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A Study on Equivalent Metrics and Equivalent Norms

Research Article

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Abstract: In this article we study equivalent metrics and then equivalent norms. Here we study the different conditions for equivalent metrics and equivalent norm and try to compare them. We try to check whether there is any similarity between the conditions for equivalent metrics and equivalent norms and we have come with some conclusion at last.

Keywords: Equivalent metric, Equivalent norm, Strong and weak norm.

1. Introduction

Metric is a notion of distance which is used in different sets. In geometry we generally find the distance between two different points. In case of analysis when we try to find the distance between the elements of non empty sets then the term metric is introduced. In preliminary part of this article I have given the definition of metric but in general we can say metric on a set is a formula which gives the distance between two points of the set provided it satisfies some conditions. We can define different metric on a set and all the metrics defined on the set are comparable. Similarly norm is a function which is used to find the size of a vector in vector space. In this article first we discuss the different results and their proofs for equivalent metric and equivalent norms. In the discussion part of the article we have given some observations comparing equivalent metrics and equivalent norms.

2. Preliminaries

In this section, we have given definition of some terms which are used in this article.

Definition 2.1. *Let X be a non empty set. Then a function $d : X \times X \to R$ is said to be a metric on X if it satisfied the following four conditions:*

(1). $0 \leq d(x, y) < \infty \ \forall x, y \in X$

(2). $d(x, y) = 0 \Leftrightarrow x = y \ \forall x, y \in X$

(3). $d(x, y) = d(y, x) \ \forall x, y \in X$ *(Symmetry)*

(4). $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$ *(Triangular inequality)*

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The ordered pair (X, d) is called a metric space. d is also called distance function. $d(x, y)$ denotes the distance between x and y for all $x, y \in X$.

Definition 2.2. *Let X be a non-empty set and d and $d^{'}$ be metrics on X. We say that $d^{'}$ is a weaker metric than d(or d is a stronger metric than $d^{'}$)if every open subset of X with respect to $d^{'}$ is also open with respect to d. Equivalently for two metrics d and $d^{'}$ on the same set X,we say $d^{'}$ is weaker metric than d(or d is a stronger metric than $d^{'}$) if $B_d(x;\delta) \subset B_{d'}(x;\delta)$, where $B_d(x;\delta)$=$\{y \in X \mid d(x,y) < \delta$ $\}$ is an open ball of radius δ centred at x with respect to metric d and $B_{d'}(x;\delta)$=$\{$ $y\in X \mid d^{'}(x,y) < \delta$ $\}$ is an open ball of radius δ centred at x with respect to metric $d^{'}$ on X.*

Definition 2.3. *Two metrics d and $d^{'}$ on a non empty set X are said to be comparable if d is either stronger or weaker than $d^{'}$.*

Definition 2.4. *Two metrics d_1 and d_2 on a set X are said to be equivalent if for every point $x_0 \in X$, every ball with center at x_0 defined with respect to d_1, $B_{d_1}(x_0;r_1) = \{x \in X : d_1(x_0,x) < r_1\}$ contains a ball with center x_0 with respect to d_2, $B_{d_2}(x_0;r_2) = \{x \in X : d_2(x_0,x) < r_2\}$ and vice-versa. In other words, two metrics d_1 and d_2 on a set X are said to be equivalent if d_1 is stronger than d_2 and d_2 is also stronger than d_1.*

Definition 2.5. *Let $V(F)$ be a vector space(F is either $\mathbb{R}$ or $\mathbb{C}$). A norm denoted by $\|.\|$ is a function from X to $\mathbb{R}$ which satisfies the following conditions:*

(1). $\|x\| \geq 0, \forall x \in X$

(2). $\|x\| = 0 \iff x = 0, \forall x \in X$

(3). $\|kx\| = |k|.\|x\|, \forall x \in X and k \in \mathbb{F}$

(4). $\|x + y\| \leq \|x\| + \|y\|, \forall x \in X.$

A vector space X together with norm function is called a normed space.

