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On Harmonic Domination Index of Graph

Shivaswamy P M[1,*]

1 Department of Mathematics, B.M.S. College of Engineering, Bengaluru, Karnataka, India.

Abstract: In this paper, we will define new index depend on domination degree called harmonic domination index. We establish this domination index for some families of graphs. We give the exact values of harmonic domination index, including the join and corona product of graphs.

MSC: 05C69, 05C10.

Keywords: Harmonic domination index, Domination degree, Minimal domination set.

1. Introduction

In this paper, we assume that G is a connected graph without loops. With the vertex set $V(G)$ and edge set $E(G)$. We denote $n = |V(G)|$ as the order and $m = |E(G)|$ as the size of a graph G. The complement of a graph G, denoted as $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ and two vertices f and g are joined by an edge in $\overline{G}$, if and only if they are not adjacent in G. There are two classified topological indices generally into two kinds: degree-based indices, and distance-based indices. The first and second $M_1(G)$ and $M_2(G)$ Zagreb indices are the old topological indices that were extensively investigated. These have been introduced by [18, 19], and are defined as:

$$M_1(G) = \sum_{v\in V(G)} d^2(v) \qquad and \qquad M_2(G) = \sum_{uv\in E(G)} d(u)d(v).$$

For more discussion, see [2–5, 7, 16, 22, 23, 26, 27]. A graph G is called connected if there is a path between any two vertices of G. Otherwise, G is called disconnected. A set $S \subseteq V(G)$ is called a dominating set of G, if for any vertex $f \in V(G) - S$ there exists a vertex $g \in S$ such that f and g are adjacent. A dominating set $S = \{f_1, f_2, ..., f_r\}$ is minimal domination set if $S - f_i$ is not a dominating set. In [21], the authors used the notation $T_m(G)$ to denote the number of minimal domination sets. In [8, 21] (2021) Hanan Ahmed et al, have defined a new degree based topological indices based on minimal dominating sets called domination topological indices and they are defined as follows:

$$DM_1(G) = \sum_{v\in V(G)} d_d^2(v),$$

$$DM_2(G) = \sum_{uv\in E(G)} d_d(u)d_d(v),$$

* E-mail: shivaswamy.pm@gmail.com

$$DM_1^* = \sum_{uv \in E(G)} [d_d(u) + d_d(v)].$$

$$DF(G) = \sum_{v \in V(G)} d_d^3(v),$$

$$DH(G) = \sum_{uv \in E(G)} [d_d(u) + d_d(v)]^2,$$

$$DF^*(G) = \sum_{uv \in E(G)} (d_d^2(u) + d_d^2(v)).$$

Where $d_d(v)$ is the domination degree of the vertex $v \in V(G)$ which is defined as:

Definition 1.1 ([21]). *For any vertex $f \in V(G)$, the domination degree denoted by $d_d(f)$ and defined as the number of minimal dominating sets of G which contains f.*

For more details of domination topological indices and their applications see ([9–12]).

Observation 1.2 ([21]). *$1 \leq d_d(v) \leq T_m(G)$, where $T_m(G)$ is the total number of minimal domination sets.*

Observation 1.3 ([21]). *Let $G(V,E)$ be a graph with $S_1, S_2, ..., S_t$ as minimal domination sets and $\gamma(G)$ is the domination number, $\Gamma(G)$ is the upper domination number of a graph G. Then $t\gamma(G) \leq \sum_{v \in V(G)} d_d(v) \leq t\Gamma(G)$.*

Definition 1.4 ([21]). *The graph G is called $k-$domination regular graph if and only if $d_d(v) = k$ for all $v \in V(G)$.*

2. The Main Results

Definition 2.1. *For a connected graph G without loops, the harmonic domination index is defined as:*

$$Dh(G) = \sum_{uv \in E(G)} \frac{2}{d_d(u) + d_d(v)}.$$

Lemma 2.2 ([21]). *$T_m(S_{r+1}) = 2$ and $T_m(K_n) = n$. And for all $v \in V(S_{r+1})$ or $v \in V(K_n)$ we get $d_d(v) = 1$.*

Proposition 2.3.

