ulam-Hyers stability of a generalized a,b,c,d mixed type functional equation of the form $g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)=\frac{a}{b}[g(x)-g(-x)]+\frac{b}{c}[g(y)-g(-y)]+\frac{c}{d}[g(z)-g(-z)]+2\left(\frac{c}{d}\right)^2[g(z)+g(-z)]+2\left(\frac{a}{b}\right)^2[g(x)+g(-x)]+2\left(\frac{b}{c}\right)^2[g(y)+g(-y)]$, where a,b,c,d are positive integers with $a\neq b\neq c\neq d\neq 0$, in BS(Banach Space) and BA (Banach Algebras)using two different methods.

MSC: 39B52, 32B72, 32B82.

Keywords: Mixed type functional equations, Banach Algebra, generalized Ulam-Hyers Stability, Fixed point.

1. Introduction and Preliminaries

The study of stability problems for functional equations is related to a question of Ulam [28] concerning the stability of group homomorphism's was affirmatively answered for Banach spaces by Hyers [14] for linear mappings and by Russians for linear mappings by considering an bounded Cauchy difference. The articles of Russians have provided a lot of influence in the development of what we now call generalized Ulam-Hyers stability of functional equations. The terminology generalized Ulam[28]and Hyers [14] stability originates from historical backgrounds. The terminologies are also applied to the case of other functional equations. Over the last seven decades, the above problem was talked by numerous authors and its solutions via various forms of functional equations like, additive, quadratic, cubic, quartic and mixed type functional equations were discussed. We refer the interested readers for more information on such problems to the monographs of [4,6,9,11,13]. One of the most famous functional equation is the additive functional equation

$$f(x+y)=f(x)+f(y). \tag{1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real - valued functions. It is often called Cauchy additive functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to

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the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1) is called an additive function. The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2}$$

and therefore the equation (2) is called quadratic functional equation. The solution and stability of the following mixed type additive-quadratic functional equations

$$f(x+2y) + f(x-2y) + 8f(y) = 2f(x) + 4f(2y) \tag{3}$$

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y) \tag{4}$$

$$f(2x \pm y \pm z) = 2f(-x \mp y \mp z) - 2f(\mp y \mp z) + f(\pm y \pm z) + 3f(x) - f(-x) \tag{5}$$

$$f(x+2y+3z) + f(x-2y+3z) + f(x+2y-3z) + f(x-2y-3z) = 4f(x) + 8[f(y) + f(-y)] + 18[f(z) + f(-z)] \tag{6}$$

were discussed by G.Zamani Eskandani [29], M.Arunkumar, J.M.Rassias [5], S. Murthy et.al., [19] and M.Arunkumar, P.Agilan [10]. Now in this paper, we introduce that the general solution and generalized Ulam-Hyers stability of a generalized a,b,c,d-type Additive Quadratic functional equation of the form

$$\begin{aligned} g\left(\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x - \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x + \frac{b}{c}y - \frac{c}{d}z\right) + g\left(-\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) = \frac{a}{b}[g(x) - g(-x)] + \frac{b}{c}[g(y) - g(-y)] \\ + \frac{c}{d}[g(z) - g(-z)] + 2\left(\frac{c}{d}\right)^2[g(z) + g(-z)] + 2\left(\frac{a}{b}\right)^2[g(x) + g(-x)] + 2\left(\frac{b}{c}\right)^2[g(y) + g(-y)] \end{aligned} \tag{7}$$

where a,b,c,d are positive integers with $a \neq b \neq c \neq d \neq 0$, in Banach spaces and Banach Algebras using two different methods.

2. General Solution of The Functional Equation (7): When g is Odd

In this section, the general solution of the functional equation (7), for odd case is discussed. Throughout this section, let us consider X and Y to be real vector spaces.

Theorem 2.1. *If an odd mapping $g : X \to Y$ satisfies the functional equation*

$$g(x+y) = g(x) + g(y) \tag{8}$$

for all $x, y \in X$, if and only if $g : X \to Y$ satisfies the functional equation

$$\begin{aligned} g\left(\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x - \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x + \frac{b}{c}y - \frac{c}{d}z\right) + g\left(-\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) = \frac{a}{b}[g(x) - g(-x)] + \frac{b}{c}[g(y) - g(-y)] \\ + \frac{c}{d}[g(z) - g(-z)] + 2\left(\frac{a}{b}\right)^2[g(x) + g(-x)] + 2\left(\frac{b}{c}\right)^2[g(y) + g(-y)] + 2\left(\frac{c}{d}\right)^2[g(z) + g(-z)] \end{aligned} \tag{9}$$

for all $x, y, z \in X$.

Proof. Let $g : X \to Y$ satisfies the functional equation (8). Setting (x, y) by $(0, 0)$ in (8), we get $g(0) = 0$. Replacing (x, y) by (x, x) and $(x, 2x)$ respectively in (8), we obtain

$$g(2x) = 2g(x) \text{ and } g(3x) = 3g(x) \tag{10}$$

for all $x \in X$. In general for any positive integer a, we have

$$g(ax) = ag(x) \tag{11}$$

for all $x \in X$. It is easy to verify from (11) that

$$g(a^2x) = a^2g(x) \text{ and } g(a^3x) = a^3g(x) \tag{12}$$

for all $x \in X$. Replacing (x, y) by $\left(\frac{a}{b}x + \frac{b}{c}y, \frac{c}{d}z\right)$ in (8) and using (8), (11) and (12), we get

$$g\left(\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) = \left(\frac{a}{b}\right)g(x) + \left(\frac{b}{c}\right)g(y) + \left(\frac{c}{d}\right)g(z) \tag{13}$$

for all $x, y, z \in X$. Again replacing z by $-z$ in equation (13) and using oddness of g, we obtain

$$g\left(\frac{a}{b}x + \frac{b}{c}y - \frac{c}{d}z\right) = \left(\frac{a}{b}\right)g(x) + \left(\frac{b}{c}\right)g(y) - \left(\frac{c}{d}\right)g(z) \tag{14}$$

for all $x, y, z \in X$. Also replacing y by $-y$ in (13) and using oddness of g, we get

$$g\left(\frac{a}{b}x - \frac{b}{c}y + \frac{c}{d}z\right) = \left(\frac{a}{b}\right)g(x) - \left(\frac{b}{c}\right)g(y) + \left(\frac{c}{d}\right)g(z) \tag{15}$$

for all $x, y, z \in X$. Finally replacing x by $-x$ in (13) and using oddness of g, we obtain

$$g\left(-\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) = -\left(\frac{a}{b}\right)g(x) + \left(\frac{b}{c}\right)g(y) + \left(\frac{c}{d}\right)g(z) \tag{16}$$

for all $x, y, z \in X$. Adding the equations (13), (14), (15) and (16), we have

$$\begin{aligned} g\left(\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x - \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x + \frac{b}{c}y - \frac{c}{d}z\right) + g\left(-\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) \\ = 2\left(\frac{a}{b}\right)g(x) + 2\left(\frac{b}{c}\right)g(y) + 2\left(\frac{c}{d}\right)g(z) \end{aligned} \tag{17}$$

for all $x, y, z \in X$. Using oddness of g in (17) and remodifying, we arrive

$$\begin{aligned} g\left(\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x - \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x + \frac{b}{c}y - \frac{c}{d}z\right) + g\left(-\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) \\ = \frac{a}{b}[g(x) - g(-x)] + \frac{b}{c}[g(y) - g(-y)] + \frac{c}{d}[g(z) - g(-z)] \end{aligned} \tag{18}$$

for all $x, y, z \in X$. Adding $2\left(\frac{a}{b}\right)^2 g(x) + 2\left(\frac{b}{c}\right)^2 g(y) + 2\left(\frac{c}{d}\right)^2 g(z)$ on both sides, remodifying and using oddness of g, we reach (9) as desired.

Conversely, $g: X \to Y$ satisfies the functional equation (9) using oddness of in (9), we arrive

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$
$$=2\left(\frac{a}{b}\right)g(x)+2\left(\frac{b}{c}\right)g(y)+2\left(\frac{c}{d}\right)g(z) \tag{19}$$

for all $x, y, z \in X$. Replacing (x, y, z) by $(x, 0, 0)$, $(0, x, 0)$ and $(0, 0, x)$, respectively in (19), we obtain

$$g\left(\frac{a}{b}x\right)=\left(\frac{a}{b}\right)g(x), g\left(\frac{b}{c}x\right)=\left(\frac{b}{c}\right)g(x) \text{ and } g\left(\frac{c}{d}x\right)=\left(\frac{c}{d}\right)g(x). \tag{20}$$

for all $x \in X$. One can easy to verify from (20) that

$$g\left(\frac{x}{\left(\frac{a}{b}\right)^i}\right)=\frac{1}{\left(\frac{a}{b}\right)^i}g(x); i=1,2,3 \tag{21}$$

for all $x \in X$. Replacing (x, y, z) by $\left(\frac{x}{\left(\frac{a}{b}\right)}, \frac{y}{\left(\frac{b}{c}\right)}, 0\right)$ in equation (19) and using oddness of g and (21), we arrive our result. □

3. General Solution of the Functional Equation (7): When g is Even

In this section, the general solution of the functional equation (7) for even case is given. Throughout this section, let us consider X and Y to be real vector spaces.

Theorem 3.1. *If an even mapping $g: X \to Y$ satisfies the functional equation*

$$g(x+y)+g(x-y)=2g(x)+2g(y) \tag{22}$$

for all $x, y \in X$ if and only if $g: X \to Y$ satisfies the functional equation

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$
$$=\frac{a}{b}[g(x)-g(-x)]+\frac{b}{c}[g(y)-g(-y)]+\frac{c}{d}[g(z)-g(-z)]$$
$$+2\left(\frac{a}{b}\right)^2[g(x)+g(-x)]+2\left(\frac{b}{c}\right)^2[g(y)+g(-y)]+2\left(\frac{c}{d}\right)^2[g(z)+g(-z)] \tag{23}$$

for all $x, y, z \in X$.