Definition 2.6. *Let two norms $\|.\|_1$ and $\|.\|_2$ defined on same vector space X are said to be comparable if either $\|x\|_1 \leq c_1\|x\|_2$ or $\|x\|_2 \leq c_2\|x\|_1$ $\forall x \in X$ and for some $c_1, c_2 \in \mathbb{R}^+$ is satisfied. If first one satisfied then $\|x\|_2$ is said to be stronger than $\|x\|_1$ and $\|x\|_1$ is weaker than $\|x\|_2$.*

Definition 2.7. *Two norms defined over a same linear space is said to be equivalent if one is weaker or stronger than the other and conversly.*

3. Results on Equivalent Metrics

Theorem 3.1. *If d and $d^{'}$ are two metrics on a non-empty set X. Then the following are equivalent:*

(1). d is stronger than $d^{'}$.

(2). For every $x \in X$, ε >0, $\exists$ δ >0 s.t. $B_d(x;\delta) \subset B_{d'}(x;\delta)$.

(3). If $x_n \to x$ in (X, d), then $x_n \to x$ in $(X, d^{'})$.

Theorem 3.2. *A metric d is stronger than $d^{'}$ if $\exists$ $\lambda >$ 0 such that $d^{'}(x, y) \leq \lambda d(x, y)$ $\forall$ $x, y \in X$.*

Proof. Let A is open in $(X, d^{'})$. To show A open in (X, d). Let, $x_0 \in A$. So, $\exists\, r > 0$ such that $B_{d'}(x_0;r) \subset A$, choose δ= $\frac{r}{2\lambda}$. Then for $y \in B_d(x_0;\delta)$ we have $d^{'}(x_0,y) \leq \lambda d(x_0,y) < \lambda\delta < \frac{r}{2} < r \Rightarrow y \in B_{d'}(x_0;r) \subset A \Rightarrow A$ is open in (X, d). □

But the converse of above result is not true, as seen from the example given below.

Example 3.3. *Let d_1 be discrete metric on $\mathbb{R}$ and d_2 be usual metric on $\mathbb{R}$. Suppose, $x_n \to x$ in $(X, d_1) \Rightarrow d(x_n, x) < \varepsilon \ \forall \ n \geq N \Rightarrow x_n = n \ \forall \ n \geq N \Rightarrow d_2(x_n, x) = \mid x_n - x \mid = 0 \ \forall n \geq N$. Therefore $x_n \to x$ in $(X, d_1) \Rightarrow x_n \to x$ in (X, d_2). Let if possible, $\exists \ \lambda >$ 0 such that $d_2(x, y) \leq \lambda d_1(x, y) \ \forall \ x, y \in \mathbb{R} \Rightarrow \mid x - y \mid \leq \lambda \ \forall \ x, y \in \mathbb{R}$, which is not true.*

Theorem 3.4. *Let, d_1 and d_2 be two metrics on a set X. If there exists two real numbers k_1 and $k_2 >$ 0 such that $k_1 d_1(x, y) \leq d_2(x, y) \leq k_2 d_1(x, y)$ for all x,y $\in X$, then the metrics d_1 and d_2 are equivalent.*

Proof. Let A is open in (X, d_1) and $x_0 \in A$. To show $\exists \ \delta > 0$ such that $B_{d_2}(x; \delta) \subset A$. As $x_0 \in A$ and A open in (X, d_1), so $\exists \ r > 0$ such that $B_{d_1}(x_0; r) \subset A$. We have, $k_1 d_1(x, y) \leq d_2(x, y) \ \forall \ x, y \in X$, $k_1 > 0$. Choose, $\delta = \frac{rk_1}{2}$. For $y \in B_{d_2}(x_0; \delta)$,

$$d_1(x_0, y) \leq \frac{1}{k_1} d_2(x_0, y) < \frac{\delta}{k_1} = \frac{r}{2} < r.$$

$\Rightarrow y \in B_{d_1}(x_0; r) \subset A$. This is true for every $y \in B_{d_2}(x_0; r)$. Therefore $B_{d_2}(x_0; \delta) \subset A$. Hence A is open in $(X, d_2) \Rightarrow d_2$ is stronger than d_1. Similarly, using $d_2(x, y) \leq k_2 d_1(x, y)$ we can show that d_1 is stronger than d_2. Hence d_1 and d_2 are equivalent metrics on X. $\square$

The above is sufficient condition for two metrics on a set to be equivalent. But not necessary as seen for discrete and usual metric on $\mathbb{R}$ in the last example.

