Int. J. Math. And Appl., 9(2)(2021), 163–167
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Smooth Fuzzy Pairwise Multifunctions

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Abstract: In this paper, we introduce and study the concepts of fuzzy lower (upper) almost continuous multifunctions where the domain of these functions is a classical bitopological space with their values as arbitrary fuzzy sets in L-fuzzy bitopological space in view of Sostak's sense.

MSC: 54A40

Keywords: Fuzzy bitopological spaces, fuzzy almpst continuous multifunctions.

1. Introduction

Since Chang [1] introduced fuzzy set theory to topology, many researches have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. In Chang's I-topology on a set X, each open set was fuzzy while the topology itself was a crisp subset of the family of all fuzzy subsets of X. From a different direction, Höhle [5] presented the notion of a fuzzy topology being viewed as an I-subset of a powerset 2^X. Then Kubiak [9] and Šostak [19] independently extended Höhle's fuzzy topology to L-subsets of L^X, which is called L-fuzzy topology (see [6, 7, 17, 19]). Papageorgiou [14] introduced and studied the notion of fuzzy multifunction and extended the concepts of fuzzy continuous functions to the fuzzy multivalued case, by introducing of fuzzy upper and fuzzy lower semi-continuous multifunctions. Mukherjee and Malakar [13] redefined the concepts of lower inverse and lower semi-continuity of a fuzzy multifunction in terms of the notion of quasi-coincidence due to Pu and Liu ([15, 16]). The concepts of fuzzy lower (upper) almost continuous and fuzzy lower (upper) almost weakly continuous fuzzy multifunctions were introduced by Mukherjee and Malakar [13], also fuzzy lower and fuzzy upper continuous multifunctions were introduced by Mahmoud [12], where their fuzzy multifunction maps each point in a classical topological space to an arbitrary fuzzy set in a fuzzy topological space in the sense of Chang [1].

2. Preliminaris

Throughout this paper, let $L = (L, \leq, \vee, \wedge, ')$ be a completely distributive lattice with an order reversing involution $'$ with the smallest element 0_L and the largest element 1_L, $L_0 = L - \{0_L\}$, $I = [0, 1]$ and $I_0 = (0, 1]$. The family of all fuzzy sets on X will be denoted by L^X ([4, 20]). The smallest element and the largest element of L^X will be denoted by 0_X and 1_X, respectively. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point x_t for $t \in L_0$ is an element of L^X such that, for $y \in X$:

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$x_t(y) = \begin{cases} t & \text{if } x = y \\ o_L & \text{if } x \neq y \end{cases}$. The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point x_t is said to be belong to a fuzzy set λ of X denoted by $x_t \in \lambda$ if $t \leq \lambda(x)$. A fuzzy point x_t is said to be quasi-coincident with a fuzzy set λ, denoted by $x_t \; q \; \lambda$ if $t \neq \lambda'(x)$, otherwise $x_t \; \bar{q} \; \lambda$. A fuzzy set λ is said to be quasi-coincident with a fuzzy set μ denoted by $\lambda \; q \; \mu$ if there exists $x \in X$ such that $\lambda(x) \neq \mu'(x)$, otherwise $\lambda \bar{q} \mu$ [11]. Also, in this paper, the indices $i, j \in \{1, 2\}$ and $i \neq j$. Let A be a subset of an ordinary bitopological space (X, τ_1, τ_2). The interior (resp. closure) of A with respect to τ_i will be denoted by $i\,\mathrm{Int}(A)$ (resp. $i\,\mathrm{Cl}(A)$). A is said to be (i, j)-regular open (resp. (i, j)-regular closed) if $A = i\,\mathrm{Int}(j\,\mathrm{Cl}(A)) = A$ (resp. $A = j\,\mathrm{Cl}(i\,\mathrm{Int}(A))$).

Definition 2.1 ([2, 18]). *A double fuzzy topology on X is a pair of maps $\tau, \tau^\star : I^X \to I$, which satisfies the following properties:*

(1). $\tau(\lambda) \leq \bar{1} - \tau^\star(\lambda)$ for each $\lambda \in I^X$.

(2). $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^\star(\lambda_1 \wedge \lambda_2) \leq \tau^\star(\lambda_1) \vee \tau^\star(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.

(3). $\tau(\bigvee\limits_{i\in\Gamma} \lambda_i) \geq \bigwedge\limits_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^\star(\bigvee\limits_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^\star(\lambda_i)$ for each $\lambda_i \in I^X, i \in \Gamma$.

The triplet $(X, \tau, \tau^\star)$ is called a double fuzzy topological space.

Remark 2.2. *$(X, \tau_1, \tau_1^\star, \sigma_1, \sigma_2^\star)$ is called a double fuzzy bitopological space.*

Definition 2.3 ([2, 18]). *Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space and $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that $r + s \leq 1$. Then the fuzzy set λ is called an (r, s)-fuzzy open if $\tau(\lambda) \geq r$ and $\tau^\star(\lambda) \leq s$, λ is called an (r, s)-fuzzy closed if, and only if $\bar{1} - \lambda$ is an (r, s)-fuzzy open set.*

Theorem 2.4 ([3, 10]). *Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. Then double fuzzy closure operator and double fuzzy interior operator of $\lambda \in I^X$ are defined by*

$$\mathrm{Cl}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\bar{1} - \mu) \geq r, \tau^\star(\bar{1} - \mu) \leq s\},$$
$$\mathrm{Int}(\lambda, r, s) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r, \tau^\star(\mu) \leq s\},$$

where $r \in I_0$ and $s \in I_1$ such that $r + s \leq 1$.