Proof. Let $g: X \to Y$ satisfies the functional equation (22). Setting (x, y) by $(0, 0)$ in (22), we get $g(0) = 0$. Replacing y by x and y by $2x$ in (22), we obtain

$$g(2x)=4g(x) \text{ and } g(3x)=9g(x) \tag{24}$$

for all $x \in X$. In general for any positive integer b, such that

$$g(bx)=b^2g(x) \tag{25}$$

for all $x \in X$. It is easy to verify from (25) that

$$g(b^2x)=b^4g(x) \text{ and } g(b^3x)=b^6g(x) \tag{26}$$

for all $x \in X$. Replacing (x, y) by $\left(\frac{a}{b}x, \frac{b}{c}y\right)$ in (22) and using (22), we get

$$g\left(\frac{a}{b}x+\frac{b}{c}y\right)+g\left(\frac{a}{b}x-\frac{b}{c}y\right)=2\left(\frac{a}{b}\right)^2 g(x)+2\left(\frac{b}{c}\right)^2 g(y) \tag{27}$$

for all $x, y \in X$. Setting (x, y) by $\left(\frac{a}{b}x, \frac{c}{d}z\right)$ in (22) and using (22), we obtain

$$g\left(\frac{a}{b}x+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{c}{d}z\right)=2\left(\frac{a}{b}\right)^2 g(x)+2\left(\frac{c}{d}\right)^2 g(z) \tag{28}$$

for all $x, z \in X$. Replacing (x, y) by $\left(\frac{b}{c}y, \frac{c}{d}z\right)$ in (22) and using (22), we have

$$g\left(\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{b}{c}y-\frac{c}{d}z\right)=2\left(\frac{b}{c}\right)^2 g(y)+2\left(\frac{c}{d}\right)^2 g(z) \tag{29}$$

for all $y, z \in X$. Adding equations (27), (28) and (29), we arrive

$$\begin{aligned}g\left(\frac{a}{b}x+\frac{b}{c}y\right)&+g\left(\frac{a}{b}x-\frac{b}{c}y\right)+g\left(\frac{a}{b}x+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{c}{d}z\right)+g\left(\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{b}{c}y-\frac{c}{d}z\right)\\&=4\left(\frac{a}{b}\right)^2 g(x)+4\left(\frac{b}{c}\right)^2 g(y)+4\left(\frac{c}{d}\right)^2 g(z)\end{aligned} \tag{30}$$

for all $x, y, z \in X$. Replacing (x, y) by $\left(\frac{a}{b}x+\frac{b}{c}y, \frac{c}{d}z\right)$in (22), we get

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)=2g\left(\frac{a}{b}x+\frac{b}{c}y\right)+2g\left(\frac{c}{d}z\right) \tag{31}$$

for all $x, y, z \in X$. Setting (x, y) by $\left(\frac{c}{d}z, \frac{a}{b}x-\frac{b}{c}y\right)$ in (22), we obtain

$$g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)=2g\left(\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{b}{c}y\right) \tag{32}$$

for all $x, y, z \in X$. Replacing (x, y) by $\left(\frac{b}{c}y, \frac{a}{b}x+\frac{c}{d}z\right)$ in (22), we get

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)=2g\left(\frac{b}{c}y\right)+2g\left(\frac{a}{b}x+\frac{c}{d}z\right) \tag{33}$$

for all $x, y, z \in X$. Setting (x, y) by $\left(\frac{b}{c}y, \frac{a}{b}x-\frac{c}{d}z\right)$ in (22), we obtain

$$g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)=2g\left(\frac{b}{c}y\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right) \tag{34}$$

for all $x, y, z \in X$. Replacing (x, y) by $\left(\frac{a}{b}x, \frac{b}{c}y+\frac{c}{d}z\right)$ in (22), we get

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y-\frac{c}{d}z\right)=2g\left(\frac{a}{b}x\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right) \tag{35}$$

for all $x, y, z \in X$. Setting (x, y) by $\left(\frac{a}{b}x, \frac{b}{c}y-\frac{c}{d}z\right)$ in (22), we have

$$g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)=2g\left(\frac{a}{b}x\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right) \tag{36}$$

for all $x, y, z \in X$. Now multiply by 2 on both sides of (30), we obtain

$$\begin{aligned}2g\left(\frac{a}{b}x+\frac{b}{c}y\right)&+2g\left(\frac{a}{b}x-\frac{b}{c}y\right)+2g\left(\frac{a}{b}x+\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)\\&=8\left(\frac{a}{b}\right)^2 g(x)+8\left(\frac{b}{c}\right)^2 g(y)+8\left(\frac{c}{d}\right)^2 g(z)\end{aligned} \tag{37}$$

for all $x,y,z \in X$. Adding $g\left(\frac{c}{b}z\right)$ on both sides of (37), we get

$$2g\left(\frac{a}{b}x+\frac{b}{c}y\right)+2g\left(\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{b}{c}y\right)+2g\left(\frac{a}{b}x+\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)=8\left(\frac{a}{b}\right)^2 g(x)+8\left(\frac{b}{c}\right)^2 g(y)+8\left(\frac{c}{d}\right)^2 g(z)+2g\left(\frac{c}{d}z\right) \tag{38}$$

for all $x,y,z \in X$. Using (31), (36) in (38), we arrive

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)+2g\left(\frac{a}{b}x+\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)$$

$$+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)=8\left(\frac{a}{b}\right)^2 g(x)+8\left(\frac{b}{c}\right)^2 g(y)+8\left(\frac{c}{d}\right)^2 g(z)+2\left(\frac{c}{d}\right)^2 g(z) \tag{39}$$

for all $x,y,z \in X$. Adding $2g\left(\frac{c}{d}z\right)$ on both sides of (39), we get

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+2g\left(\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{b}{c}y\right)+2g\left(\frac{a}{b}x+\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)$$

$$+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)=8\left(\frac{a}{b}\right)^2 g(x)+8\left(\frac{b}{c}\right)^2 g(y)+10\left(\frac{c}{d}\right)^2 g(z)+2g\left(\frac{c}{d}z\right) \tag{40}$$

for all $x,y,z \in X$. Using (32) in (40), we arrive

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{c}{b}z+\frac{a}{b}x-\frac{c}{d}z\right)+g\left(\frac{c}{d}z-\frac{a}{b}x+\frac{b}{c}y\right)$$

$$+2g\left(\frac{a}{b}x+\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)$$

$$=8\left(\frac{a}{b}\right)^2 g(x)+8\left(\frac{b}{c}\right)^2 g(y)+12\left(\frac{c}{d}\right)^2 g(z) \tag{41}$$

for all $x,y,z \in X$. Adding $2g\left(\frac{b}{c}y\right)$ on both sides of (41), we get

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+2g\left(\frac{b}{c}y\right)+2g\left(\frac{a}{b}x+\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)$$

$$=8\left(\frac{a}{b}\right)^2 g(x)+8\left(\frac{b}{c}\right)^2 g(y)+2g\left(\frac{b}{c}y\right)+12\left(\frac{c}{d}\right)^2 g(z) \tag{42}$$

for all $x,y,z \in X$. Using (33) in (42), we arrive

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)$$

$$=8\left(\frac{a}{b}\right)^2 g(x)+10\left(\frac{b}{c}\right)^2 g(y)+12\left(\frac{c}{d}\right)^2 g(z) \tag{43}$$

for all $x,y,z \in X$. Adding $2g\left(\frac{b}{c}y\right)$ on both sides of (43), we obtain

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+2g\left(\frac{a}{b}x-\frac{c}{d}z\right)+2g\left(\frac{b}{c}y\right)$$

$$+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)=8\left(\frac{a}{b}\right)^2g(x)+10\left(\frac{b}{c}\right)^2g(y)+2g\left(\frac{b}{c}y\right)+12\left(\frac{c}{d}\right)^2g(z) \quad (44)$$

for all $x, y, z \in X$. Using (34) in (44), we arrive

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)$$

$$+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)$$

$$=8\left(\frac{a}{b}\right)^2g(x)+12\left(\frac{b}{c}\right)^2g(y)+12\left(\frac{c}{d}\right)^2g(z) \quad (45)$$

for all $x, y, z \in X$. Adding $2g\left(\frac{a}{b}x\right)$ on both sides of (45), we get

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)$$

$$+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{a}{b}x\right)+2g\left(\frac{b}{c}y+\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)$$

$$=8\left(\frac{a}{b}\right)^2g(x)+12\left(\frac{b}{c}\right)^2g(y)+12\left(\frac{c}{d}\right)^2g(z)+2g\left(\frac{a}{b}x\right) \quad (46)$$

for all $x, y, z \in X$. Using (35) in (46), we arrive

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y-\frac{c}{d}z\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)$$

$$=10\left(\frac{a}{b}\right)^2g(x)+12\left(\frac{b}{c}\right)^2g(y)+12\left(\frac{c}{d}\right)^2g(z)+2g\left(\frac{a}{b}x\right) \quad (47)$$

for all $x, y, z \in X$. Adding $2g\left(\frac{a}{b}x\right)$ on both sides of (47), we have

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$

$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y-\frac{c}{d}z\right)+2g\left(\frac{a}{b}x\right)+2g\left(\frac{b}{c}y-\frac{c}{d}z\right)$$

$$=10\left(\frac{a}{b}\right)^2g(x)+12\left(\frac{b}{c}\right)^2g(y)+12\left(\frac{c}{d}\right)^2g(z)+2g\left(\frac{a}{b}x\right) \quad (48)$$

for all $x, y, z \in X$. Using (36) in (48), we arrive

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$
$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$
$$+g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(-\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)$$
$$=12\left(\frac{a}{b}\right)^2 g(x)+12\left(\frac{b}{c}\right)^2 g(y)+12\left(\frac{c}{d}\right)^2 g(z) \tag{49}$$

for all $x, y, z \in X$. Using evenness of g in (49), we have

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$
$$=4\left(\frac{a}{b}\right)^2 g(x)+4\left(\frac{b}{c}\right)^2 g(y)+4\left(\frac{c}{d}\right)^2 g(z) \tag{50}$$

for all $x, y, z \in X$. Using evenness of g in (50) one can get,

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$
$$=2\left(\frac{a}{b}\right)^2[g(x)+g(-x)]+2\left(\frac{b}{c}\right)^2[g(y)+g(-y)]+2\left(\frac{c}{d}\right)^2[g(z)+g(-z)] \tag{51}$$

for all $x, y, z \in X$. Adding $\frac{a}{b}g(x)+\frac{b}{c}g(y)+\frac{c}{b}g(z)$ on both sides of (51) and using evenness of g, we desired our result. Conversely, $g : X \to Y$ satisfies the functional equation (23). Using evenness of g in (23), we have

$$g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)$$
$$=4\left(\frac{a}{b}\right)^2 g(x)+4\left(\frac{b}{c}\right)^2 g(y)+4\left(\frac{c}{d}\right)^2 g(z) \tag{52}$$

