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On K^* Quasi-n-Class (Q) Operators

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Abstract: In this paper, we introduce a new type of operator called K^* Quasi-n-Class (Q). We study some basic properties of this class and its relation to other classes.

Keywords: K Quasi-n-normal, K^* Quasi-n-normal, K Quasi-n-Class (Q), K^* Quasi-n-Class (Q), n power class (Q), Class (Q) operators.

1. Introduction

Throughout this paper, H is a seperable complex Hilbert space, B(H) is the Banach algebra of all bounded linear operators. A bounded linear operator $T \in B(H)$ is said to be class (Q) if $T^{*2}T^2 = (T^*T)^2$. This concept was introduced and studied by Jibril in 2010 [1]. In 2018, Manikandan and Veluchamy extended this to the class of $(N+K)$-power class (Q) [4], while Paramesh and Nirmala introduced and studied the class on n-power class (Q) in 2019 [5]. Revathi and Maheswari studied M-quasi class (Q) [6].

Definition 1.1. *An operator $T \in B(H)$ is said to be:*

(1). Quasi class (Q) if $T(T^{*2}T^2) = (T^*T)^2)T$.

(2). K Quasi Class (Q) if $T^k(T^{*2}T^2) = (T^*T)^2)T^k$ for some positive integer k.

(3). M quasi-class (Q) if $T(T^{*2}T^2) = M(T^*T)^2)T$.

(4). K quasi-n-class (Q) if $T^k(T^{*2}T^{2n}) = (T^*T^n)^2T^k$ for some positive integers n and k.

(5). K^* Quasi -n - Class (Q) $(T^*)^k(T^{*2}T^{2n}) = (T^*T^n)^2)(T^*)^k$ for some positive integers n and k.

Remark 1.2. *We remark that, if T is self adjoint and $n = k = 1$, then T is quasi class (Q). If T is self adjoint and $n = 1$, then T is K quasi class (Q).*

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2. Main Results

Theorem 2.1. *Let $T \in B(H)$ be K^* Quasi-n-Class (Q), then;*

(1). T^P is K^* Quasi-n-Class (Q) for $P \geq 1$.

(2). λT is K^* Quasi-n-Class (Q) for each λ coming from the set of real.

(3). T^{-1} is K^* Quasi-n-Class (Q) provided it exists.

Proof.

(1). Suppose T is K^* Quasi-n-Class (Q), we then need to prove that T^p is equally K^* Quasi-n-Class (Q) by mathematical induction. Now suppose that T is K^* Quasi-n-Class (Q), then the result holds for $M = 1$.

$$(T^*)^k(T^{*2}T^{2n}) = (T^*T^n)^2)(T^*)^k \tag{1}$$

Suppose the result holds for $P = M$.

$$((T^*)^k(T^{*2}T^{2n})(^M = ((T^*T^n)^2)(T^*)^k)^M. \tag{2}$$

Then we prove that it holds for $P = M + 1$, hence we have;

$$\begin{aligned}((T^*)^k(T^{*2}T^{2n})(^{M+1} &= ((T^*T^n)^2)(T^*)^k)^{M+1}\\ &= ((T^*)^k(T^{*2}T^{2n})(^{M+1}\\ &= ((T^*T^n)^2)(T^*)^k)^M(T^*)^k)(T^*T^n)^2)\\ &= ((T^*T^n)^2)(T^*)^k)^M(T^*T^n)^2)(T^*)^k)\\ &= ((T^*T^n)^2)(T^*)^k)^{M+1}.\end{aligned}$$

hence the result holds for $P = M + 1$ and therefore T^P is K^* Quasi-n-Class (Q) for each $P \geq 1$.

