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A Note On Composition Operators Between Orlicz Spaces

Research Article

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Abstract: In this paper, we characterize the boundedness of composition operators between any two Orlicz spaces.

Keywords: Boundedness, composition operators, Orlicz spaces.

1. Introduction

Let $X = (X, \Sigma, \mu)$ be a σ- finite complete measure space. Any nonsingular measurable transformation τ induces a composition operator C_τ from $L^0(X)$ to itself defined by

$$C_\tau f(x) = f(\tau(x)), \quad x \in X, \quad f \in L^0(X),$$

where $L^0(X)$ denotes the linear space of all equivalence classes of all real valued Σ-measurable functions on X, where we identify any two functions that are equal μ-almost everywhere on X. A nondecreasing continuous convex function $\phi : [0,\ \infty) \to [0,\ \infty)$ for which $\phi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$ is said to be an *Orlicz function*. For any $f \in L^0(X)$, we define the *modular*

$$I_\phi(f) = \int_X \phi(|f(x)|)d\mu(x)$$

and the *Orlicz space*

$$L^\phi(\mu) = \left\{f \in L^0(X) \mid I_\phi(\lambda f) < \infty \ \text{ for some } \ \lambda = \lambda(f) > 0\right\}.$$

This space is a Banach space with two norms: the *Luxemburg – Nakano norm*

$$\|f\|_\phi = \inf\{\lambda > 0 \mid I_\phi(f/\lambda) \le 1\}$$

and the *Orlicz norm* in the Amemiya form

$$\|f\|_\phi^0 = \inf_{k>0}(1 + I_\phi(kf))/k.$$

Composition operators on Orlicz spaces have been studied in [3–5, 14, 15] and references therein. In this paper, we study comopositions operators between any two Orlicz spaces. The techniques used in this paper essentially depend on the conditions of embedding of one Orlicz space into another. See, [12, Page 45] and [18] for details.

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2. Boundedness of Composition Operators

In this section, we characterize the boundedness of composition operators between any two Orlicz spaces. Boundedness of composition operators on Orlicz spaces have been discussed in [3–5, 14].

Theorem 2.1. *Let (X,Σ,μ) be a σ-finite nonatomic measure space and $\tau : X \to X$ be a nonsingular measurable transformation such that $\tau(X) = X$. Let $g_\tau = \frac{d\mu\circ\tau^{-1}}{d\mu}$ be the Raydon-Nikodym derivative of $\mu\circ\tau^{-1}$ with respect to μ . Then the composition operator $C_\tau : L^{\phi_1}(X) \longrightarrow L^{\phi_2}(X)$ is bounded if and only if there exist $a, b > 0$ and $0 \le h \in L^1(X)$ such that $\phi_2(au)g_\tau(x) \le b\phi_1(u) + h(x)$ for almost all $x \in X$ and for all $u \ge 0$.*

Proof. Suppose that the given condition holds. Let $0 \ne f \in L^{\phi_1}(X)$, then $I_{\phi_1}\left(\frac{f}{\|f\|_{\phi_1}}\right) \le 1$. Let $M \ge 1$ be a real number satisfying $M(b + \|h\|_1) \ge 1$, where $\|h\|_1 = \int_X h(x)d\mu(x)$. Then

$$\begin{aligned}
I_{\phi_2}\left(\frac{C_\tau f}{(M(b+\|h\|_1)\|f\|_{\phi_1})/a}\right) &= \int_X \phi_2\left(\frac{a|C_\tau f(x)|}{M(b+\|h\|_1)\|f\|_{\phi_1}}\right) d\mu(x)\\
&\le \frac{1}{M(b+\|h\|_1)}\int_X \phi_2\left(\frac{a|f(\tau(x))|}{\|f\|_{\phi_1}}\right) d\mu(x)\\
&= \frac{1}{M(b+\|h\|_1)}\int_{\tau(X)} \phi_2\left(\frac{a|f(y)|}{\|f\|_{\phi_1}}\right) d(\mu\circ\tau^{-1})(y)\\
&= \frac{1}{M(b+\|h\|_1)}\int_X \phi_2\left(\frac{a|f(y)|}{\|f\|_{\phi_1}}\right) d(\mu\circ\tau^{-1})(y)\\
&= \frac{1}{M(b+\|h\|_1)}\int_X \phi_2\left(\frac{a|f(y)|}{\|f\|_{\phi_1}}\right) g_\tau(y)d\mu(y)\\
&\le \frac{1}{M(b+\|h\|_1)}\int_X \left(b\ \phi_1\left(\frac{|f(y)|}{\|f\|_{\phi_1}}\right) + h(y)\right) d\mu(y)\\
&\le 1.
\end{aligned}$$

Thus $\|C_\tau f\|_{\phi_2} \le \frac{M}{a}(b+\|h\|_1)\|f\|_{\phi_1}$. This shows that C_τ is bounded. This proves the converse part.