3. Double Fuzzy Almost Pairwise Continuous Multifunctions

Definition 3.1. *Let (X, τ_1, τ_2) be an ordinary bitopological space and $(Y, \sigma_1, \sigma_1^\star, \sigma_2, \sigma_2^\star)$ be an L-fuzzy bitoplogical space. By $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_1^\star, \sigma_2, \sigma_2^\star)$ we mean that F is a fuzzy multifunction between X and Y, and we call it fuzzy multifunction.*

Definition 3.2. *A fuzzy multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_1^\star, \sigma_2, \sigma_2^\star)$ is called:*

(1). double fuzzy lower (upper) pairwise almost continuous at some point $x_0 \in X$ if for every $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$, $r \in I_0$ and $x_0 \in F^-(\mu)$ ($x_0 \in F^+(\mu)$), there exists $U \in \tau_i(x_0)$ such that $U \subset F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s))$ ($U \subset F^+(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s))$).

(2). double fuzzy lower (upper) pairwise almost continuous if F is double fuzzy lower (upper) pairwise almost continuous at each $x_0 \in X$.

Theorem 3.3. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_1^\star, \sigma_2, \sigma_2^\star)$ be a fuzzy multifunction. Then the following statements are equivalent:*

(1). F is double fuzzy lower pairwise almost continuous.

(2). $F^-(\mu) \subset i\,\mathrm{Int}(F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)))$ for every $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$, $r \in I_0$.

(3). $F^-(\mu) \in \tau_i$ for every (r, s)-(i, j)-fuzzy regular open set $\mu \in I^Y$.

(4). $F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)) \in \tau_i$ for every $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$, $r \in I_0$.

(5). $F^+(\eta) \supset j\,\mathrm{Cl}(F^+(i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s)))$ for every $\eta \in I^Y$ with $\sigma_i(\bar{1} - \eta) \geq r$, $r \in I_0$.

(6). $F^+(\eta)$ is τ_i-closed in X for every (r, s)-(i, j)-fuzzy regular closed set $\eta \in I^Y$.

(7). For each $x \in X$ and each net $\{S_n : n \in (D, >)\}$ in X converging to x and for any (r, s)-(i, j)-fuzzy regular open set $\mu \in I^Y$ with $F(x)\ q\ \mu$, the net is eventually in $F^-(\mu)$.

Proof. (1) $\Rightarrow$ (2): Let $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$. Let $x \in F^-(\mu)$ be arbitrary. Then there exists $U \in \tau_i(x)$ such that $U \subset F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s))$. Consequently $x \in U \subset i\,\mathrm{Int}(F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)))$. Thus $F^-(\mu) \subset i\,\mathrm{Int}(F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)))$.

(2) $\Rightarrow$(3): Let $\mu \in I^Y$ be (r, s)-(i, j)-fuzzy regular open set. Then $\mu = i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s))$ and therefore $\sigma_i(\mu) \geq r$. By (2), we have $F^-(\mu) \subset i\,\mathrm{Int}(F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s))) = i\,\mathrm{Int}(F^-(\mu))$. Hence $F^-(\mu) \in \tau_i$.

(3) $\Rightarrow$ (4): Let $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$. Then $i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s))$ is (r, s)-(i, j)-fuzzy regular open then by (3), $F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)) \in \tau_i$.

(4) $\Rightarrow$ (1): Let $x \in X$ be arbitrary and let $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$ and $x \in F^-(\mu)$. Then by (4), $F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)) = U \in \tau_i$ (say). Also since $\mu \leq i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)$ and $F(x)\ q\ \mu$ we have $F(x)\ q\ (i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s))$. Thus, $x \in F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)) = U$. Hence F is double fuzzy lower pairwise almost continuous.

(2) $\Rightarrow$ (5): Let $\eta \in I^Y$ with $\sigma_i(\bar{1} - \eta) \geq r$. Then by (2), $F^-(\bar{1} - \eta) \subset i\,\mathrm{Int}(F^-(i\,\mathrm{Int}(j\,\mathrm{Cl}(\bar{1} - \eta, r, s), r, s))) = i\,\mathrm{Int}(X - F^-((i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s))))$. Then we have $X - F^+(\eta) = F^-(\bar{1} - \eta) \subset j\,\mathrm{Int}(X - F^-((i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s)))) = j\,\mathrm{Int}(X - F^+(i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s))) = X - (i\,\mathrm{Cl}(F^+(i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s))))$. Thus, $F^+(\eta) \supset i\,\mathrm{Cl}(F^+(i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s)))$.