Theorem 3.5. *Every metric d on X has uncountably many equivalent metrics.*

Proof. Consider, d is a metric on set X. For any positive real number ε, define $d_\varepsilon(x, y) = \frac{d(x, y)}{\varepsilon}$. Then,

(1). $0 \leq d_\varepsilon(x, y) < \infty \ \forall \ x, y \in X$.

(2). $d_\varepsilon(x, y) = 0 \Leftrightarrow \frac{d(x, y)}{\varepsilon} = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$.

(3). $d_\varepsilon(x, y) = \frac{d(x, y)}{\varepsilon} = \frac{d(y, x)}{\varepsilon} = d_\varepsilon(y, x)$.

(4). $d_\varepsilon(x, y) = \frac{d(x, y)}{\varepsilon} \leq \frac{d(x, z) + d(z, y)}{\varepsilon} = d_\varepsilon(x, z) + d_\varepsilon(z, y)$.

Hence all conditions for metric are satisfied by d_ε. So, d_ε is metric for each positive real number ε.
<u>*Claim*</u>: For all, $x_0 \in$ X we must show $B_d(x_0; r) = B_{d_\varepsilon}(x_0; \frac{r}{\varepsilon})$.
Let $x \in B_d(x_0; r) \Rightarrow d(x_0, x) < r \Rightarrow \frac{d(x_0, x)}{\varepsilon} < \frac{r}{\varepsilon}$, for positive real number $\varepsilon \Rightarrow x \in B_{d_\varepsilon}(x_0; \frac{r}{\varepsilon})$. Therefore $B_d(x_0; r) \subseteq B_{d_\varepsilon}(x_0; \frac{r}{\varepsilon})$. Again, $x \in B_{d_\varepsilon}(x_0; \frac{r}{\varepsilon}) \Rightarrow d_\varepsilon(x_0, x) < \frac{r}{\varepsilon} \Rightarrow d(x_0, x) < r \Rightarrow x \in B_d(x_0; r)$. Therefore $B_{d_\varepsilon}(x_0; \frac{r}{\varepsilon}) \subseteq B_d(x_0; r)$. Hence, d and d_ε are equivalent metrics. Since, number of real positive number ε is uncountably many so d_ε's are uncountably many. So a metric d has uncountably many equivalent metrics. $\square$

Theorem 3.6. *If X is a finite set, then all metrics on X are equivalent.*

Proof. Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set and d and d^* be two metrics on X. Let

$$K_1 = \max\{d(x_i, x_j) : 1 \leq i, j \leq n, i \neq j\}, \qquad k_1 = \min\{d(x_i, x_j) : 1 \leq i, j \leq n, i \neq j\},$$

$$K_2 = \max\{d^*(x_i, x_j) : 1 \leq i, j \leq n, i \neq j\}, \qquad k_2 = \min\{d^*(x_i, x_j) : 1 \leq i, j \leq n, i \neq j\}$$

Therefore K_1, K_2, k_1 and $k_2 > 0$. $d(x_i, x_j) \leq K_1$ and $d^*(x_i, x_j) \geq k_2 \Rightarrow \frac{d(x_i, x_j)}{d^*(x_i, x_j)} \leq \frac{K_1}{k_2} \Rightarrow d(x_i, x_j) \leq \lambda d^*(x_i, x_j) \ \forall \ i \neq j$ and $\lambda = \frac{K_1}{k_2}$. If $i = j$, then $d(x_i, x_j) = 0 = \lambda d^*(x_i, x_j) \Rightarrow d$ is weaker than d^*. Similarly, d^* is weaker than d. Therefore d and d^* are equivalent. $\square$

4. Results on Equivalent Norms

Theorem 4.1. *Let* $\|.\|_1$ *and* $\|.\|_2$ *be two norms on a vector space* X*. Then* $\|.\|_1$ *and* $\|.\|_2$ *are equivalent iff* $\exists\ c_1$ *and* $c_2 > 0$ *such that* $c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1$ *for all* $x \in X$*.*

