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On Extension of D-Operator Operators

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Abstract: In this paper, we generalize the class of D-Operator by extending this study to n-D-Operator and investigate the basic properties of this class. We also show that this class is closed under strong operator topology.

Keywords: D-Operator, Class (Q), Almost Class (Q), (α, β)-Class (Q), Normal operators, n-Normal, n-D-Operator operators.
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1. Introduction

Throughout this paper, H is a separable complex Hilbert space, $B(H)$ is the Banach algebra of all bounded linear operators. n-normal if $T^*T^n = T^nT^*$, $T \in B(H)$ is normal if $T^*T = TT^*$, quasinormal if $T(T^*T) = (T^*T)T$. D-Operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$ [1], class (Q) if $T^{*2}T^2 = (T^*T)^2$ [5], n-power class (Q) if $T^{*2}(T^n)^2 = (T^*T^n)^2$ [6], n-D-Operator if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, for any positive integer n. We note that n-D-Operator is D-Operator when $n = 1$.

2. Main Results

Definition 2.1. *Let $T \in B(H)$ be Drazin invertible. Then an operator T is called n-D-Operator, denoted by, [nD], if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, for any positive integer n.*

Proposition 2.2. *Let $T \in [nD]$, then the following holds;*

(1). $\lambda T \in [nD]$ for every scalar λ.

(2). $S \in [nD]$ for every $S \in B(H)$ that is unitarily equivalent to T.

(3). The restriction T/M of T to any closed subspace M of H which reduces T is in $[nD]$.

(4). $(T^D)^n \in [nD]$.

Proof.

(1). The proof is trivial.

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(2). Since S is unitarily equivalent to T, there exists a unitary operator $U \in B(H)$ such that $S = UTU^*$. Hence;

$$\begin{aligned}
S^{*2n}(S^D)^{2n} &= (UT^*U^*)^2(U(T^D)^nU^*)^2 \\
&= (UT^*U^*)(UT^*U^*)(U(T^D)^nU^*)(U(T^D)^nU^*) \\
&= UT^*T^*(T^D)^n(T^D)^nU^* \\
&= UT^{*2}(T^D)^{2n}U^* \\
&= U(T^*(T^D)^n)^2U^* \\
&= UT^*(T^D)^nT^*(T^D)^nU^* \\
&= (UT^*U^*)(U(T^D)^nU^*)(UT^*U^*)(U(T^D)^nU^*) \\
&= S^*(S^D)^nS^*(S^D)^n \\
&= (S^*(S^D)^n)^2.
\end{aligned}$$

Thus $S \in [nD]$.

(3).
$$\begin{aligned}
(T/M)^{*2}((T/M)^D)^{2n} &= (T/M)^*(T/M)^*((T/M)^D)^n((T/M)^D)^n \\
&= (T^*/M)(T^*/M)((T^D)^n/M)((T^D)^n/M) \\
&= (T^*T^*/M)((T^D)^nT^D)^n/M) \\
&= (T^{*2}/M)((T^D)^{2n}/M) \\
&= (T^{*2}(T^D)^{2n})/M \\
&= (T^*(T^D)^nT^*(T^D)^n)/M \\
&= ((T^*(T^D)^n)/M)((T^*(T^D)^n)/M) \\
&= ((T^*/M)((T^D)^n/M)(T^*/M)((T^D)^n/M)) \\
&= ((T^*/M)((T^D)^n/M)^2 \\
&= ((T/M)^*((T/M)^D)^n)^2.
\end{aligned}$$

Hence $T/M \in [nD]$.