(2). $((\lambda T)^*)^k(\lambda T)^{*2}(\lambda T)^{2n} = (\lambda T^*)^k(\lambda^2T^{*2})(\lambda^{2n}T^{2n})$ Implies $((\lambda T)^*)^k(\lambda T)^{*2}(\lambda T)^{2n} = ((\lambda T)^*(\lambda T)^n)^2((\lambda T)^*)^k$. Hence

$$\begin{aligned}&= \lambda^k(T^*)^k(\lambda^2T^{*2}))(\lambda^{2n}T^{2n})\\ &= \lambda^k\lambda^2\lambda^{2n}(T^*)^k(T^{*2})(T^{2n})\\ &= \lambda^k\lambda^2\lambda^{2n}(T^*T^n)^2(T^*)^k\\ &= \lambda^2T^{*2}\lambda^{2n}\lambda^k(T^*)^k\\ &= (\lambda T)^{*2}(\lambda T)^{2n}((\lambda T)^*)^k.\\ &= ((\lambda T)^*(\lambda T)^n)^2((\lambda T)^*)^k.\end{aligned}$$

λT is K^* Quasi-n-Class (Q).

$$\begin{aligned}
(3).\ ((T^{-1})^*)^k(T^{-1})^{*2}(T^{-1})^{2n} &= ((T^*)^{-1})^k(T^{*2})^{-1}(T^{2n})^{-1} \\
&= ((T^*)^k)^{-1}(T^{*2}T^{2n})^{-1} \\
&= ((T^*)^k(T^{*2}T^{2n}))^{-1} \\
&= (T^{*2}T^{2n}(T^*)^k)^{-1} \\
&= (T^{*2}T^{2n})^{-1}((T^*)^k)^{-1} \\
&= (T^{*2})^{-1}(T^{2n})^{-1}((T^*)^{-1})^k \\
&= (T^{-1})^{*2}(T^{-1})^{2n}((T^*)^{-1})^k \\
&= ((T^{-1})^*(T^{-1})^n)^2((T^*)^{-1})^k
\end{aligned}$$

Implies $((T^{-1})^*)^k(T^{-1})^{*2}(T^{-1})^{2n} = ((T^{-1})^*(T^{-1})^n)^2((T^*)^{-1})^k$. Thus T^{-1} is K^* Quasi-n-Class (Q). □

Remark 2.2. *Let T_α, $T_\beta \in B(H)$ be two K^* Quasi-n-Class (Q), then $T_\alpha + T_\beta$ is not necessarily K^* Quasi-n-Class (Q).*

Example 2.3. *Let* $T_\alpha = \begin{pmatrix} 2i & 4 \\ 0 & -2i \end{pmatrix}$ *and* $T_\beta = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow T_\alpha + T_\beta = \begin{pmatrix} 2i+4 & 4 \\ 0 & -2i+2 \end{pmatrix}$. *Suppose $n = k = 1$, then;*

$$\begin{aligned}
((T_\alpha+T_\beta)^*)^1((T_\alpha+T_\beta)^{*2}(T_\alpha+T_\beta)^2) &= \begin{pmatrix} 4+2i & 0 \\ 4 & 2-2i \end{pmatrix}\left\{\begin{pmatrix} 12+16i & 0 \\ 24 & -8i \end{pmatrix}\begin{pmatrix} 12+16i & 24 \\ 0 & -8i \end{pmatrix}\right\} \\
&= \begin{pmatrix} 4+2i & 0 \\ 4 & 2-2i \end{pmatrix}\begin{pmatrix} 384i & 288+64i \\ 288+384i & 512 \end{pmatrix} \\
&= \begin{pmatrix} 1536i-768 & 1024+832i \\ 1344+1728i & 2176-768i \end{pmatrix} \qquad (3)
\end{aligned}$$