1. *In the star graph S_{r+1} with $r+1$ vertices $Dh(S_{r+1}) = r$.*
2. *For K_n, we have $Dh(K_n) = \frac{n(n-1)}{2}$.*
3. *For $S_{r+1,s+1}$, with $s+r+2$ vertices, we have $Dh(S_{r+1,s+1}) = \frac{r+s+1}{2}$.*

Lemma 2.4 ([21]). *$T_m(K_{r,s}) = rs + 2$, with $r \geq 2,\ s \geq 2$ and $d_d(v) = \begin{cases} r+1 & \\ s+1 & \end{cases}$ for all $v \in V(K_{r,s})$*

Theorem 2.5. *If $G \cong K_{r,s}$, with $r \geq 2,\ s \geq 2$, then $Dh(G) = \frac{2rs}{r+s+2}$.*

Proof. Using Lemma 2.4, we get

$$Dh(G) = \sum_{uv \in E(G)} \frac{2}{d_d(u) + d_d(v)} = \sum_{uv \in E(G)} \frac{2}{r+s+2} = \frac{2rs}{r+s+2}$$

□

The Windmill graph is the graph Wd_r^s is obtained by taking s copies of the complete graph K_r with a vertex in common.

Lemma 2.6 ([21]). *Suppose G is Wd_r^s. Then $T_m(Wd_r^s) = (r-1)^s+1$. And* $d_d(v) = \begin{cases} 1, & \textit{if } v \textit{ is the center vertex;} \\ (r-1)^{s-1}, & \textit{otherwise.} \end{cases}$

Theorem 2.7. *Let G be the Windmill graph Wd_r^s, then*

$$Dh(G) = \frac{s(r-1)}{1+(r-1)^{s-1}} + \frac{s\big((r-1)^2-(r-1)\big)}{2}.$$

Proof. Suppose E_1 is the set of all edges which are connected to the center vertex. E_2 is the set of all edges of the complete graph.

$$\begin{aligned} Dh(G) &= \sum_{uv\in E(G)} \frac{2}{d_d(u)+d_d(v)} = \sum_{uv\in E_1(G)} \frac{2}{d_d(u)+d_d(v)} + \sum_{uv\in E_2(G)} \frac{2}{d_d(u)+d_d(v)} \\ &= \sum_{uv\in E_1(G)} \frac{2}{1+(r-1)^{s-1}} + \sum_{uv\in E_2(G)} \frac{2}{(r-1)^{s-1}+(r-1)^{s-1}} \\ &= \frac{2}{1+(r-1)^{s-1}}|E_1| + \frac{2}{2(r-1)^{s-1}}|E_2| \\ &= \frac{2}{1+(r-1)^{s-1}}\big(s(r-1)\big) + \frac{2}{2(r-1)^{s-1}}\left(\frac{s(r-1)(r-1)-1}{2}\right) \\ &= \frac{s(r-1)}{1+(r-1)^{s-1}} + \frac{s\big((r-1)^2-(r-1)\big)}{2}. \end{aligned}$$

□

Proposition 2.8. *Suppose G is $r-$domination-regular graph, then $Dh(G) = \frac{|E(G)|}{k}$.*

Definition 2.9. *For any graphs G and H their cartesian product $G \times H$ is defined as [15] the graph on the vertex set $V(G) \times V(H)$ with vertices $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are adjacent by an edge if and only if either ($[f_1 = g_1$ and $\{f_2, g_2\} \in E(H)]$) or ($[f_2 = g_2$ and $\{f_1, g_1\} \in E(G)]$).*

Definition 2.10. *The book graph B_r is a cartesian product of a star S_{r+1} and single edge P_2 [20].*