Setting (x, y, z) by $(x, 0, 0)$, $(0, x, 0)$ and $(0, 0, x)$in (52), we obtain

$$g\left(\frac{a}{b}x\right)=\left(\frac{a}{b}\right)^2 g(x); \qquad g\left(\frac{b}{c}y\right)=\left(\frac{b}{c}\right)^2 g(x) \quad \text{and} \quad g\left(\frac{c}{d}z\right)=\left(\frac{c}{d}\right)^2 g(x) \tag{53}$$

for all $x \in X$. It is easy to verify from (53), that

$$g\left(\frac{x}{\left(\frac{a}{b}\right)^i}\right)=\frac{1}{\left(\frac{a}{b}\right)^i}g(x), i=1,2,3 \tag{54}$$

for all $x \in X$. Replacing (x, y, z) by $\left(\frac{x}{\left(\frac{a}{b}\right)}, \frac{y}{\left(\frac{b}{c}\right)}, 0\right)$ in (52) and using evenness of g and (54), we desired our result. $\square$

For sections 4, 5 and 6, let us consider X and Y to a normed space and a Banach space. Define a mapping $Dg : X \to Y$ by

$$Dg(x,y,z)=g\left(\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x-\frac{b}{c}y+\frac{c}{d}z\right)+g\left(\frac{a}{b}x+\frac{b}{c}y-\frac{c}{d}z\right)$$
$$+g\left(-\frac{a}{b}x+\frac{b}{c}y+\frac{c}{d}z\right)-\frac{a}{b}[g(x)-g(-x)]-\frac{b}{c}[g(y)-g(-y)]-\frac{c}{d}[g(z)-g(-z)]$$
$$-2\left(\frac{a}{b}\right)^2[g(x)+g(-x)]-2\left(\frac{b}{c}\right)^2[g(y)+g(-y)]-2\left(\frac{c}{d}\right)^2[g(z)+g(-z)]$$

for all $x, y, z \in X$.

4. Stability Results for (7): Odd Case-Direct Method

In this section, we present the generalized Ulam-Hyers stability of the functional equation (7) for odd case.

Theorem 4.1. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k\to\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{kj}} = 0 \tag{55}$$

for all $x, y, z \in X$. Let $g_a : X \to Y$ be an odd function satisfying the inequality

$$\| Dg_a(x, y, z) \| \le \alpha(x, y, z) \tag{56}$$

for all $x, y, z \in X$. There exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (7) and

$$\| g_a(x) - A(x) \| \le \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} \tag{57}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{g_a\left(\left(\frac{a}{b}\right)^{kj} x\right)}{\left(\frac{a}{b}\right)^{kj}} \tag{58}$$

for all $x \in X$.

Proof. Assume that $j = 1$. Replacing (x, y, z) and $(x, 0, 0)$ in (56) and using oddness of g_a, we get

$$\left\| 2g_a\left(\frac{a}{b}x\right) - 2\left(\frac{a}{b}\right) g_a(x) \right\| \le \alpha(x, 0, 0) \tag{59}$$

for all $x \in X$. It follows from (59) that

$$\left\| \frac{g_a\left(\frac{a}{b}x\right)}{\left(\frac{a}{b}\right)} - g_a(x) \right\| \le \frac{\alpha}{2\left(\frac{a}{b}\right)}(x, 0, 0) \tag{60}$$

for all $x \in X$. Replacing x by $\left(\frac{a}{b}\right) x$ in (60) and dividing by $\left(\frac{a}{b}\right)$, we obtain

$$\left\| \frac{g_a\left(\frac{b}{c}x\right)}{\left(\frac{b}{c}\right)} - \frac{g_a\left(\frac{a}{b}x\right)}{\left(\frac{a}{b}\right)} \right\| \le \frac{\alpha}{2\left(\frac{b}{c}\right)} \left(\left(\frac{a}{b}\right) x, 0, 0\right) \tag{61}$$

for all $x \in X$. It follows from (60) and (61) that

$$\left\| \frac{g_a\left(\frac{b}{c}x\right)}{\left(\frac{b}{c}\right)} - g_a(x) \right\| \le \frac{1}{2\left(\frac{a}{b}\right)} \left[\alpha(x, 0, 0) + \frac{\alpha}{\left(\frac{a}{b}\right)} \left(\left(\frac{a}{b}\right) x, 0, 0\right)\right] \tag{62}$$

for all $x \in X$. Generalizing, we have

$$\left\| g_a(x) - \frac{g_a\left(\left(\frac{a}{b}\right)^k x\right)}{\left(\frac{a}{b}\right)^k} \right\| \le \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=0}^{n-1} \frac{\alpha\left(\left(\frac{a}{b}\right)^k x, 0, 0\right)}{\left(\frac{a}{b}\right)^k} \le \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^k x, 0, 0\right)}{\left(\frac{a}{b}\right)^k} \tag{63}$$

for all $x \in X$. In order to prove convergence of the sequence

$$\left\{ \frac{g_a \left(\frac{a}{b}\right)^k x}{\left(\frac{a}{b}\right)^k} \right\},$$

Replace x by $\left(\frac{a}{b}\right)^l x$ and dividing $\left(\frac{a}{b}\right)^l$ (63), for any $k, l > 0$, to deduce

$$\begin{aligned} \left\| \frac{g_a\left(\left(\frac{a}{b}\right)^l x\right)}{\left(\frac{a}{b}\right)^l} - \frac{g_a\left(\left(\frac{a}{b}\right)^{k+l} x\right)}{\left(\frac{a}{b}\right)^{k+l}} \right\| &= \frac{1}{2^l} \left\| g_a\left(\left(\frac{a}{b}\right)^l x\right) - \frac{g_a\left(\left(\frac{a}{b}\right)^k \cdot \left(\frac{a}{b}\right)^l x\right)}{\left(\frac{a}{b}\right)^k} \right\| \\ &\leq \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=0}^{n-1} \frac{\alpha\left(\left(\frac{a}{b}\right)^{k+l} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{k+l}} \qquad (64) \\ &\leq \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{k+l} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{k+l}} \to 0 \ \ as \ \ l \to \infty \qquad (65) \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{g_a\left(\left(\frac{a}{b}\right)^k x\right)}{\left(\frac{a}{b}\right)^k} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{k \to \infty} \frac{g_a\left(\left(\frac{a}{b}\right)^k x\right)}{\left(\frac{a}{b}\right)^k}, \qquad \forall \ x \in X.$$

Letting $k \to \infty$ in (63), we see that (57) holds for all $x \in X$. To prove that A satisfies (7), replacing (x, y, z) by $\left(\left(\frac{a}{b}\right)^k x, \left(\frac{a}{b}\right)^k y, \left(\frac{a}{b}\right)^k z\right)$ and dividing $\left(\frac{a}{b}\right)^k$ in (56), we obtain

$$\frac{1}{\left(\frac{a}{b}\right)^k} \left\| Dg_a\left(\left(\frac{a}{b}\right)^k x, \left(\frac{a}{b}\right)^k y, \left(\frac{a}{b}\right)^k z\right) \right\| \leq \frac{1}{\left(\frac{a}{b}\right)^k} \alpha\left(\left(\frac{a}{b}\right)^k x, \left(\frac{a}{b}\right)^k y, \left(\frac{a}{b}\right)^k z\right)$$

for all $x, y, z \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$DA(x, y, z) = 0.$$

Hence A satisfies (7) for all $x, y, z \in X$. To show that A is unique, let $B(x)$ be another additive mapping satisfying (7) and (57), then

$$\begin{aligned} \| \ A(x) - B(x) \ \| &= \frac{1}{\left(\frac{a}{b}\right)^l} \left\| \ A\left(\left(\frac{a}{b}\right)^l x\right) - B\left(\left(\frac{a}{b}\right)^l x\right) \right\| \\ &\leq \frac{1}{\left(\frac{a}{b}\right)^l} \left\| \ A\left(\left(\frac{a}{b}\right)^l x\right) - g_a\left(\left(\frac{a}{b}\right)^l x\right) \right\| + \left\| \ g_a\left(\left(\frac{a}{b}\right)^l x\right) - B\left(\left(\frac{a}{b}\right)^l x\right) \right\| \\ &\leq \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{k+l} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{k+l}} \to 0 \ \ as \ \ l \to \infty \end{aligned}$$

for all $x \in X$. Hence A is unique. Now, replacing x by $\frac{x}{\left(\frac{a}{b}\right)}$ in (59), we get

$$\left\| 2g_a(x) \ - 2\left(\frac{a}{b}\right) g_a\left(\frac{x}{\left(\frac{a}{b}\right)}\right) \right\| \leq \alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right) \qquad (66)$$

for all $x \in X$. It follows from (66) that

$$\left\| g_a(x) \ - \left(\frac{a}{b}\right) g_a\left(\frac{x}{\left(\frac{a}{b}\right)}\right) \right\| \leq \frac{1}{2} \alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right) \qquad (67)$$

for all $x \in X$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem is true. This completes the proof of the theorem. $\square$

The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of (7).

Corollary 4.2. *Let λ and s be a nonnegative real numbers. Let an odd function $g_a : X \to Y$ satisfying the inequality*

$$\| Dg_a(x,y,z) \| \leq \begin{cases} \lambda & \\ \lambda \left(\| x \|^s + \| y \|^s + \| z \|^s \right), & s \neq 1; \\ \lambda \left(\| x \|^s \| y \|^s \| z \|^s + \left\{ \| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s} \right\} \right), & s \neq \frac{1}{3}; \end{cases} \tag{68}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\| g_a(x) - A(x) \| \leq \begin{cases} \frac{\lambda}{2\left| \left(\frac{a}{b}\right) - 1 \right|}, \\ \frac{\lambda \| x \|^s}{2\left| \left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^s \right|}, \\ \frac{\lambda \| x \|^{3s}}{2\left| \left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^{3s} \right|}, \end{cases} \tag{69}$$

for all $x \in X$.

Proof. If we replace

$$\alpha(x,y,z) = \begin{cases} \lambda; \\ \lambda \left(\| x \|^s + \| y \|^s + \| z \|^s \right); \\ \lambda \left(\| x \|^s \| y \|^s \| z \|^s + \left\{ \| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s} \right\} \right); \end{cases} \tag{70}$$

in Theorem 4.1, we arrive (69). □

5. Stability Result for (7): Even Case-Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (7) for even case.

Theorem 5.1. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that,*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z \right)}{\left(\frac{a}{b}\right)^{kj}} \text{ converges to } \mathbb{R} \text{ and } \lim_{k \to \infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z \right)}{\left(\frac{a}{b}\right)^{kj}} = 0 \tag{71}$$

for all $x, y, z \in X$. Let $g_q : X \to Y$ be an even function satisfying the inequality,

$$\| Dg_q(x,y,z) \| \leq \alpha(x,y,z) \tag{72}$$

for all $x, y, z \in X$. Then there exists unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (7) and

$$\| g_q(x) - Q(x) \| \leq \frac{1}{4\left(\frac{b}{c}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0 \right)}{\left(\frac{a}{b}\right)^{2kj}} \tag{73}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k \to \infty} \frac{g_q\left(\left(\frac{a}{b}\right)^{kj} x \right)}{\left(\frac{a}{b}\right)^{2kj}} \tag{74}$$

for all $x \in X$.