Similarly

$$\begin{aligned}
((T_\alpha+T_\beta)^*)^1((T_\alpha+T_\beta)^*(T_\alpha+T_\beta)^1)^2 &= \begin{pmatrix} 2i+4 & 0 \\ 4 & -2i+2 \end{pmatrix}\left\{\begin{pmatrix} 2i+4 & 0 \\ 4 & -2i+2 \end{pmatrix}\begin{pmatrix} 2i+4 & 4 \\ 0 & -2i+2 \end{pmatrix}\right\}^2 \\
&= \begin{pmatrix} 2i+4 & 0 \\ 4 & -2i+2 \end{pmatrix}\begin{pmatrix} 80+640i & 275i+384 \\ 352i+384 & 384 \end{pmatrix} \\
&= \begin{pmatrix} 2720i-960 & 986+1868i \\ 1792+2496i & 332i+1704 \end{pmatrix} \qquad (4)
\end{aligned}$$

Hence $(1) \neq (2)$, *thus $T_\alpha + T_\beta$ is not K^* Quasi-n-Class (Q).*

Theorem 2.4. *Let T_α, $T_\beta \in B(H)$ be two K^* Quasi-n-Class (Q) operators with $T_\alpha T_\beta = T_\alpha^* T_\beta^* = T_\alpha^* T_\beta = 0$, then $T_\alpha + T_\beta$ is K^* Quasi-n-Class (Q).*

Proof.

$$\begin{aligned}
((T_\alpha+T_\beta)^*)^k((T_\alpha+T_\beta)^{*2}(T_\alpha+T_\beta)^{2n}) &= (T_\alpha^*+T_\beta^*)^k(T_\alpha^{*2}+T_\beta^{*2})((T_\alpha^{2n}+T_\beta^{2n})) \\
&= ((T_\alpha^*)^k + k(T_\alpha^*)^{k-1}T_\beta^* + \cdots + (T_\beta^*)^k)((T_\alpha^{*2}+T_\beta^{*2})(T_\alpha^{2n} + nT_\alpha^{2n-2}T_\beta^2 + \cdots + T_\beta^{2n})) \\
&= ((T_\alpha^*)^k + (T_\beta^*)^k)((T_\alpha^{*2}+T_\beta^{*2})(T_\alpha^{2n}+T_\beta^{2n}))
\end{aligned}$$

$$= ((T_\alpha^*)^k + (T_\beta^*)^k)(T_\alpha^{*2}T_\alpha^{2n} + T_\alpha^{*2}T_\beta^{2n} + T_\beta^{*2}T_\alpha^{2n} + T_\beta^{*2}T_\beta^{2n})$$
$$= ((T_\alpha^*)^k + (T_\beta^*)^k)((T_\alpha^{*2}T_\alpha^{2n} + T_\beta^{*2}T_\beta^{2n})$$
$$= (((T_\alpha^*)^k T_\alpha^{*2}T_\alpha^{2n} + (T_\alpha^*)^k T_\beta^{*2}T_\beta^{2n} + (T_\beta^*)^k T_\alpha^{*2}T_\alpha^{2n} + (T_\beta^*)^k T_\beta^{*2}T_\beta^{2n})$$
$$= ((T_\alpha^*)^k T_\alpha^{*2}T_\alpha^{2n} + (T_\beta^*)^k T_\beta^{*2}T_\beta^{2n})$$

T_α^* and T_β^* being K^* Quasi-n- Class (Q), we have;

$$((T_\alpha^* T_\alpha^n)^2 (T_\alpha^*)^k) + ((T_\beta^* T_\beta^n)^2 (T_\beta^*)^k)$$

Thus $T_\alpha + T_\beta$ is K^* Quasi-n-Class (Q). □

Theorem 2.5. *Let T_α, $T_\beta \in B(H)$ be two K^* Quasi-n-Class (Q) operators, then $T_\alpha T_\beta$ is K^* Quasi-n-Class (Q) operator for $T_\alpha T_\beta^* = T_\beta^* T_\alpha$ and $T_\alpha T_\beta = T_\beta T_\alpha$.*

Proof.