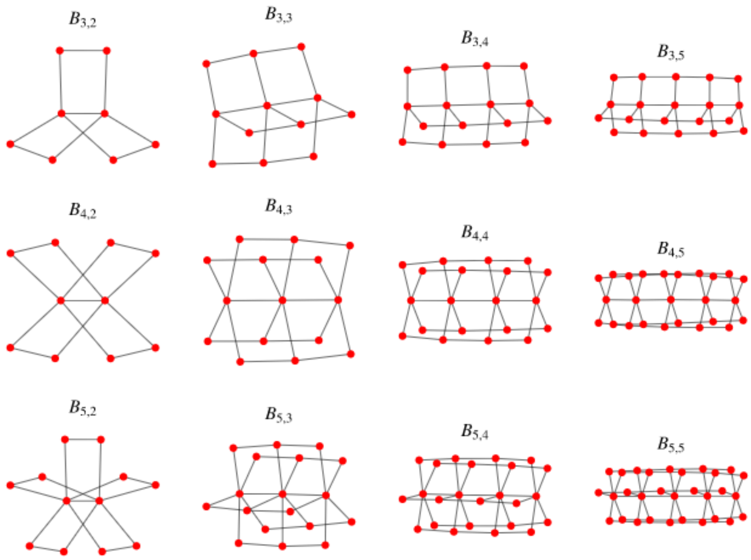

Figure 1. **Book graph**

Lemma 2.11 ([21]). *Suppose $G \cong B_r$, with $r \geq 3$. Then $T_m(G) = 2^r + 3$. And for $v \in V(B_r)$, we have:*

$$d_d(v) = \begin{cases} 3, & \text{if } v \text{ is the center vertex;} \\ 2^{r-1}+1, & \text{otherwise.} \end{cases}$$

Theorem 2.12.

$$Dh(B_r) = \frac{r}{2^{r-1}+1} + \frac{1}{3} + \frac{4r}{4+2^{r-1}}.$$

Proof. Based on the domination degree of the vertices of G, we get three types of edges in G. The first type, E_1 denotes the set of r edges $(f_i g_i)$ with initial and terminal vertices of the same domination degree $2^{r-1}+1$. The second type E_2 denotes the set containing only one edge (fg) with the same domination degree of initial and terminal vertices which equals 3, and the third type E_3 denotes the set of $2r$ edges of initial vertices of the domination degree 3 and terminal vertices of domination degree $2^{r-1}+1$. Hence,

$$\begin{aligned} Dh(B_r) &= \sum_{uv \in E(B_r)} \frac{2}{d_d(u)+d_d(v)} \\ &= \sum_{uv \in E_1} \frac{2}{d_d(u)+d_d(v)} + \sum_{uv \in E_2} \frac{2}{d_d(u)+d_d(v)} + \sum_{uv \in E_3} \frac{2}{d_d(u)+d_d(v)} \\ &= \sum_{uv \in E_1} \frac{2}{(2^{r-1}+1)+(2^{r-1}+1)} + \sum_{uv \in E_2} \frac{2}{3+3} + \sum_{uv \in E_3} \frac{2}{3+(2^{r-1}+1)} \\ &= \frac{r}{2^{r-1}+1} + \frac{1}{3} + \frac{4r}{4+2^{r-1}}. \end{aligned}$$

□

Lemma 2.13 ([21]). *Let $G \cong K_{n_1,n_2,\ldots,n_k}$, with $n_1 \geq 2$, $n_2 \geq 2, \ldots, n_k \geq 2$. Then $T_m(G) = \sum_{i=2}^{k} n_1 n_i + \sum_{i=3}^{k} n_2 n_i + \ldots + n_{k-1} n_k + k$.*

Theorem 2.14. *Suppose $G \cong K_{n_1,n_2,\ldots,n_k}$ with $n_1 \geq 2$, $n_2 \geq 2, \ldots, n_k \geq 2$, then*

$$Dh(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)+2}.$$

Proof. Suppose $G \cong K_{n_1,n_2,\ldots,n_k}$, with $n_1 \geq 2$, $n_2 \geq 2, \ldots, n_k \geq 2$. Note that if $G \cong K_{n_1,n_2,\ldots,n_k}$ so, for any vertex $v \in G$ we have $d_d(v) = d(v)+1$, and $|E(G)| = T_m(G) - k$. So by the definition of harmonic index we get: $Dh(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)+2}$. □

Lemma 2.15 ([21]). *Let G be any connected graph with n_1 vertices and m_1 edges. Let $H \cong G \circ K_{n_2}$, where K_{n_2}. There are $(n_2+1)^{n_1}$ minimal domination sets in H, and $d_d(v) = (n_2+1)^{n_1-1}$.*