Proof. Assume that $j = 1$, replacing (x, y, z) by $(x, 0, 0)$ in (72) and using of evenness of g_q, we get

$$\left\| 4g_q\left(\left(\frac{a}{b}\right)x\right) - 4\left(\frac{b}{c}\right)g_q(x)\right\| \leq \alpha(x, 0, 0) \tag{75}$$

for all $x \in X$. The rest of the proof is similar to that Theorem 4.1. □

The following corollary is an immediate consequence of Theorem 5.1 concerning the stability of (7).

Corollary 5.2. *Let λ and s be a nonnegative real numbers. Let an even function $g_q : X \to Y$ satisfying the inequality*

$$\| Dg_q(x, y, z) \| \leq \begin{cases} \lambda; & \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 2; \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{2}{3}; \end{cases} \tag{76}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that,

$$\| g_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{4\left|\left(\frac{a}{b}\right)^2 - 1\right|}, \\ \frac{\lambda \| x \|^s}{4\left|\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^s\right|}, \\ \frac{\lambda \| x \|^{3s}}{4\left|\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^{3s}\right|}, \end{cases} \tag{77}$$

for all $x \in X$.

6. Stability Results for (7): Mixed Case-Direct Method

In this section, we intoduce the generalized Ulam-Hyers stability of the functional equation (7) for mixed case.

Theorem 6.1. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function satisfying (55) and (71) for all $x, y, z \in X$. Let $g : X \to Y$ be a function satisfying the inequality*

$$\| Dg(x, y, z) \| \leq \alpha(x, y, z) \tag{78}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (7) and

$$\begin{aligned} \| g(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[& \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} \right) \\ & + \frac{1}{4\left(\frac{a}{b}\right)^2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} \right) \Bigg] \end{aligned} \tag{79}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ is defined in (58) and (74) respectively for all $x \in X$.

Proof. Let $g_o(x) = \frac{g_a(x) - g_a(-x)}{2}$ for all $x \in X$. Then $g_o(0) = 0$ and $g_o(-x) = -g_o(x)$ for all $x \in X$. Hence,

$$\| Dg_o(x, y, z) \| \leq \frac{1}{2}\{\| Dg_o(x, y, z) \| + \| Dg_o(-x, -y, -z) \|\} \leq \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2} \tag{80}$$

for all $x, y, z \in X$. By Theorem 4.1, we have

$$\| g_o(x) - A(x) \| \leq \frac{1}{4\left(\frac{a}{b}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} \right) \tag{81}$$

for all $x \in X$. Also let, $g_e(x) = \frac{g_q(x)+g_q(-x)}{2}$ for all $x \in X$. Then $g_e(0) = 0$ and $g_e(-x) = g_e(x)$ for all $x \in X$. Hence,

$$\| Dg_e(x, y, z) \| \leq \frac{1}{2} \{\| Dg_q(x, y, z) \| + \| Dg_q(-x, -y, -z) \|\} \leq \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2} \tag{82}$$

for all $x, y, z \in X$. By Theorem 5.1, we have

$$\| g_e(x) - Q(x) \| \leq \frac{1}{8\left(\frac{a}{b}\right)^2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} \right) \tag{83}$$

for all $x \in X$. Define

$$g(x) = g_e(x) + g_o(-x) \tag{84}$$

for all $x \in X$. It follows from (81), (83) and (84), we arrive

$$\begin{aligned}
\| g(x) - A(x) - Q(x) \| &= \| g_e(x) + g_o(-x) - A(x) - Q(x) \| \\
&\leq \|g_o(-x) - A(x)\| + \| g_e(x) - Q(x) \| \\
&\leq \frac{1}{4\left(\frac{a}{b}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} \right) \\
&\quad + \frac{1}{8\left(\frac{a}{b}\right)^2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} \right)
\end{aligned} \tag{85}$$

for all $x \in X$. Hence the theorem is proved. □

Using Corollaries 4.2 and 5.2, we have the following corollary concerning the stability of (7).

Corollary 6.2. *Let λ and s be a nonnegative real numbers. Let a function $g : X \to Y$ satisfying the inequality*

$$\| Dg(x, y, z) \| \leq \begin{cases} \lambda; & \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 1, 2; \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{86}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|g(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{2}\left[\frac{1}{\left|\left(\frac{a}{b}\right)-1\right|} + \frac{1}{2\left|\left(\frac{a}{b}\right)^2-1\right|}\right], \\ \frac{\lambda\| x \|^s}{2}\left[\frac{1}{\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^s\right|} + \frac{1}{2\left|\left(\frac{a}{b}\right)^2-\left(\frac{a}{b}\right)^s\right|}\right], \\ \frac{\lambda\| x \|^{3s}}{2}\left[\frac{1}{\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^s\right|} + \frac{1}{2\left|\left(\frac{a}{b}\right)^2-\left(\frac{a}{b}\right)^{3s}\right|}\right], \end{cases} \tag{87}$$

for all $x \in X$.

For Sections 7, 8 and 9, let us consider W and B to a normed space and a Banach space, respectively. Define a mapping $Dg : W \to B$ by

$$\begin{aligned} Dg(x,y,z) &= g\left(\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x - \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x + \frac{b}{c}y - \frac{c}{d}z\right) \\ &\quad + g\left(-\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) - \frac{a}{b}[g(x) - g(-x)] - \frac{b}{c}[g(y) - g(-y)] - \frac{c}{d}[g(z) - g(-z)] \\ &\quad - 2\left(\frac{a}{b}\right)^2 [g(x) + g(-x)] - 2\left(\frac{b}{c}\right)^2 [g(y) + g(-y)] - 2\left(\frac{c}{d}\right)^2 [g(z) + g(-z)] \end{aligned}$$

for all $x, y, z \in W$.

7. Fixed Point Stability Results of (7)

The following theorems are useful to prove our fixed point stability results.

Theorem 7.1 (Banach contraction principle [37]). *Let (X, d) be a complete metric spaces and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is,*

(A_1) $d(Tx, Ty) \leq d(x, y)$, *for some (Lipschtiz constant) $L < 1$, then,*

(i) The mapping T has one and only fixed point $x^* = T(x^*)$.

(ii) The fixed point for each given element x^* is globally contractive that is

(A_2) $\lim_{\left(\frac{a}{b}\right)\to\infty} T^{\left(\frac{a}{b}\right)}x = x^*$, *for any starting point $x \in X$.*

(iii) One has the following estimation inequalities,

(A_3) $d\left(T^{\left(\frac{a}{b}\right)}x, x^*\right) \leq \frac{1}{1-L} d\left(T^{\left(\frac{a}{b}\right)}x, T^{\left(\frac{a}{b}\right)+1}x\right), \qquad \forall \left(\frac{a}{b}\right) \geq 0, \qquad \forall x \in X.$

(A_4) $d(x, x^*) = \frac{1}{1-L} d(x, x^*), \qquad \forall x \in X.$

Theorem 7.2 (The Alternative Fixed Point [37]). *Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschtiz constant L, then for each given element $x \in X$ either,*

(B_1) $d\left(T^{\left(\frac{a}{b}\right)}x, T^{\left(\frac{a}{b}\right)+1}x\right) = \infty, \qquad \forall \left(\frac{a}{b}\right) \geq 0.$

(B_2) *There exists a natural number $\left(\frac{a}{b}\right)_0$ such that,*

(i) $d(T^{\left(\frac{a}{b}\right)}x, T^{\left(\frac{a}{b}\right)+1}x) < \infty$ for all $\forall \left(\frac{a}{b}\right) \geq 0$.

(ii) The sequence $\left\{T^{\left(\frac{a}{b}\right)}x\right\}$ is convergent to a fixed point y^* of T,

(iii) y^* is the unique fixed point of T in the set $Y = \left\{y \in Y; d\left(T^{\left(\frac{a}{b}\right)_o}x, y\right) < \infty\right\}$.

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

8. Fixed Point Stability of (7): Odd Case-Fixed Point Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (7) for odd case using fixed point method.

Theorem 8.1. *Let $g_a : W \to B$ be an odd mapping for which there exists a function $\alpha : W^3 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{ik} x, \left(\frac{a}{b}\right)^{ik} y, \left(\frac{a}{b}\right)^{ik} z\right)}{\left(\frac{a}{b}\right)^{ik}} = 0 \tag{88}$$

where

$$\eta_i = \begin{cases} \left(\frac{a}{b}\right), & i = 0; \\ \frac{1}{\left(\frac{a}{b}\right)}, & i = 1; \end{cases}$$

such that the functional inequality

$$\| Dg_a(x,y,z) \| \le \alpha(x,y,z) \tag{89}$$

for all $x,y,z \in W$. If there exist $L = L(i)$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right)$$

has the property,

$$\frac{1}{\eta_i}\beta\left(\eta_i x\right) = L\beta\left(x\right) \tag{90}$$

for all $x \in W$. Then there exists a unique additive function $A : W \to B$ satisfying the functional equation (7) and

$$\| g_a(x) - A(x) \| \le \frac{L^{1-i}}{1-L}\beta(x) \tag{91}$$

holds for all $x \in W$.