$$\begin{aligned}
((T_\alpha T_\beta)^*)^k((T_\alpha T_\beta)^{*2}(T_\alpha T_\beta)^{2n}) &= ((T_\alpha T_\beta)^*)^k((T_\beta T_\alpha)^{*2}(T_\alpha T_\beta)^{2n}) \\
&= (T_\alpha^* T_\beta^*)^k((T_\alpha^{*2}T_\beta^{*2})(T_\alpha^{2n}T_\beta^{2n})) \\
&= (T_\alpha^*)^k((T_\beta^*)^k T_\alpha^{*2})(T_\beta^{*2}T_\alpha^{2n})T_\beta^{2n} \\
&= (T_\alpha^*)^k(T_\alpha^{*2}(T_\beta^*)^k)(T_\alpha^{2n}T_\beta^{*2})T_\beta^{2n} \\
&= ((T_\alpha^*)^k T_\alpha^{*2})((T_\beta^*)^k T_\alpha^{2n})(T_\beta^{*2})T_\beta^{2n}) \\
&= ((T_\alpha^*)^k T_\alpha^{*2}T_\alpha^{2n})((T_\beta^*)^k T_\beta^{*2}T_\beta^{2n}) \\
&= (T_\alpha^{*2}T_\alpha^{2n}(T_\alpha^*)^k)(T_\beta^{*2}T_\beta^{2n}(T_\beta^*)^k) \\
&= (T_\alpha^{*2}T_\alpha^{2n})((T_\alpha^*)^k T_\beta^{*2})(T_\beta^{2n}(T_\beta^*)^k) \\
&= T_\alpha^{*2}(T_\alpha^{2n}T_\beta^{*2})((T_\alpha^*)^k T_\beta^{2n})(T_\beta^*)^k) \\
&= (T_\alpha^{*2}T_\beta^{*2})((T_\alpha^{2n}T_\beta^{2n})(((T_\alpha^*)^k(T_\beta^*)^k) \\
&= (T_\beta T_\alpha)^{*2}(T_\alpha T_\beta)^{2n}((T_\alpha^*)(T_\beta^*))^k \\
&= (T_\alpha T_\beta)^{*2}(T_\alpha T_\beta)^{2n}((T_\alpha T_\beta)^*)^k \\
&= ((T_\alpha T_\beta)^*(T_\alpha T_\beta)^n)^2((T_\alpha T_\beta)^*)^k.
\end{aligned}$$

Hence

$$((T_\alpha T_\beta)^*)^k((T_\alpha T_\beta)^{*2}(T_\alpha T_\beta)^{2n}) = ((T_\alpha T_\beta)^*(T_\alpha T_\beta)^n)^2((T_\alpha T_\beta)^*)^k.$$

Thus $T_\alpha T_\beta$ is K^* Quasi-n-Class (Q). □

Theorem 2.6. *Let $T \in B(H)$ be K^* Quasi-n-Class (Q), then T^* is K^* Quasi-n-Class (Q) provided $T^{*2}T^{2n} = T^{2n}T^{*2}$.*

Proof.

$$\begin{aligned}
((T^*)^*)^k)((T^*)^{*2}(T^*)^{2n}) &= ((T^*)^k)^*((T^{*2})(T^{2n})^*) \\
&= ((T^*)^k)^*(T^{2n}T^{*2})^* \\
&= ((T^{2n}T^{*2})(T^*)^k)^*
\end{aligned}$$

$$= ((T^*)^k(T^{2n}T^{*2}))^*$$
$$= ((T^*)^{*2}(T^*)^{2n})((T^*)^k)^*$$
$$= ((T^*)^*(T^*)^n)^2((T^*)^*)^K$$

Implies

$$((T^*)^*)^k)((T^*)^{*2}(T^*)^{2n}) = ((T^*)^*(T^*)^n)^2((T^*)^*)^K$$

Thus T^* is K^* Quasi-n-Class (Q). $\square$

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