Lemma 4.2. *Let* $\{x_1, x_2, ..., x_n\}$ *be a linearly independent set of vectors in a normed space* X *(of any dimension). Then there is a number* $c > 0$ *such that for every choice of scalars* $a_1, a_2, ..., a_n$ *we have* $\|a_1x_1 + a_2x_2 + ... + a_nx_n\| \geq c(|a_1| + |a_2| + ... + |a_n|)$ *for* $c > 0$*.*

Theorem 4.3. *On a finite dimensional vector space* X*, any norm* $\|.\|_1$ *is equivalent to any other norm* $\|.\|_2$ *defined on this.*

Proof. Let dimX=n and $\{e_1, e_2, ..., e_n\}$ any basis for X. Then every $x \in X$ has a unique represantation $x{=}\alpha_1e_1 + \alpha_2e_2 + ... + \alpha_ne_n$. where α 's are scalars. Now by Lemma 4.1 there is a positive constant c such that $\|x\|_2 \geq c(|a_1| + |a_2| + ... + |a_n|)$ for $c > 0$. By triangular inequality $\|x\|_1 = \|\alpha_1e_1 + \alpha_2e_2 + ... + \alpha_ne_n\|_1 \leq \sum_{j=1}^{n}|\alpha_j|\|e_j\|_1 \leq k\sum_{j=1}^{n} n|a_j| = \frac{k}{c}.c\sum_{j=1}^{n}|a_j| \leq \frac{k}{c}\|x\|_2$. Therefore $\|x\|_1 \leq \frac{k}{c}\|x\|_2$ implies $\frac{c}{k}\|x\|_1 \leq \|x\|_2$. After interchanging the rules of $\|.\|_1$ and $\|.\|_2$ in the preceding argument we have a $c^{'} > 0$ and some $k > 0$ such that $\|x\|_2 \leq \frac{k}{c^{'}}\|x\|_1$. Therefore we have $\frac{c}{k}\|x\|_1 \leq \|x\|_2 \leq \frac{k}{c^{'}}\|x\|_1$ implies $\|x\|_1 \leq \frac{k}{c}\|x\|_2 \leq \frac{k^2}{c.c^{'}}\|x\|_1$. Thus using Theorem 4.1 the norms $\|x\|_1$ and $\|x\|_2$ are equivalent. □

5. Discussion

After studying the various results for equivalent metric and equivalent norms, we observe the following.

On the set of real numbers $\mathbb{R}$ any two metrics may not be equivalent. For example usual metric and discrete metric are not equivalent. But on $\mathbb{R}$ any two norms are always equivalent as $\mathbb{R}$ is finite dimensional. The property of completeness of a metric may not be shared by an equivalent metric. For example: d and $d^{'}$ are defined on (0,1] as $d(x, y) = |x - y|$ and $d^{'}(x, y) = |\frac{1}{x} - \frac{1}{y}|$. Here d and $d^{'}$ are equivalent. But (0,1] is complete with respect to $d^{'}$ but not with respect to d.But in case if $\|x\|_1$ and $\|x\|_2$ are equivalent then $(X, \|x\|_1)$ complete implies $(X, \|x\|_2)$ complete and vice-versa. Thus we can say the property of completeness need not to be shared by two equivalent metrics. However, for two equivalent norms completeness of one implies the completeness of other.Therefore the properties which holds in case of equivalent metrics may not hold in equivalent norms and vice-versa. For metrics d_1 and d_2 on X; they are equivalent if there exists k_1 and $k_2 > 0$ such that

$$k_1d_1(x, y) \leq d_2(x, y) \leq k_2d_1(x, y)$$

for all x,y ∈ X. This condition is sufficient but not necessary. But however for two norms $\|x\|_1$ and $\|x\|_2$ on a linear space, the two norms are equivalent iff $\exists\ c_1$ and $c_2 > 0$ such that $c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1$ for all $x \in X$. Although these two conditions for equivalent metrics and equivalent norms look alike they are different as in case of norms the condition is necessary as well as sufficient.

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