Theorem 2.16. *Let G be a graph with n_1 vertices and m_1 edges. Let K_{n_2} be a complete graph of order n_2. Then*

$$Dh(G \circ K_{n_2}) = \frac{2m_1 + n_1 n_2 (n_2-1) + 2n_1 n_2}{(n_2+1)^{n_1-1}}.$$

Proof. Note that $|V(G \circ K_{n_2})| = n_1 + n_1 n_2$. And $G \circ K_{n_2}$ is $(n_2+1)^{n_1-1}$domination regular graph. Also, based on domination degree of the vertices of $G \circ K_{n_2}$, there are three types of edges in $G \circ K_{n_2}$. first type the edges of G, second type the edges of K_{n_2} and let E_1 denote the set of the edges that connect one vertex from G and one vertex from K_{n_2}. Hence

$$Dh(G \circ K_{n_2}) = \sum_{uv \in E(G \circ K_{n_2})} \frac{2}{d_d(u)+d_d(v)} = \sum_{uv \in E(G)} \frac{2}{d_d(u)+d_d(v)} + \sum_{uv \in E(K_{n_2})} \frac{2}{d_d(u)+d_d(v)} + \sum_{uv \in E_1} \frac{2}{d_d(u)+d_d(v)}$$

$$
\begin{aligned}
&= \sum_{uv\in E(G)} \frac{2}{(n_2+1)^{n_1-1}+(n_2+1)^{n_1-1}} + \sum_{uv\in E(K_{n_2})} \frac{2}{(n_2+1)^{n_1-1}+(n_2+1)^{n_1-1}} \\
&+ \sum_{uv\in E_1} \frac{2}{(n_2+1)^{n_1-1}+(n_2+1)^{n_1-1}} \\
&= \frac{|E(G)|}{(n_2+1)^{n_1-1}} + \frac{|E(K_{n_2})|n_1}{(n_2+1)^{n_1-1}} + \frac{|E_1|}{(n_2+1)^{n_1-1}} \\
&= \frac{m_1}{(n_2+1)^{n_1-1}} + \frac{n_1n_2(n_2-1)}{2(n_2+1)^{n_1-1}} + \frac{n_1n_2}{(n_2+1)^{n_1-1}} \\
&= \frac{2m_1+n_1n_2(n_2-1)+2n_1n_2}{(n_2+1)^{n_1-1}}. \qquad \square
\end{aligned}
$$

Lemma 2.17 ([21]). *Suppose $H \cong G \circ \overline{K_{n_2}}$ where G be a graph of order n_1. Then,*

$$T_m(H) = \sum_{i=0}^{n_1} \binom{n_1}{i}.$$

Theorem 2.18. *Let G be a graph with n_1 vertices and m_1 edges. Let $H \cong G \circ \overline{K_{n_2}}$. Then,*

$$Dh(H) = \frac{n_1n_2+m_1}{T_m(H)-2^{n_1-1}}.$$

Proof. It is clear that $H \cong G \circ \overline{K_{n_2}}$ is domination regular graph. And every $v \in V(H)$ is contained in every minimal dominating sets of H except $\binom{n_1-1}{0}+\binom{n_1-1}{1}+\ldots+\binom{n_1-1}{n_1-2}+\binom{n_1-1}{n_1-1}=2^{n_1-1}$ minimal dominating sets. Hence, $d_{d\,H}(v) = T_m(H)-2^{n_1-1}$ and by using the definition of harmonic domination index we get:

$$
\begin{aligned}
Dh(H) &= \sum_{uv\in E(G)} \frac{2}{d_d(u)+d_d(v)} = \sum_{uv\in E(G)} \frac{2}{(T_m(H)-2^{n_1-1})+(T_m(H)-2^{n_1-1})} \\
&= \frac{n_1n_2+m_1}{T_m(H)-2^{n_1-1}}. \qquad \square
\end{aligned}
$$

A join G_1+G_2 of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 is the graph on the vertex set $V_1\cup V_2$ and the edge set $E_1\cup E_2\cup\{u_1u_2 : u_1\in V_1, u_2\in V_2\}$ [14].