Proof. Consider the set

$$X = \{P/P : W \to B,\ P(0) = 0\}$$

and introduce the generalized metric on X.

$$d(p,q) = \inf\left\{K \in (0,\infty) : \| p(x) - q(x) \| \le K\beta(x), x \in W\right\}.$$

It is easy to see that (X,d) is complete. Define $T : X \to X$ by

$$Tp(x) = \frac{1}{\eta_i}p(\eta_i x)$$

for all $x \in W$. Now $p,q \in X$,

$$\begin{aligned} d(p,q) \le K &\Rightarrow \| p(x) - q(x) \| \le K\beta(x), && x \in W; \\ &\Rightarrow \left\| \frac{1}{\eta_i}p(\eta_i x) - \frac{1}{\eta_i}q(\eta_i x) \right\| \le \frac{1}{\eta_i}K\beta(\eta_i x), && x \in W; \\ &\Rightarrow \| Tp(x) - Tq(x) \| \le LK\beta(x), && x \in W; \\ &\Rightarrow d(Tp,Tq) \le LK. \end{aligned}$$

This implies $d(Tp,Tq) \le Ld(p,q)$ for all $p,q \in X$. (i,e.,) T is strictly contractive mapping on X with Lipschtiz constant L. It is follows from (59) that

$$\left\| 2g_a\left(\frac{a}{b}x\right) - 2\left(\frac{a}{b}\right)g_a(x) \right\| \le \alpha(x,0,0) \tag{92}$$

for all $x \in W$. It is follows from (92) that

$$\left\| g_a(x) - \frac{g_a\left(\frac{a}{b}x\right)}{\left(\frac{a}{b}\right)} \right\| \leq \frac{1}{2\left(\frac{a}{b}\right)} \alpha(x,0,0) \tag{93}$$

for all $x \in W$. Using (90), for the case $i = 0$, it reduces to

$$\left\| g_a(x) - \frac{1}{\left(\frac{a}{b}\right)} g_a\left(\frac{a}{b}x\right) \right\| \leq \frac{1}{\left(\frac{a}{b}\right)} \beta(x) \tag{94}$$

for all $x \in W$. (i.e.,) $d(g_a, Tg_a) \leq \frac{1}{\left(\frac{a}{b}\right)} \Rightarrow d(g_a, Tg_a) \leq \frac{1}{\left(\frac{a}{b}\right)} = L = L^1 < \infty$. Again replacing $x = \frac{x}{\left(\frac{a}{b}\right)}$ in (92), we get

$$\left\| g_a\left(x\right) - \left(\frac{a}{b}\right) g_a\left(\frac{x}{\left(\frac{a}{b}\right)}\right) \right\| \leq \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right) \tag{95}$$

for all $x \in W$. Using (90) for the case $i = 1$, it reduces to,

$$\left\| g_a\left(x\right) - \left(\frac{a}{b}\right) g_a\left(\frac{x}{\left(\frac{a}{b}\right)}\right) \right\| \leq \beta(x) \tag{96}$$

for all $x \in W$. (i.e.,) $d(g_a, Tg_a) \leq 1 \Rightarrow d(g_a, Tg_a) \leq 1 = L^0 < \infty$. In above case, we arrive

$$d(g_a, Tg_a) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds. By $(B_2(ii))$, it follows that there exists a fixed point A of T in X, such that

$$A(x) = \lim_{k\to\infty} \frac{g_a\left(\eta_i^k x\right)}{\eta_i^k}, \qquad \forall x \in W. \tag{97}$$

In order to prove $A : W \to B$ is additive. Replacing (x, y, z) by $(\eta_i^k x, \eta_i^k y, \eta_i^k z)$ in (93) and dividing by η_i^k, it follows from (88) and (97), we see that A satisfies (7) for all $x, y, z \in W$. Hence A satisfies the functional equation (7). By $(B_2\ (iii))$, A is the unique fixed point of T in the set,

$$Y = \{g_a \in X : d\left(Tg_a, A\right) < \infty\}.$$

Using the fixed point alternative result, A is the unique function such that, $\| g_a(x) - A(x) \| \leq K\beta(x)$ for all $x \in W$, and $k > 0$. Finally by $(B_2(iv))$, we obtain

$$d\left(g_a, A\right) \leq \frac{1}{1-L} d\left(g_a, Tg_a\right)$$

(i.e.,) $d\left(g_a, A\right) \leq \frac{L^{1-i}}{1-L}$. Hence, we conclude that

$$\| g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x)$$

for all $x \in W$. This completes the proof of the theorem. $\square$

Corollary 8.2. *Let $g_a : W \to B$ be an odd mapping and there exists a real numbers λ and s such that,*

$$\| Dg_a(x,y,z) \| \leq \begin{cases} \lambda, \\ \lambda\left\{\| x \|^s + \| y \|^s + \| z \|^s\right\}; \\ \lambda\left\{\| x \|^s \ \| y \|^s \ \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right\}; \end{cases} \tag{98}$$

for all $x, y, z \in W$. There exists a unique additive mapping $A : W \to B$ such that

$$\| g_a(x) - A(x) \| \leq \begin{cases} \frac{\lambda}{2\left|\left(\frac{a}{b}\right)-1\right|}, & \\ \frac{\lambda \| x \|^s}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^s\right|}, & s \neq 1 \\ \frac{\lambda \| x \|^{3s}}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^{3s}\right|}, & s \neq \frac{1}{3} \end{cases} \tag{99}$$

for all $x \in W$.

Proof. Setting

$$\alpha(x, y, z) = \begin{cases} \lambda, \\ \lambda \left\{\| x \|^s + \| y \|^s + \| z \|^s\right\}; \\ \lambda \left\{\| x \|^s \ \| y \|^s \ \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right\}; \end{cases}$$

for all $x, y, z \in W$. Now

$$\begin{aligned} \frac{\alpha\left(\eta_i^k x, \eta_i^k y, \eta_i^k z\right)}{\eta_i^k} &= \begin{cases} \frac{\lambda}{\eta_i^k}, \\ \frac{\lambda}{\eta_i^k} \left\{\| \eta_i^k x \|^s + \| \eta_i^k y \|^s + \|\eta_i^k z \|^s\right\}; \\ \frac{\lambda}{\eta_i^k} \left\{\| \eta_i^k x \|^s \ \| \eta_i^k y \|^s \ \|\eta_i^k z \|^s + \left\{\| \eta_i^k x \|^{3s} + \|\eta_i^k y \|^{3s} + \|\eta_i^k z \|^{3s}\right\}\right\}; \end{cases} \\ &= \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \end{cases} \end{aligned} \tag{100}$$

i.e., (88) is holds. But we have

$$\beta(x) = \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right).$$

Hence,

$$\beta(x) = \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right) = \begin{cases} \frac{\lambda}{2} \\ \frac{\lambda}{2\left(\frac{a}{b}\right)^s} \| x \|^s \\ \frac{\lambda}{2\left(\frac{c}{d}\right)^s} \| x \|^{3s} \end{cases}$$

Also,

$$\frac{1}{\eta_i}\beta\left(\eta_i x\right) = \begin{cases} \frac{\lambda}{2\eta_i} \\ \frac{\lambda}{2\eta_i} \| \eta_i x \|^s \\ \frac{\lambda}{2\eta_i} \| \eta_i x \|^{3s} \end{cases} = \begin{cases} \eta_i^{-1}\beta(x) \\ \eta_i^{s-1}\beta(x) \\ \eta_i^{3s-1}\beta(x) \end{cases} \tag{101}$$

Hence the inequality (90) holds

either $L = \left(\frac{a}{b}\right)^{-1}$ for $s = 0$ if $i = 0$ and $L = \frac{1}{\left(\frac{a}{b}\right)^{-1}}$ for $s = 0$ if $i = 1$.

either $L = \left(\frac{a}{b}\right)^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{\left(\frac{a}{b}\right)^{s-1}}$ for $s > 1$ if $i = 1$.

either $L = \left(\frac{a}{b}\right)^{3s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{\left(\frac{a}{b}\right)^{3s-1}}$ for $s > 1$ if $i = 1$.

Now from (95), we prove the following cases:

Case: 1 $L = \left(\frac{a}{b}\right)^{-1}, \quad i = 0$

$$\| g_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(\left(\frac{a}{b}\right)^{-1}\right)^{1-0}}{1-\left(\frac{a}{b}\right)^{-1}} \frac{\lambda}{2} = \frac{\lambda}{2\left(\left(\frac{a}{b}\right)-1\right)}. \tag{102}$$

Case: 2 $L = \left(\frac{1}{\left(\frac{a}{b}\right)}\right)^{-1}, \quad i = 1$

$$\| g_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(\left(\frac{a}{b}\right)\right)^{1-1}}{1-\left(\frac{a}{b}\right)} \frac{\lambda}{2} = \frac{\lambda}{2\left(1-\left(\frac{a}{b}\right)\right)}. \tag{103}$$

Case: 3 $L = \left(\frac{a}{b}\right)^{s-1}, \quad s < 1, \quad i = 0$

$$\| d_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(\left(\frac{a}{b}\right)^{s-1}\right)^{1-0}}{1-\left(\frac{a}{b}\right)^{s-1}} \frac{\lambda}{2\left(\frac{a}{b}\right)^{s}} \| x \|^s = \frac{\lambda \| x \|^s}{2\left(\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^{s}\right)}. \tag{104}$$

Case: 4 $L = \left(\frac{1}{\left(\frac{a}{b}\right)}\right)^{s-1}, \quad s > 1, \quad i = 1$

$$\| g_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(\left(\frac{a}{b}\right)^{1-s}\right)^{1-1}}{1-\left(\frac{a}{b}\right)^{1-s}} \frac{\lambda}{2\left(\frac{a}{b}\right)^{s}} \| x \|^s = \frac{\lambda \| x \|^s}{2\left(\left(\frac{a}{b}\right)^{s}-\left(\frac{a}{b}\right)\right)}. \tag{105}$$

Case: 5 $L = \left(\frac{a}{b}\right)^{3s-1}, \quad s < \frac{1}{3}, \quad i = 0$

$$\| g_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(\left(\frac{c}{d}\right)^{3s-1}\right)^{1-0}}{1-\left(\frac{a}{b}\right)^{3s-1}} \frac{\lambda}{2\left(\frac{a}{b}\right)^{3s}} \| x \|^{3s} = \frac{\lambda \| x \|^{3s}}{2\left(\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^{3s}\right)}. \tag{106}$$

Case: 6 $L = \left(\frac{1}{\left(\frac{a}{b}\right)}\right)^{3s-1}, \quad s > \frac{1}{3}, \quad i = 1$

$$\| g_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(\left(\frac{a}{b}\right)^{1-3s}\right)^{1-1}}{1-\left(\frac{a}{b}\right)^{1-3s}} \frac{\lambda}{2\left(\frac{a}{b}\right)^{3s}} \| x \|^{3s} = \frac{\lambda \| x \|^{3s}}{2\left(\left(\frac{a}{b}\right)^{3s}-\left(\frac{a}{b}\right)\right)}. \tag{107}$$

Hence the proof of the corollary. □

9. Fixed Point Stability of (7): Even Case-Fixed Point Method

In this section, we give the generalized Ulam-Hyers stability of the functional equation (7), for even case.