(4). Suppose $T \in [nD]$, then; $T^{*2n}(T^D)^{2n} = (T^*(T^D)^n)^2$, hence $T^*T^*(T^D)^n(T^D)^n = T^*(T^D)^nT^*(T^D)^n$, taking adjoint on both sides $((T^*)^D)^n((T^*)^D)^nTT = ((T^*)^D)^nT((T^*)^D)^nT$. Thus $(((T^D)^n)^*)^2T^2 = (((T^D)^n)^*)T)^2$. Hence $(T^D)^n \in [nD]$. □

Proposition 2.3. *The set of all n-D-Operators is a closed subset of $B(H)$ on H.*

Proof. Let $\langle T_q \rangle$ be a sequence of $[nD]$ operators with $T_q \longrightarrow T$. We have to show that $T \in [nD]$. Now $T_q \longrightarrow T$ implies $T_q^* \longrightarrow T^*$ and $(T_q^D)^n \longrightarrow (T^D)^n$. Thus $T_q^*(T_q^D)^n \longrightarrow T^*(T^D)^n$ gives

$$(T_q^*(T_q^D)^n)^2 \longrightarrow (T^*(T^D)^n)^2 \tag{1}$$

Similarly, $T_q^{*2} \longrightarrow T^{*2}$ and $(T_q^D)^{2n} \longrightarrow (T^D)^{2n}$, thus

$$T_q^{*2}(T_q^D)^{2n} \longrightarrow T^{*2}(T^D)^{2n} \tag{2}$$

Hence from (1) and (2), we have;

$$\begin{aligned}\| T^{*2}(T^D)^{2n} - (T^*(T^D)^n)^2 \| &= \| T^{*2}(T^D)^{2n} - T_q^{*2}(T_q^D)^{2n} + T_q^{*2}(T_q^D)^{2n} - (T^*(T^D)^n)^2 \| \\ &\leq \| T^{*2}(T^D)^{2n} - T_q^{*2}(T_q^D)^{2n} \| + \| T_q^{*2}(T_q^D)^{2n} - (T^*(T^D)^n)^2 \| \\ &= \| T^{*2}(T^D)^{2n} - T_q^{*2}(T_q^D)^{2n} \| + \| T_q^{*2}((T_q^D)^n)^2 - (T^*(T^D)^n)^2 \|\end{aligned}$$

$\longrightarrow 0$ as $q \longrightarrow \infty$ and thus, $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, hence $T \in [nD]$. □

Proposition 2.4. *Let $S, T \in [nD]$. If $[S, T] = [S, T^*] = 0$, then $TS \in [nD]$.*

Proof. $[S, T] = [S, T^*] = 0 \Rightarrow [S, T] = [S^D, T] = [S^*, T^D] = 0$ with $S, T \in [nD]$, we have $S^{*2}(S^D)^{2n} = (S^*(S^D)^n)^2$ and $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, hence

$$\begin{aligned}(TS)^{*2}((TS)^D)^{2n} &= (TS)^*(TS)^*(TS)^D(TS)^D \\ &= S^*T^*S^*T^*(T^D)^n(S^D)^n(T^D)^n(S^D)^n \\ &= S^*S^*(S^D)^n(S^D)^nT^*T^*(T^D)^n(T^D)^n \\ &= S^{*2}T^{*2}(S^D)^{2n}(T^D)^{2n} \\ &= S^*S^*T^*T^*(S^D)^n(T^D)^n(S^D)^n(T^D)^n \\ &= S^*T^*S^*T^*(S^D)^n(T^D)^n(S^D)^n(T^D)^n \\ &= (TS)^*(TS)^*((TS)^D)^n)^2.\end{aligned}$$

Hence $TS \in [nD]$. □

Proposition 2.5. *Let $S, T \in [nD]$. If $TS = ST = 0$, then $S + T \in [nD]$.*

Proof. $S, T \in [nD] \Rightarrow S^{*2}(S^D)^{2n} = (S^*(S^D)^n)^2$ and $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$. $TS = ST = 0 \Rightarrow T^*S^* = S^*T^*$, which further implies $((S+T)^D)^n = (S^D)^n + (T^D)^n$. Thus,