We now prove the direct part. Write $X = \bigcup_{i=1}^{\infty} X_i$, where $\{X_i\}_{i=1}^{\infty}$ is a pairwise disjoint sequence of measurable subsets of X with $\mu(X_i) < \infty$ for every $i = 1, 2, \ldots$. For every $r \in \mathbb{Q}^+$, define

$$f_{r,i}(x) = \begin{cases} r & \text{if } x \in X_i \\ 0 & \text{otherwise.} \end{cases}$$

Consider the function

$$h_n(x) = \sup_{u\ge 0}\left(\phi_2(2^{-n}u)g_\tau(x) - 2^n\phi_1(u)\right).$$

We first show that

$$h_n(x) = \sup_{\substack{r\in\mathbb{Q}^+\\ i\in\mathbb{N}}}\left(\phi_2(2^{-n}f_{r,i}(x))g_\tau(x) - 2^n\phi_1(f_{r,i}(x))\right).$$

Choose any $x \in X$ such that $g_\tau(x) \ne 0$ and let $x \in X_i$. Take a sequence $\{u_k\}$ of nonnegative numbers such that

$$\phi_2(2^{-n}u_k)g_\tau(x) - 2^n\phi_1(u_k) \ge h_n(x) - 2^{-k}.$$

Since ϕ_1 and ϕ_2 are continuous, there exists $r_k \in \mathbb{Q}^+$ satisfying $\phi_2(2^{-n}r_k)g_\tau(x) \ge \phi_2(2^{-n}u_k)g_\tau(x) - 2^{-k}$ and $\phi_1(r_k) \le \phi_1(u_k) + 2^{-n-k}$. Then

$$\begin{aligned}
\phi_2(2^{-n}f_{r_k,i}(x))g_\tau(x) - 2^n\phi_1(f_{r_k,i}(x)) &= \phi_2(2^{-n}r_k)g_\tau(x) - 2^n\phi_1(r_k)\\
&\ge \phi_2(2^{-n}u_k)g_\tau(x) - 2^{-k} - 2^n\phi_1(u_k) - 2^{-k}\\
&\ge h_n(x) - 3\cdot 2^{-k}.
\end{aligned}$$

Therefore,

$$\begin{aligned} h_n(x) &\leq \phi_2(2^{-n} f_{r_k,i}(x)) g_\tau(x) - 2^n \phi_1(f_{r_k,i}(x)) + 3 \cdot 2^{-k} \\ &\leq \sup_{\substack{r \in \mathbb{Q}^+ \\ i \in \mathbb{N}}} \left(\phi_2(2^{-n} f_{r,i}(x)) g_\tau(x) - 2^n \phi_1(f_{r,i}(x)) \right) + 3 \cdot 2^{-k}. \end{aligned}$$

Taking $k \to \infty$, we obtain the desired equality. We can rewrite it in the form

$$h_n(x) = \sup_{k \in \mathbb{N}} \left(\phi_2(2^{-n} f_k(x)) g_\tau(x) - 2^n \phi_1(f_k(x)) \right) \tag{1}$$

where (f_k) is any rearrangement of $(f_{r,i})$ with $f_1 = f_{0,i}$. From (1), it is clear that h_n are measurable and $h_n(x) \geq 0$ for each $x \in X$. To complete the proof, we need only to show that $\int_X h_n(x) d\mu(x) < \infty$ for some n. Suppose this is not true. Denote

$$b_{m,n}(x) = \max_{1 \leq k \leq m} \left(\phi_2(2^{-n} f_k(x)) g_\tau(x) - 2^n \phi_1(f_k(x)) \right).$$