Theorem 9.1. *Let $g_q : W \to B$ be an even mapping for which there exists a function $\alpha : W^3 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\alpha\left(\eta_i^k x, \eta_i^k y, \eta_i^k z\right)}{\eta_i^{2k}} = 0 \tag{108}$$

where

$$\eta_i = \begin{cases} \left(\frac{a}{b}\right), & i = 0; \\ \frac{1}{\left(\frac{a}{b}\right)}, & i = 1; \end{cases}$$

such that the functional inequality with

$$\| Dg_q(x, y, z) \| \leq \alpha(x, y, z) \tag{109}$$

for all $x, y, z \in W$. If there exist $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right)$$

has the property,

$$\frac{1}{\eta_i^2}\beta\left(\eta_i x\right) = L\beta\left(x\right) \tag{110}$$

for all $x \in W$. Then there exists a unique quadratic function $Q : W \to B$ satisfying the functional equation (7) and

$$\| g_q(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) \tag{111}$$

holds for all $x \in W$.

Proof. Consider the set $X = \{P/P : W \to B,\ P(0) = 0\}$ and introduce the generalized metric on X.

$$d(p, q) = \inf\left\{K \in (0, \infty) : \| p(x) - q(x) \| \leq K\beta(x), x \in W\right\}$$

It is easy to see that (X, d) is complete. Define $T : X \to X$ by $Tp(x) = \frac{1}{\eta_i^2}p(\eta_i x)$ for all $x \in W$. Now $p, q \in X$,

$$\begin{aligned}
d(p, q) \leq K &\Rightarrow \| p(x) - q(x) \| \leq K\beta(x); && x \in W\\
&\Rightarrow \left\| \frac{1}{\eta_i^2}p(\eta_i x) - \frac{1}{\eta_i^2}q(\eta_i x) \right\| \leq \frac{1}{\eta_i^2}K\beta(\eta_i x); && x \in W\\
&\Rightarrow \left\| \frac{1}{\eta_i^2}p(\eta_i x) - \frac{1}{\eta_i^2}q(\eta_i x) \right\| \leq LK\beta(x); && x \in W\\
&\Rightarrow \| Tp(x) - Tq(x) \| \leq LK\beta(x); && x \in W\\
&\Rightarrow d(Tp, Tq) \leq LK.
\end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p, q)$ for all $p, q \in X$. (i,e.,) T is strictly contractive mapping on X with Lipschtiz constant L. Replacing (x, y, z) by $(x, 0, 0)$ in (109) and using evenness of g, we get

$$\left\| 4g\left(\frac{a}{b}x\right) - 4\left(\frac{a}{b}\right)^2 g(x) \right\| \leq \alpha(x, 0, 0) \tag{112}$$

$$\left\| g(x) - \frac{g\left(\frac{a}{b}x\right)}{\left(\frac{a}{b}\right)^2} \right\| \leq \frac{1}{4\left(\frac{a}{b}\right)^2}\alpha(x, 0, 0) \tag{113}$$

for all $x \in W$, using (114) for the case $i = 0$, it reduces to

$$\left\| g_q(x) - \frac{1}{\left(\frac{a}{b}\right)^2}g_q\left(\frac{a}{b}x\right) \right\| \leq \frac{1}{2\left(\frac{a}{b}\right)^2}\beta(x) \tag{114}$$

for all $x \in W$, (i.e.,) $d(g_q, Tg_q) \leq \frac{1}{\left(\frac{a}{b}\right)} \Rightarrow d(g_q, Tg_q) \leq \frac{1}{2\left(\frac{a}{b}\right)} = L = L^1 < \infty$. Again replacing $x = \frac{x}{\left(\frac{a}{b}\right)}$ in (112), we get

$$\left\| g_q\left(x\right) - \left(\frac{a}{b}\right)^2 g_q\left(\frac{x}{\left(\frac{a}{b}\right)}\right) \right\| \leq \frac{1}{4}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right) \tag{115}$$

for all $x \in W$, using (110) for the case $i = 1$, it reduces to

$$\left\| g_q\left(x\right) - \left(\frac{a}{b}\right)^2 g_q\left(\frac{x}{\left(\frac{a}{b}\right)}\right) \right\| \leq \frac{1}{2}\beta(x); \tag{116}$$

for all $x \in W$, (i.e.,) $d(g_q, Tg_q) \leq \frac{1}{2} < 1 \Rightarrow d(g_q, Tg_q) \leq 1 \Rightarrow L^0 < \infty$. In above cases, we arrive

$$d(g_q, Tg_q) \leq L^{1-i}.$$

The rest of the proof is similar to that of Theorem 7.1 This completes the proof of the theorem. ☐

Corollary 9.2. *Let $g_q : W \to B$ be an even mapping and there exists a real numbers λ and s such that,*

$$\| Dg_q(x,y,z) \| \leq \begin{cases} \lambda, & \\ \lambda \left\{ \| x \|^s + \| y \|^s + \| z \|^s \right\}, & s \neq 2; \\ \lambda \left\{ \| x \|^s \ \|y\|^s \ \| z \|^s + \left\{ \| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s} \right\} \right\}, & s \neq \frac{1}{3}; \end{cases} \tag{117}$$

for all $x, y, z \in W$. There exists a unique quadratic mapping $Q : W \to B$ such that

$$\| g_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{4\left| \left(\frac{a}{b}\right)^2 - 1 \right|}, \\ \frac{\lambda \| x \|^s}{4\left| \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^s \right|}, \\ \frac{\lambda \| x \|^{3s}}{4\left| \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^{3s} \right|}, \end{cases} \tag{118}$$

for all $x \in W$.

10. Fixed Point Stability of (7): Mixed Case-Fixed Point Method

In this section, we discuss the generalized Ulam-Hyers stability of the functional equation (7) for the mixed case.

Theorem 10.1. *Let $g : W \to B$ be a mapping for which there exists a function $\alpha : W^3 \to [0, \infty)$ with the condition (88) and (108) where*

$$\eta_i = \begin{cases} \left(\frac{a}{b}\right), & i = 0; \\ \frac{1}{\left(\frac{a}{b}\right)}, & i = 1; \end{cases}$$

such that the functional inequality with

$$\| Dg(x,y,z) \| \leq \alpha(x,y,z) \tag{119}$$

for all $x, y, z \in W$. If there exist $L = L(i)$ such that the function for all $x \in W$, such that the function

$$x \to \beta(x) = \frac{1}{2} \alpha \left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0 \right),$$

has the properties (90) and (110) for all $x \in W$. Then there exists a unique additive function $A : W \to B$ and a unique quadratic function $Q : W \to B$ satisfying the functional equation (7) and

$$\| g(x) - A(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L} \left[\beta(x) + \beta(-x) \right] \tag{120}$$

holds for all $x \in W$.

Proof. It follows from (80) and Theorem 7.1, that

$$\| g_o(x) - A(x) \| \leq \frac{1}{2} \frac{L^{1-i}}{1-L} \left[\beta(x) + \beta(-x) \right] \tag{121}$$

Similarly, it is follows from (82) and Theorem 8.1, that

$$\| g_e(x) - Q(x) \| \leq \frac{1}{2} \frac{L^{1-i}}{1-L} \left[\beta(x) + \beta(-x) \right] \tag{122}$$

for all $x \in W$. Define

$$g(x) = g_o(x) + g_e(x) \tag{123}$$

for all $x \in W$. From (121), (122) and (123), we have

$$\begin{aligned}\| g(x) - A(x) - Q(x) \| &\leq \| g_e(x) + g_o(x) - A(x) - Q(x) \| \\ &\leq \| g_o(x) - A(x) \| + \| g_e(x) - Q(x) \| \\ &\leq \frac{L^{1-i}}{1-L} [\beta(x) + \beta(-x)]\end{aligned}$$

for all $x \in W$. Hence the theorem is proved. □

Using Corollaries 7.2 and 8.2 we have the following corollary concerning the stability of (7).

Corollary 10.2. *Let $g : W \to B$ be a mapping and there exists a real numbers λ and s such that,*

$$\| Dg(x, y, z) \| \leq \begin{cases} \lambda, & \\ \lambda \{\| x \|^s + \| y \|^s + \| z \|^s\}, & s \neq 1, 2 \\ \lambda \left\{\| x \|^s \ \| y \|^s \ \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right\}, & s \neq \frac{1}{3}, \frac{2}{3} \end{cases} \tag{124}$$

for all $x, y, z \in W$. There exists a unique additive mapping $A : W \to B$ and a unique quadratic mapping $Q : W \to B$ such that,

$$\| g(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{2\left|\left(\frac{a}{b}\right)-1\right|} + \frac{\lambda}{4\left|\left(\frac{a}{b}\right)^2-1\right|}, \\ \frac{\lambda\| x \|^s}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^s\right|} + \frac{\lambda\| x \|^s}{4\left|\left(\frac{a}{b}\right)^2-\left(\frac{a}{b}\right)^s\right|}, \\ \frac{\lambda\| x \|^{3s}}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^{3s}\right|} + \frac{\lambda\| x \|^{3s}}{4\left|\left(\frac{a}{b}\right)^2-\left(\frac{a}{b}\right)^{3s}\right|}, \end{cases} \tag{125}$$

for all $x \in W$.

Banach Algebra Stability Results for (7)

For sections 10, 11 and 12, let us consider X and Y to a normed Algebra and a Banach Algebra, respectively. Define a mapping $Dg : X \to Y$ by,

$$\begin{aligned}Dg(x, y, z) = {} & g\left(\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x - \frac{b}{c}y + \frac{c}{d}z\right) + g\left(\frac{a}{b}x + \frac{b}{c}y - \frac{c}{d}z\right) \\ & + g\left(-\frac{a}{b}x + \frac{b}{c}y + \frac{c}{d}z\right) - \left(\frac{a}{b}\right)[g(x) - g(-x)] - \left(\frac{b}{c}\right)[g(y) - g(-y)] - \left(\frac{c}{d}\right)[g(z) - g(-z)] \\ & - 2\left(\frac{a}{b}\right)^2 [g(x) + g(-x)] - 2\left(\frac{b}{c}\right)^2 [g(y) + g(-y)] - 2\left(\frac{c}{d}\right)^2 [g(z) + g(-z)]\end{aligned}$$

for all $x, y, z \in X$.

11. Stability Results for (7): Odd Case-Direct method

In this section, we present the generalized Ulam-Hyers stability of the functional equation (7), when g is odd.

Definition 11.1. *Let X be Banach Algebra. A mapping $A : X \to X$ is said to be Additive derivation if the Additive function A satisfies,v*

$$A(ab) = A(a)b + aA(b) \tag{126}$$

for all $a, b \in X$. Also the additive derivation for three variables satisfies

$$A(abc) = A(a)bc + aA(b)c + abA(c) \tag{127}$$

for all $a, b, c \in X$.