Lemma 2.19 ([21]). *Let G_1 and G_2 be any non complete graphs of n_1, n_2 vertices respectively, such that G_1 and G_2 do not have any vertex of full degree. Then, $T_m(G_1+G_2)=T_m(G_1)+T_m(G_2)+n_1n_2$, and*

$$d_{d\ G_1+G_2}(v) = \begin{cases} d_{d\,G_1}(v)+n_2, & \text{if } v\in V(G_1); \\ d_{d\,G_2}(v)+n_1, & \text{if } v\in V(G_2). \end{cases}$$

Theorem 2.20. *Suppose G_1 and G_2 are any non complete graphs of n_1, n_2 vertices and m_1, m_2 edges respectively, such that G_1 and G_2 do not have any vertex of full degree. Then*

$$
\begin{aligned}
Dh(G_1+G_2) &= Dh(G_2)(1-n_1)+Dh(G_1)(1-n_2) \\
&+ \left[\frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_1)} + \frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_2)} + \frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_3)} + \ldots + \frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_{n_2})}\right] \\
&+ \left[\frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_1)} + \frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_2)} + \frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_3)} + \ldots + \frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_{n_2})}\right] \\
&+ \ldots \\
&+ \left[\frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_1)} + \frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_2)} + \frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_3)} + \ldots + \frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_{n_2})}\right]
\end{aligned}
$$

Proof.

$$Dh(G_1+G_2) = \sum_{uv\in E(G_1+G_2)} \frac{2}{d_{dG_1+G_2}(u)+d_{dG_1+G_2}(v)}$$

$$= \overbrace{\sum_{uv\in E(G_1)} \frac{2}{d_{dG_1+G_2}(u)+d_{dG_1+G_2}(v)}}^{1} + \overbrace{\sum_{uv\in E(G_2)} \frac{2}{d_{dG_1+G_2}(u)+d_{dG_1+G_2}(v)}}^{2}$$

$$+ \overbrace{\sum_{\substack{u\in V(G_1)\\ v\in V(G_2)}} \frac{2}{d_{dG_1+G_2}(u)+d_{dG_1+G_2}(v)}}^{3}$$

$$\overbrace{\sum_{uv\in E(G_1)} \frac{2}{d_{dG_1+G_2}(u)+d_{dG_1+G_2}(v)}}^{1} = \sum_{uv\in E(G_1)} \frac{2}{(d_{dG_1}(u)+n_2)+(d_{dG_1}(v)+n_2)}$$

$$= \sum_{uv\in E(G_1)} \frac{2}{d_{dG_1}(u)+d_{dG_1}(v)+2n_2}$$

$$= \sum_{uv\in E(G_1)} \frac{2}{d_{dG_1}(u)+d_{dG_1}(v)+2n_2} \times \frac{-2n_2}{-2n_2}$$

$$= \sum_{uv\in E(G_1)} \frac{2-2n_2}{d_{dG_1}(u)+d_{dG_1}(v)}$$

$$= \sum_{uv\in E(G_1)} \frac{2}{d_{dG_1}(u)+d_{dG_1}(v)} - n_2 \sum_{uv\in E(G_1)} \frac{2}{d_{dG_1}(u)+d_{dG_1}(v)}$$

$$= Dh(G_1)(1-n_2)$$

$$\overbrace{\sum_{uv\in E(G_2)} \frac{2}{d_{dG_1+G_2}(u)+d_{dG_1+G_2}(v)}}^{2} = \sum_{uv\in E(G_2)} \frac{2}{(d_{dG_2}(u)+n_1)+(d_{dG_2}(v)+n_1)}$$

$$= \sum_{uv\in E(G_2)} \frac{2}{d_{dG_2}(u)+d_{dG_2}(v)+2n_1}$$

$$= \sum_{uv\in E(G_2)} \frac{2}{d_{dG_2}(u)+d_{dG_2}(v)+2n_1} \times \frac{-2n_1}{-2n_1}$$

$$= \sum_{uv\in E(G_2)} \frac{2-2n_1}{d_{dG_2}(u)+d_{dG_2}(v)}$$

$$= \sum_{uv\in E(G_2)} \frac{2}{d_{dG_2}(u)+d_{dG_2}(v)} - n_1 \sum_{uv\in E(G_2)} \frac{2}{d_{dG_2}(u)+d_{dG_2}(v)}$$