(5) $\Rightarrow$ (6): Let η be (r, s)-(i, j)-fuzzy regular closed set in Y. Then $\eta = i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s)$ and $\sigma_i(\bar{1} - \eta) \geq r$. By (5) we have $F^+(\eta) \supset i\,\mathrm{Cl}(F^+(i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta, r, s), r, s))) = i\,\mathrm{Cl}(F^+(\eta))$. Hence $F^+(\eta)$ is τ_i-closed set in X.

(6) $\Leftrightarrow$ (3): Let μ be an (r, s)-(i, j)-fuzzy regular open set in Y. Then $\bar{1} - \mu$ is (r, s)-fuzzy regular closed. By using (6) we have $F^+(\bar{1} - \mu) = X - F^-(\mu)$ is τ_i-closed set in X. Hence $F^-(\mu)$ is τ_i-open set in X. The converse is clear.

(3) $\Leftrightarrow$ (7): Suppose that there exists (r, s)-(i, j)-fuzzy regular open set μ of Y such that $F^-(\mu)$ is not τ_i-open in X. Then there exists $x \in F^-(\mu)$ such that for any τ_i-open set U containing x, $U \nsubseteq F^-(\mu)$. Let B_x denote the τ_i-open set containing $x \in X$. For each $U_x^\alpha \in B_x$ there exists $S_\alpha \in U_x^\alpha$ such that $S_\alpha \notin F^-(\mu)$. Then $B_x = D$ (say) is a directed set under set inclusion and the net $\{S_\alpha : S_\alpha \in U_x^\alpha \in B_x$ and $S_\alpha \notin F^-(\mu)\}$ obviously converges to x in X, but $S_\alpha \notin F^-(\mu)$ for all α, which contradicts with (7). The converse is obvious. ☐

Theorem 3.4. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_1^\star, \sigma_2, \sigma_2^\star)$ be a fuzzy multifunction. Then the following statements are equivalent:*

(1). F is double fuzzy upper almost continuous.

(2). $F^+(\mu) \subset i\,\mathrm{Int}(F^+(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)))$ for every $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$, $r \in I_0$.

(3). $F^+(\mu) \in \tau_i$ for every (r, s)-(i, j)-fuzzy regular open set $\mu \in I^Y$.

(4). $F^+(i\,\mathrm{Int}(j\,\mathrm{Cl}(\mu, r, s), r, s)) \in \tau_i$ for every $\mu \in I^Y$ with $\sigma_i(\mu) \geq r$, $r \in I_0$.

(5). $F^-(\eta) \supset i\,\mathrm{Cl}(F^-(i\,\mathrm{Cl}(j\,\mathrm{Int}(\eta,r,s),r,s))))$ for every $\eta \in I^Y$ with $\sigma_i(\bar{1}-\eta) \geq r$, $r \in I_0$.

(6). $F^-(\eta)$ is τ_i-closed in X for every (r,s)-(i,j)-fuzzy regular closed set $\eta \in I^Y$.

(7). For each $x \in X$ and each net $\{S_n : n \in (D,>)\}$ in X converging to x and for any (r,s)-(i,j)-fuzzy regular open set $\mu \in I^Y$ with $F(x) \leq \mu$, the net is eventually in $F^+(\mu)$.

Proof. It is similar to the prove of Theorem 3.3. □

Theorem 3.5. *A fuzzy multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_1^\star,\sigma_2,\sigma_2^\star)$ is double fuzzy lower (upper) almost pairwise continuous if and only if for any (r,s)-(i,j)-fuzzy semiopen set λ in Y, $i\,\mathrm{Cl}(F^+(\lambda)) \subset F^+(i\,\mathrm{Cl}(\mu,r,s))$ $(i\,\mathrm{Cl}(F^-(\lambda)) \subset F^-(i\,\mathrm{Cl}(\mu,r,s)))$.*

Proof. Let F be fuzzy lower almost pairwise continuous and let λ be (r,s)-(i,j)-fuzzy semiopen set in Y, $r \in I_0$. Then $\lambda \leq i\,\mathrm{Cl}(j\,\mathrm{Int}(\lambda,r,s),r,s) = \eta$ (say). So η is (r,s)-(i,j)-fuzzy regular closed in Y. By Theorem 3.3, we have $F^+(\eta)$ is τ_i-closed in X and hence $i\,\mathrm{Cl}(F^+(\lambda)) \subset i\,\mathrm{Cl}(F^+(\eta)) = F^+(\eta) = F^+(i\,\mathrm{Cl}(j\,\mathrm{Int}\,\lambda,r,s),r,s)) \subset F^+(i\,\mathrm{Cl}(\lambda,r,s))$. Conversely, since every (r,s)-(i,j)-fuzzy regular closed set is (r,s)-(i,j)-fuzzy semiopen, for any (r,s)-fuzzy regular closed set λ in Y we have $i\,\mathrm{Cl}(F^+(\lambda)) \subset F^+(i\,\mathrm{Cl}(\lambda,r,s)) = F^+(\lambda)$. Consequently, $F^+(\lambda)$ is τ_i-closed in X and hence by Theorem 3.3, F is double fuzzy lower almost continuous. The proof of fuzzy upper almost continuous is similar. □

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