Theorem 11.2. *Let $j = \pm 1$. Let $g_a : X \to Y$ be a odd mapping for which there exists function $\alpha, \beta : X^3 \to [0, \infty)$ with the condition*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k\to\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{kj}} = 0 \tag{128}$$

$$\sum_{k=0}^{\infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k\to\infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{kj}} = 0 \tag{129}$$

such that the functional inequalities

$$\| Dg_a(x, y, z) \| \leq \alpha(x, y, z) \tag{130}$$

and

$$\| g_a(xyz) - g_a(x)\, yz - xg_a(y)z - xyg_a(z) \| \leq \beta(x, y, z) \tag{131}$$

for all $x, y, z \in X$. Then there exists a unique Additive Derivation mapping $A : X \to Y$ satisfying the functional equation (7) and

$$\| g_a(x) - A(x) \| \leq \frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} \tag{132}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{n\to\infty} \frac{g_a\left(\left(\frac{a}{b}\right)^{kj} x\right)}{\left(\frac{a}{b}\right)^{kj}} \tag{133}$$

for all $x \in X$.

Proof. It follows from Theorem 4.1 that A is a unique additive mapping and satisfies (7) for all $x, y, z \in X$. It follows from (131) that

$$\begin{aligned}
\| A(xyz) - A(x)\, yz - xA(y)z - xyA(z) \| &\leq \frac{1}{\left(\frac{a}{b}\right)^k} \left\| g_a\left(\left(\frac{a}{b}\right)^k (xy\,z)\right) - g_a\left(\left(\frac{a}{b}\right)^k x\right)\left(\left(\frac{a}{b}\right)^k y \left(\frac{a}{b}\right)^k z\right) \right. \\
&\quad \left. - \left(\frac{a}{b}\right)^k x\, g_a\left(\left(\frac{a}{b}\right)^k y\right)\left(\frac{a}{b}\right)^k z - \left(\frac{a}{b}\right)^k x \left(\frac{a}{b}\right)^k y\, g_a\left(\left(\frac{a}{b}\right)^k z\right) \right\| \\
&\leq \frac{1}{\left(\frac{a}{b}\right)^k} \beta\left(\left(\frac{a}{b}\right)^k x, \left(\frac{a}{b}\right)^k y, \left(\frac{a}{b}\right)^k z\right) \to 0 \text{ as } k \to \infty
\end{aligned}$$

for all $x, y, z \in X$. Hence, the mapping $A : X \to Y$ is a unique Additive Derivation satisfying (132). □

This following corollary is a immediate consequence of Theorem 10.2 concerning the stability of (7).

Corollary 11.3. *Let $g_a : X \to Y$ be a odd mapping and there exists a real numbers λ and s such that*

$$\| Dg_a(x, y, z) \| \leq \begin{cases} \lambda; & \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 1 \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{1}{3} \end{cases} \tag{134}$$

$$\| g_a(xyz) - g_a(x)yz - xg_a(y)z - xyg_a(z) \| \leq \begin{cases} \lambda; \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); \end{cases} \tag{135}$$

for all $x, y, z \in X$. Then there exists a unique additive derivation $A : X \to Y$ such that

$$\| g_a(x) - A(x) \| \leq \begin{cases} \frac{\lambda}{2\left|\left(\frac{a}{b}\right)-1\right|}, \\ \frac{\lambda \| x \|^s}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^s\right|}, \\ \frac{\lambda \| x \|^{3s}}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^{3s}\right|}, \end{cases} \tag{136}$$

for all $x \in X$.

12. Stability Results for (7): Even Case-Direct Method

In this section, we discuss the generalized Ulam-Hyers stability of the functional equation (7), when g is even.

Definition 12.1. *Let X be Banach Algebra. A mapping $Q : X \to X$ is said to be quadratic derivation if the quadratic function Q satisfies*

$$Q(ab) = Q(a)b^2 + a^2Q(b) \tag{137}$$

for all $a, b \in X$. Also the quadratic derivation for three variables satisfies

$$Q(abc) = Q(a)b^2c^2 + a^2Q(b)c^2 + a^2b^2Q(c) \tag{138}$$

for all $a, b, c \in X$.

Theorem 12.2. *Let $j = \pm 1$. Let $g_q : X \to Y$ be an even mapping for which there exists functions $\alpha, \beta : X^3 \to [0, \infty)$ with the conditions*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k\to\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} = 0 \tag{139}$$

$$\sum_{k=0}^{\infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k\to\infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} = 0 \tag{140}$$

such that the functional inequalities

$$\| Dg_q(x, y, z) \| \leq \alpha(x, y, z) \tag{141}$$

and

$$\left\| g_q(xyz) - y^2z^2g_q(x) - x^2g_q(y)z^2 - x^2y^2g_q(z) \right\| \leq \beta(x, y, z) \tag{142}$$

for all $x, y, z \in X$. There exists a unique quadratic derivation $Q : X \to Y$ satisfying the functional equation (7) and

$$\| g_q(x) - Q(x) \| \leq \frac{1}{4\left(\frac{a}{b}\right)^2} \sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} \tag{143}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{g_q\left(\left(\frac{a}{b}\right)^{kj} x\right)}{\left(\frac{a}{b}\right)^{2kj}} \tag{144}$$

for all $x \in X$.

Proof. It follows from Theorem 5.1 that Q is a unique quadratic mapping and satisfies (7) for all $x, y, z \in X$. It follows from (142) that

$$\begin{aligned}\| Q(xyz) - Q(x)\, y^2 z^2 - x^2 Q(y) z^2 - x^2 y^2 Q(z) \| &\leq \frac{1}{\left(\frac{a}{b}\right)^{2k}} \left\| g_q\left(\left(\frac{a}{b}\right)^k (xy\, z)\right) - g_q\left(\left(\frac{a}{b}\right)^k x\right)\left(\frac{a}{b}\right)^k y \left(\frac{a}{b}\right)^k z \right. \\ &\quad \left. - \left(\frac{a}{b}\right)^k x\, g_q\left(\left(\frac{a}{b}\right)^k y\right)\left(\frac{a}{b}\right)^k z - \left(\frac{a}{b}\right)^k x \left(\frac{a}{b}\right)^k y\, g_q\left(\left(\frac{a}{b}\right)^k z\right)\right\| \\ &\leq \frac{1}{\left(\frac{a}{b}\right)^{2k}} \beta\left(\left(\frac{a}{b}\right)^k x, \left(\frac{a}{b}\right)^k y, \left(\frac{a}{b}\right)^k z\right) \to 0 \;\; as \;\; k \to \infty\end{aligned}$$

for all $x, y, z \in X$. Hence, the mapping $Q : X \to Y$ is a unique Quadratic Derivation satisfying (143). □

Corollary 12.3. *Let $g_q : X \to Y$ be a even mapping and there exists a real numbers λ and s such that,*

$$\| Dg_q(x, y, z) \| \leq \begin{cases} \lambda; & \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 2; \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{2}{3}; \end{cases} \tag{145}$$

$$\| g_q(xyz) - y^2 z^2 g_q(x) - x^2 g_q(y) z^2 - x^2 y^2 g_q(z) \| \leq \begin{cases} \lambda; \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); \end{cases} \tag{146}$$

for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to Y$ such that

$$\| g_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{4\left|\left(\frac{a}{b}\right)^2 - 1\right|} \\ \frac{\lambda \| x \|^s}{4\left|\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^s\right|}; \\ \frac{\lambda \| x \|^{3s}}{4\left|\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^{3s}\right|}; \end{cases} \tag{147}$$

for all $x \in X$.

13. Stability Results for (7): Mixed Case-Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (7), when g is Mixed case.

Theorem 13.1. *Let $j = \pm 1$. Let $g : X \to Y$ be a mapping for which there exists functions $\alpha, \beta : X^3 \to [0, \infty)$ with the conditions (127), (129) and (138), (140) such that the functional inequalities,*

$$\| Dg(x, y, z) \| \leq \alpha(x, y, z) \tag{148}$$

(131) *and* (142) *for all $x, y, z \in X$. There exists a unique additive derivation mapping $A : X \to Y$ and a unique quadratic derivation mapping $Q : X \to Y$ satisfying the functional equation* (7) *and*

$$\begin{aligned}\| g(x) - A(x) - Q(x) \| \leq \frac{1}{2} &\left[\frac{1}{2\left(\frac{a}{b}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{kj}}\right)\right] \\ &+ \frac{1}{4\left(\frac{a}{b}\right)^2} \left[\sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}} + \frac{\alpha\left(-\left(\frac{a}{b}\right)^{kj} x, 0, 0\right)}{\left(\frac{a}{b}\right)^{2kj}}\right)\right]\end{aligned} \tag{149}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ are defined in (132) *and* (143) *respectively for all $x \in X$.*

Proof. The proof follows by Theorems 6.1, using Theorems 10.2 and 11.2. $\square$

Using Corollaries 10.3 and 11.3 we have the following corollary concerning the stability of (7).

Corollary 13.2. *Let $g : X \to Y$ be a mapping and there exists a real numbers λ and s such that*

$$\| Dg(x,y,z) \| \leq \begin{cases} \lambda; & \\ \lambda \left\{ \| x \|^s + \| y \|^s + \| z \|^s \right\}; & s \neq 1,2; \\ \lambda \left\{ \| x \|^s \, \| y \|^s \, \| z \|^s + \left\{ \| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s} \right\} \right\}; & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{150}$$

and (135), (146) for all $x, y, z \in X$. There exists a unique additive derivation $A : X \to Y$ and a unique quadratic derivation $Q : X \to Y$ such that

$$\| g(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{2\left| \left(\frac{a}{b}\right) - 1 \right|} + \frac{\lambda}{4\left| \left(\frac{a}{b}\right)^2 - 1 \right|}; \\ \frac{\lambda \| x \|^s}{2\left| \left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^s \right|} + \frac{\lambda \| x \|^s}{4\left| \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^s \right|}; \\ \frac{\lambda \| x \|^{3s}}{2\left| \left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^{3s} \right|} + \frac{\lambda \| x \|^{3s}}{4\left| \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^{3s} \right|}; \end{cases} \tag{151}$$

for all $x \in X$.

14. Stability Results for (7): Odd Case-Fixed Point Method

In this section, we give the generalized Ulam-Hyers stability of the functional equation (7), when g is odd case.