$$= Dh(G_2)(1-n_1)$$

$$\overbrace{\sum_{\substack{u\in V(G_1)\\ v\in V(G_2)}} \frac{2}{d_{dG_1+G_2}(u)+d_{dG_1+G_2}(v)}}^{3} = \left[\frac{2}{(d_{dG_1}(u_1)+n_2)+(d_{dG_2}(v_1)+n_1)} + \frac{2}{(d_{dG_1}(u_1)+n_2)+(d_{dG_2}(v_2)+n_1)}\right.$$

$$\left. + \frac{2}{(d_{dG_1}(u_1)+n_2)+(d_{dG_2}(v_3)+n_1)} + \ldots + \frac{2}{(d_{dG_1}(u_1)+n_2)+(d_{dG_2}(v_{n_2})+n_1)}\right]$$

$$+ \left[\frac{2}{(d_{dG_1}(u_2)+n_2)+(d_{dG_2}(v_1)+n_1)} + \frac{2}{(d_{dG_1}(u_2)+n_2)+(d_{dG_2}(v_2)+n_1)}\right.$$

$$\left. + \frac{2}{(d_{dG_1}(u_2)+n_2)+(d_{dG_2}(v_3)+n_1)} + \ldots + \frac{2}{(d_{dG_1}(u_2)+n_2)+(d_{dG_2}(v_{n_2})+n_1)}\right]$$

$$+ \ldots + \left[\frac{2}{(d_{dG_1}(u_{n_1})+n_2)+(d_{dG_2}(v_1)+n_1)} + \frac{2}{(d_{dG_1}(u_{n_1})+n_2)+(d_{dG_2}(v_2)+n_1)}\right.$$

$$\left. + \frac{2}{(d_{dG_1}(u_{n_1})+n_2)+(d_{dG_2}(v_3)+n_1)} + \ldots + \frac{2}{(d_{dG_1}(u_{n_1})+n_2)+(d_{dG_2}(v_{n_2})+n_1)}\right]$$

$$= \left[\frac{2}{d_{dG_1}(u_1)+d_{dG_2}(v_1)+(n_2+n_1)} \times \frac{-(n_2+n_1)}{-(n_2+n_1)}\right.$$

$$
\begin{aligned}
&+\frac{2}{d_{dG_1}(u_1)+d_{dG_2}(v_2)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\\
&+\frac{2}{d_{dG_1}(u_1)+d_{dG_2}(v_3)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\\
&+\ldots+\frac{2}{d_{dG_1}(u_1)+d_{dG_2}(v_{n_2})+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\Bigg]\\
&+\Bigg[\frac{2}{d_{dG_1}(u_2)+d_{dG_2}(v_1)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\\
&+\frac{2}{d_{dG_1}(u_2)+d_{dG_2}(v_2)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\Bigg]\\
&+\frac{2}{d_{dG_1}(u_2)+d_{dG_2}(v_3)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\\
&+\ldots+\frac{2}{d_{dG_1}(u_2)+d_{dG_2}(v_{n_2})+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\Bigg]\\
&+\ldots+\Bigg[\frac{2}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_1)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\\
&+\frac{2}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_2)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\Bigg]\\
&+\frac{2}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_3)+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\\
&+\ldots+\frac{2}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_{n_2})+(n_2+n_1)}\times\frac{-(n_2+n_1)}{-(n_2+n_1)}\Bigg]\\
&=\Bigg[\frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_1)}+\frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_2)}\\
&+\frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_3)}+\ldots+\frac{2-(n_2+n_1)}{d_{dG_1}(u_1)+d_{dG_2}(v_{n_2})}\Bigg]\\
&+\Bigg[\frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_1)}+\frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_2)}\\
&+\frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_3)}+\ldots+\frac{2-(n_2+n_1)}{d_{dG_1}(u_2)+d_{dG_2}(v_{n_2})}\Bigg]\\
&+\cdots+\Bigg[\frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_1)}+\frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_2)}\\
&+\frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_3)}+\ldots+\frac{2-(n_2+n_1)}{d_{dG_1}(u_{n_1})+d_{dG_2}(v_{n_2})}\Bigg]
\end{aligned}
$$

□

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