Then $b_{m,n}$ are measurable, $b_{m,n}(x) \geq 0$ and $b_{m,n}(x)$ is a nondecreasing sequence tending to $h_n(x)$ as $m \to \infty$ for every $x \in X$. Thus for any n, there exists m_n such that $\int_X b_{m_n,n}(x) d\mu(x) \geq 2^n$. Writing $b_n = b_{m_n,n}$, we have $\int_X b_n(x) d\mu(x) \geq 2^n$ for $n = 1, 2, \ldots$. Let

$$B_{n,k} = \left\{ x \in X \mid \phi_2(2^{-n} f_k(x)) g_\tau(x) - 2^n \phi_1(f_k(x)) = b_n(x) \right\}$$

and

$$B_n = X \setminus (B_{n,1} \cup B_{n,2} \cup \ \ldots \cup B_{n,m_n}).$$

Then $\mu(B_n) = 0$.

Let

$$\tilde{f}_n(x) = \begin{cases} 0 & \text{if } x \in B_{n,1} \cup B_n \\ f_k(x) & \text{if } x \in B_{n,k} \setminus \bigcup_{j=1}^{k-1} B_{n,j}, \ k = 2, 3, \ \ldots, m_n. \end{cases}$$

Then

$$\begin{aligned} b_n(x) &= \phi_2(2^{-n} \tilde{f}_n(x)) g_\tau(x) - 2^n \phi_1(\tilde{f}_n(x)) \\ &\geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_X \phi_2(2^{-n} \tilde{f}_n(x)) g_\tau(x) d\mu(x) &= 2^n \int_X \phi_1(\tilde{f}_n(x)) d\mu(x) + \int_X b_n(x) d\mu(x) \\ &\geq \int_X b_n(x) d\mu(x) \\ &\geq 2^n. \end{aligned}$$

Thus by Lemma 8.3 [12], with $a_n(x) = \phi_2(2^{-n} \tilde{f}_n(x)) g_\tau(x)$ and $\alpha_n = 1$, we obtain an increasing sequence $\{n_k\}$ and a sequence $\{A_k\}$ of pairwise disjoint measurable sets such that

$$\int_{A_k} \phi_2(2^{-n_k} \tilde{f}_{n_k}(x)) g_\tau(x) d\mu(x) = 1, \quad k = 1, 2, \ \ldots$$

Put

$$f(x)=\begin{cases}\tilde{f}_{n_k}(x) & \text{if } x\in A_k\\ 0 & \text{otherwise.}\end{cases}$$

Then for any $\lambda>0$, we obtain

$$\begin{aligned}\int_X \phi_2\left(\lambda C_\tau f(x)\right)d\mu(x) &= \int_X \phi_2\left(\lambda f(\tau(x))\right)d\mu(x)\\ &= \int_{\tau(X)} \phi_2\left(\lambda f(y)\right)d(\mu\circ\tau^{-1})(y)\\ &= \int_X \phi_2(\lambda f(y))g_\tau(y)d\mu(y)\\ &= \sum_{k=1}^{\infty}\int_{A_k}\phi_2(\lambda\tilde{f}_{n_k}(y))g_\tau(y)d\mu(y)\\ &\geq \sum_{k=p}^{\infty}\int_{A_k}\phi_2(2^{-n_k}\tilde{f}_{n_k}(y))g_\tau(y)d\mu(y)\\ &= \infty,\end{aligned}$$

where p is so large that $2^{-n_p}\leq\lambda$, and

$$\begin{aligned}\int_X \phi_1(f(x))d\mu(x) &= \sum_{k=1}^{\infty}2^{-n_k}\left(\int_{A_k}\phi_2(2^{-n_k}\tilde{f}_{n_k}(x))g_\tau(x)d\mu(x)-\int_{A_k}b_{n_k}(x)d\mu(x)\right)\\ &\leq \sum_{k=1}^{\infty}2^{-n_k}\int_{A_k}\phi_2(2^{-n_k}\tilde{f}_{n_k}(x))g_\tau(x)d\mu(x)\\ &= \sum_{k=1}^{\infty}2^{-n_k}\\ &\leq 1.\end{aligned}$$

Thus, $f\in L^{\phi_1}(X)$ but $C_\tau(f)\notin L^{\phi_2}(X)$, which is a contradiction. Hence, $\int_X h_n(x)d\mu(x)<\infty$ for some n. This completes the proof. □

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