Theorem 14.1. *Let $j = \pm 1$. Let $g_a : X \to Y$ be a odd mapping for which there exists functions $\alpha, \beta : X^3 \to [0, \infty)$ with the conditions*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z \right)}{\left(\frac{a}{b}\right)^{kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k \to \infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z \right)}{\left(\frac{a}{b}\right)^{kj}} = 0 \tag{152}$$

$$\sum_{k=0}^{\infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z \right)}{\left(\frac{a}{b}\right)^{kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k \to \infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z \right)}{\left(\frac{a}{b}\right)^{kj}} = 0 \tag{153}$$

where η is defined in (88) satisfying the functional inequalities

$$\| Dg_a(x,y,z) \| \leq \alpha(x,y,z) \tag{154}$$

and

$$\| g_a(xyz) - g_a(x)\,yz - xg_a(y)z - xyg_a(z) \| \leq \beta(x,y,z) \tag{155}$$

for all $x, y, z \in X$. Then there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2} \alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0 \right)$$

as the property

$$\frac{1}{\eta_i} \beta\left(\eta_i x\right) = L\beta(x) \tag{156}$$

for all $x \in X$. Then there exists a unique Additive Derivation mapping $A : X \to Y$ satisfying the functional equation (7) and

$$\| gf_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L} \beta(x) \tag{157}$$

for all $x \in X$.

Proof. It follows from Theorem 7.1 that A is a unique additive mapping and satisfies (7) for all $x \in X$. It follows from (152), (153) and (155) that

$$\begin{aligned}\| A(xyz) - A(x)\,yz - xA(y)z - xyA(z) \| &\leq \frac{1}{\left(\frac{a}{b}\right)^k}\left\| g_a\left(\left(\frac{a}{b}\right)^k (xy\,z)\right) - g_a\left(\left(\frac{a}{b}\right)^k x\right)\left(\left(\frac{a}{b}\right)^k y\,\left(\frac{a}{b}\right)^k z\right)\right.\\ &\left. - \left(\frac{a}{b}\right)^k x\, g_a\left(\left(\frac{a}{b}\right)^k y\right)\left(\frac{a}{b}\right)^k z - \left(\frac{a}{b}\right)^k x\,\left(\frac{a}{b}\right)^k y\, g_a\left(\left(\frac{a}{b}\right)^k z\right)\right\|\\ &\leq \frac{1}{\left(\frac{a}{b}\right)^k}\beta\left(\left(\frac{a}{b}\right)^k x,\, \left(\frac{a}{b}\right)^k y,\, \left(\frac{a}{b}\right)^k z\right) \to 0 \;\; as \;\; k \to \infty.\end{aligned}$$

Thus the mapping $A : X \to Y$ is unique additive derivation mapping satisfying (7). $\square$

Corollary 14.2. *Let $g_a : X \to Y$ be a odd mapping and there exists a real numbers λ and s such that,*

$$\| Dg_a(x, y, z) \| \leq \begin{cases} \lambda; \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 1; \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{1}{3}; \end{cases} \tag{158}$$

$$\| g_a(xyz) - g_a(x)yz - xg_a(y)z - xyg_a(z) \| \leq \begin{cases} \lambda; \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 1; \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{1}{3}; \end{cases} \tag{159}$$

for all $x, y, z \in X$. Then there exists a unique additive derivation $A : X \to Y$ such that

$$\| g_a(x) - A(x) \| \leq \begin{cases} \frac{\lambda}{2\left|\left(\frac{a}{b}\right)-1\right|}, \\ \frac{\lambda \| x \|^s}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^s\right|}, \\ \frac{\lambda \| x \|^{3s}}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^{3s}\right|}, \end{cases} \tag{160}$$

for all $x \in X$.

15. Stability Results for (7): Even Case-Fixed Point Method

In this section, we discuss the generalized Ulam-Hyers stability of the functional equation (7), when g is even case.

Theorem 15.1. *Let $j = \pm 1$. Let $g_a : X \to Y$ be a even mapping for which there exists functions $\alpha, \beta : X^3 \to [0, \infty)$ with the conditions*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} \;\; converges\ in\ \mathbb{R}\ and \;\; \lim_{k\to\infty} \frac{\alpha\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} = 0 \tag{161}$$

$$\sum_{k=0}^{\infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} \;\; converges\ in\ \mathbb{R}\ and \;\; \lim_{k\to\infty} \frac{\beta\left(\left(\frac{a}{b}\right)^{kj} x, \left(\frac{a}{b}\right)^{kj} y, \left(\frac{a}{b}\right)^{kj} z\right)}{\left(\frac{a}{b}\right)^{2kj}} = 0 \tag{162}$$

where η_i is defined in (88) satisfying the functional inequalities

$$\| Dg_q(x, y, z) \| \leq \alpha(x, y, z) \tag{163}$$

and

$$\left\| g_q(xyz) - g_q(x)\,y^2 z^2 - x^2 g_q(y) z^2 - x^2 y^2 g_q(z) \right\| \leq \beta(x, y, z) \tag{164}$$

for all $x, y, z \in X$. Then there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right)$$

as the property

$$\frac{1}{\eta_i^2}\beta\left(\eta_i x\right) = L\beta(x) \tag{165}$$

for all $x \in X$. Then there exists a unique Quadratic Derivation mapping $Q : X \to Y$ satisfying the functional equation (7) and

$$\| g_q(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x) \tag{166}$$

for all $x \in X$.

Proof. It follows from Theorem 8.1 that Q is a unique quadratic mapping and satisfies (7) for all $x \in X$. It follows from (161), (162) and (164) that

$$\begin{aligned}\| Q(xyz) - Q(x)\, y^2 z^2 - x^2 Q(y) z^2 - x^2 y^2 Q(z) \| &\leq \frac{1}{\left(\frac{a}{b}\right)^{2k}}\left\| g_q\left(\left(\frac{a}{b}\right)^k (xy\,z)\right) - g_q\left(\left(\frac{a}{b}\right)^k x\right)\left(\frac{a}{b}\right)^k y\left(\frac{a}{b}\right)^k z \right.\\ &\left. - \left(\frac{a}{b}\right)^k x\, g_q\left(\left(\frac{a}{b}\right)^k y\right)\left(\frac{a}{b}\right)^k z - \left(\frac{a}{b}\right)^k x\left(\frac{a}{b}\right)^k y\, g_q\left(\left(\frac{a}{b}\right)^k z\right)\right\|\\ &\leq \frac{1}{\left(\frac{a}{b}\right)^{2k}}\beta\left(\left(\frac{a}{b}\right)^k x, \left(\frac{a}{b}\right)^k y, \left(\frac{a}{b}\right)^k z\right) \to 0 \;\; as \;\; k \to \infty.\end{aligned}$$

Thus the mapping $Q : X \to Y$ is unique quadratic derivation mapping satisfying (7). □

Proof. Let $g_q : X \to Y$ be a even mapping and there exists a real numbers λ and s such that,

$$\| Dg_q(x, y, z) \| \leq \begin{cases} \lambda; & \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 1; \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{1}{3}; \end{cases} \tag{167}$$

$$\| g_q(xyz) - g_q(x) y^2 z^2 - x^2 g_q(y) z^2 - x^2 y^2 g_q(z) \| \leq \begin{cases} \lambda; & \\ \lambda\left(\| x \|^s + \| y \|^s + \| z \|^s\right); & s \neq 1; \\ \lambda\left(\| x \|^s \| y \|^s \| z \|^s + \left\{\| x \|^{3s} + \| y \|^{3s} + \| z \|^{3s}\right\}\right); & s \neq \frac{1}{3}; \end{cases} \tag{168}$$

for all $x, y, z \in X$. Then there exists a unique quadratic derivation $Q : X \to Y$ such that

$$\| g_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{4\left|\left(\frac{a}{b}\right)^2 - 1\right|}; \\ \frac{\lambda \| x \|^s}{4\left|\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^s\right|}; \\ \frac{\lambda \| x \|^{3s}}{4\left|\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^{3s}\right|}; \end{cases} \tag{169}$$

for all $x \in X$. □

16. Stability Results for (7): Mixed Case-Fixed Point Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (7), when g is mixed case.

Theorem 16.1. *Let $g : X \to Y$ be a mapping for which there exist a function $\alpha, \beta : X^3 \to [0, \infty)$ with the condition, (152), (153) and (161), (162) where η is defined in (88) such that the functional inequality*

$$\| \; Dg(x, y, z) \; \| \leq \alpha(x, y, z) \tag{170}$$

and (155), (164) for all $x, y, z \in X$. If there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{a}{b}\right)}, 0, 0\right),$$

has the properties (156) and (165) for all $x \in X$. Then there exists a unique additive derivation mapping $A : X \to Y$ and a unique quadratic derivation mapping $Q : X \to Y$ satisfying the functional equation (7) and

$$\|g(x) - A(x) - Q(x) \,\| \leq \frac{L^{1-i}}{1-L}\left[\beta(x) + \beta(-x)\right] \tag{171}$$

holds for all $x \in X$.

Corollary 16.2. *Let $g : X \to Y$ be a mapping and there exists a real numbers λ and s such that*

$$\| \; Dg(x, y, z) \; \| \leq \begin{cases} \lambda; & \\ \lambda\left\{\| \, x \, \|^s + \| \, y \, \|^s + \| \, z \, \|^s\right\}; & s \neq 1, 2; \\ \lambda\left\{\| \, x \, \|^s \; \| \, y \, \|^s \; \| \, z \, \|^s + \left\{\| \, x \, \|^{3s} + \| \, y \, \|^{3s} + \| \, z \, \|^{3s}\right\}\right\}; & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{172}$$

and (155), (164) for all $x, y, z \in X$. There exists a unique additive derivation mapping $A : X \to Y$ and a unique quadratic derivation mapping $Q : X \to Y$ such that

$$\| \; g(x) - A(x) - Q(x) \; \| \leq \begin{cases} \frac{\lambda}{2\left|\left(\frac{a}{b}\right)-1\right|} + \frac{\lambda}{4\left|\left(\frac{a}{b}\right)^2-1\right|} \\ \frac{\lambda\| \, x \, \|^s}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^s\right|} + \frac{\lambda\| \, x \, \|^s}{4\left|\left(\frac{a}{b}\right)^2-\left(\frac{a}{b}\right)^s\right|}; \\ \frac{\lambda\| \, x \, \|^{3s}}{2\left|\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^{3s}\right|} + \frac{\lambda\| \, x \, \|^{3s}}{4\left|\left(\frac{a}{b}\right)^2-\left(\frac{a}{b}\right)^{3s}\right|}; \end{cases} \tag{173}$$

for all $x \in X$.

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