$$\begin{aligned}&= (S+T)^{*2}((S+T)^D)^{2n} \\ &= (S+T)^*(S+T)^*((S+T)^D)^n((S+T)^D)^n \\ &= (S^* + T^*)(S^* + T^*)(S^D + T^D)^n(S^D + T^D)^n \\ &= (S^{*2} + T^{*2})((S^D)^{2n} + (T^D)^{2n}) \\ &= S^{*2}(S^D)^{2n} + T^{*2}(T^D)^{2n} \\ &= (S^*(S^D)^n)^2 + (T^*(T^D)^n)^2 \\ &= (S^*(S^D)^n + T^*(T^D)^n)(S^*(S^D)^n + T^*(T^D)^n) \\ &= (S^* + T^*)((S^D)^n + (T^D)^n)(S^* + T^*)((S^D)^n + (T^D)^n) \\ &= ((S+T)^*((S+T)^D)^n)^2.\end{aligned}$$

Hence $S + T \in [nD]$. □

Theorem 2.6. *Let $T_{\alpha_1}, T_{\alpha_2}, \ldots, T_{\alpha_q} \in [nD]$, then it follows that;*

(1). $T_{\alpha_1} \oplus T_{\alpha_2} \oplus \cdots \oplus T_{\alpha_q} \in [nD]$.

(2). $T_{\alpha_1} \otimes T_{\alpha_2} \otimes \cdots \otimes T_{\alpha_q} \in [nD]$.

Proof.

(1). $T_{\alpha_j} \in [nD]$ for all $\alpha_j = 1, 2, \ldots, \alpha_q$ implies; $T^{*2}_{\alpha_j}(T^D_{\alpha_j})^{2n} = (T^*_{\alpha_j}(T^D_{\alpha_j})^n)^2$, thus

$$\begin{aligned}
(T_{\alpha_1} \oplus T_{\alpha_2} \oplus \cdots \oplus T_{\alpha_j})^{*2}((T_{\alpha_1} \oplus T_{\alpha_2} \oplus \cdots \oplus T_{\alpha_j})^D)^{2n} &= T^{*2}_{\alpha_1}(T^D_{\alpha_1})^{2n} \oplus T^{*2}_{\alpha_2}(T^D_{\alpha_2})^{2n} \oplus \cdots \oplus T^{*2}_{\alpha_j}(T^D_{\alpha_j})^{2n} \\
&= (T^*_{\alpha_1}(T^D_{\alpha_1})^n)^2 \oplus (T^*_{\alpha_2}(T^D_{\alpha_2})^n)^2 \oplus \cdots \oplus (T^*_{\alpha_j}(T^D_{\alpha_j})^n)^2 \\
&= T^*_{\alpha_1}(T^D_{\alpha_1})^n T^*_{\alpha_1}(T^D_{\alpha_1})^n \oplus T^*_{\alpha_2}(T^D_{\alpha_2})^n T^*_{\alpha_2}(T^D_{\alpha_2})^n \oplus \ldots \\
&\quad \oplus T^*_{\alpha_j}(T^D_{\alpha_j})^n T^*_{\alpha_j}(T^D_{\alpha_j})^n \\
&= T^*_{\alpha_1}(T^D_{\alpha_1})^n \oplus T^*_{\alpha_2}(T^D_{\alpha_2})^n \oplus \cdots \oplus T^*_{\alpha_j}(T^D_{\alpha_j})^n \\
&= ((T^*_{\alpha_1} \oplus T^*_{\alpha_2} \oplus \cdots \oplus T^*_{\alpha_j})((T^D_{\alpha_1})^n \oplus (T^D_{\alpha_2})^n \oplus \cdots \oplus (T^D_{\alpha_j})^n)) \\
&= ((T_{\alpha_1} \oplus T_{\alpha_2} \oplus \cdots \oplus T_{\alpha_j})^*((T_{\alpha_1} \oplus T_{\alpha_2} \oplus \cdots \oplus T_{\alpha_j})^D)^n)^2
\end{aligned}$$

(2). The proof for (2) follows similarly. □

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