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An Analytical Solution for Transient Asymmetric Heat Conduction in a Multilayer Elliptic Annulus and its Associated Thermal Stresses

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Abstract: A closed-form solution is presented for the transient heat conduction problem in the elliptic annulus with multiple layers in the radial direction, but time-independent volumetric heat sources in each layer. The solution is obtained applying an integral transform technique analogous to Vodicka's approach considering series expansion function in terms of an eigenfunction to solve the heat conduction partial differential equation in elliptical coordinates. The results are obtained as a series solution in terms of Mathieu functions and hold convergence test. The temperature and thermal stresses for the three-layer elliptic region have been computed numerically and exhibited in a graphical form, by using dimensionless values and therefore discussed.

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Keywords: Multilayer elliptic annulus, heat conduction, integral transform, thermal stresses.

1. Introduction

Composite materials consisting of several layers in contact with different properties and having distinct boundaries between them emerged as new prospects for the modern technology. It offers many challenging problems for theoretical and experimental studies. Thermal analysis of multilayer composite media is widely used in real physical and modern engineering applications. For example, semicircular fibre insulated heaters, multilayer insulation materials and nuclear fuel rods due to the added advantage of combining physical, mechanical, and thermal properties of different materials, composite regions such as multilayer slabs, cylinders and spheres. Even multilayer transient heat conduction finds applications in thermodynamics, fuel cells and electrochemical reactors. Multilayer components find a wide range of applications in various automotive, spaces, chemical, civil and nuclear industries. Many of these applications require a detailed knowledge of transient temperature and heat-flux distribution within the composite layers. Temperature distribution in such components, with the presence of boundary conditions, may be obtained by using both analytical and numerical techniques. The analytical methods are such as separation of variables, the Laplace-transform method, the method of finite integral transforms, the Eigen function-expansion and Green's function methods. Nonetheless, numerical solutions are preferred and prevalent in practice, due to either unavailability or higher mathematical complexity of the corresponding exact solutions. Many researchers have been solved the transient heat conduction problem in composite layers. For instance, Ölçer [1] presented an analytical study

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for the distribution of three-dimensional unsteady temperatures in a hollow right circular cylinder of finite length. Torabi [2] investigates the analytical solution for transient temperature and thermal stresses within three circular composite disks. The combination of the separation of variables method and Duhamel's theorem are adopted to solve the energy partial differential equations. Wankhede and Bhonsale [3] determined the solution of heat conduction on multilayered composite plates, cylinders or spheres consisting of k-layers using a new integral transform which is more analogues compared to the classical method of Sturm-Liouville system. Salt [4, 5] investigated the response of a two-dimensional multi-layer composite slab, to a sudden temperature change. The solution was analytically examined in two and three-layer composite slabs. Singh [6] discusses the methodology as well as possible application in nuclear reactors of analytical solutions of two-dimensional multilayer heat conduction in spherical and cylindrical coordinates. Singh et al. [7, 8] and Jain et al. [9, 10] have studied two-dimensional multilayer transient conduction problems in spherical and cylindrical coordinates. They have used the combination of separation of variables and eigenfunction expansion methods to solve the two-dimensional multilayer transient heat conduction in spherical coordinates, in polar coordinates with multilayers in the radial direction, and in a multilayer annulus. They have used the method of partial solutions to obtain the temperature distributions. In the method of partial solutions, the nonhomogeneous transient problem is split into a homogeneous transient problem and a nonhomogeneous steady-state problem. Then, the method of separation of variables is used to solve the homogeneous transient problem whereas the eigenfunction expansion method is used to solve the nonhomogeneous steady-state problem. Kayhani et al. [11] presented a steady analytical solution for heat conduction in a cylindrical multilayer composite laminate in which the fibre direction may vary between layers. The analytical solution is obtained for general linear boundary conditions that are suitable for various conditions including combinations of conduction, convection, and radiation both inside and outside the cylinder. The Sturm-Liouville theorem is used to derive an appropriate Fourier transformation for this problem. An exact analytical solution for unsteady conductive heat transfer in a cylindrical multilayer composite laminate has been studied by Norouzi et al. [12] using Separation of variables method. In an another investigation, Norouzi et al. [13] obtained an exact analytical solution for steady conductive heat transfer in multilayer spherical fibre reinforced composite laminates under the general linear boundary conditions. Recently, Gaikwad and Wange [14] obtained an analytical solution using separation of variables method for the three-dimensional heat conduction equations in a multilayered sphere with time-independent volumetric heat sources in each layer. Monte [15] solved transient heat conduction problems in one-dimensional multi-layer solids applying conventional techniques based on Vodicka's approach. A 'natural' analytic technique has been employed for solving M-layer unsteady heat conduction of composite media. Chiba [16] analytically obtained the second-order statistics of the temperature and thermal stresses in an annular disc with spatially random heat transfer coefficients on the upper and lower surfaces. Vasilenko and Urusova [17] considered the problem of a stressed state in elliptic plates for a rigid contour fixation. Vasilenko [18] propose an approach for solving the problem of determining the temperature fields and stresses in orthotropic elliptic plates whose principal axes of elasticity and thermal conduction do not coincide with the axes of the ellipse. However, a literature survey showed that the authors above have not yet been developed an analytical solution for the unsteady temperature distribution in a multilayer composite elliptical annular geometry with time-independent heat sources are switched on in each layer under specified initial temperature.

The main objective of this study is a new analytical method to derive the solutions by establishing Sturm-Liouville integral transform. Vodicka's method [19], which is a type of integral transform method, and a perturbation method are employed to obtain the analytical solutions for the statistics. Additionally, the method we propose in this paper can also be applied to solid discs, strips and plates. The success of this novel research mainly lies with the new mathematical procedures which present a much simpler approach for optimization of the design in terms of material usage in the aerospace, automobile, chemical, power, and civil engineering, biomedical industry, thermodynamic and solidification processes, and high-density microelectronics as well as for production of fiber-insulated heaters, multilayer insulators, nuclear fuel rods, fuel cells, electrochemical reactors, building structures and particularly for the determination of the thermoelastic behavior in elliptical plate engaged as the foundation of pressure vessels, furnaces, etc.

2. Formulation of the Problem

Consider a k- layer composite elliptic annulus is occupying space $D : \{(\xi, \eta, z) \in R^3 : \xi_i < \xi < \xi_{i+1},\ 0 < \eta < 2\pi, 1 \leq i \leq k\}$ indicates that in elliptical coordinates system is the most appropriate choice for reference frame, which is related to the rectilinear coordinate (x, y, z) by the relation

$$x = c\cosh\xi\cos\eta, y = c\sinh\xi\sin\eta, c = (a^2 - b^2)^{1/2} \tag{1}$$

where, c is the semi-focal length of the ellipse, a and b are semi-major and semi-minor axis, respectively. Let k_i and λ_i be the thermal diffusivity and thermal conductivity of the i^{th} layer. At $t = 0$, each i^{th} layer is at zero temperature, and time-independent heat sources $g_i(\xi, \eta)$ are switched on for $t > 0$. We assume that each rigid layer in each annulus composite body may be subjected to any combination of temperature and heat flux boundary conditions.

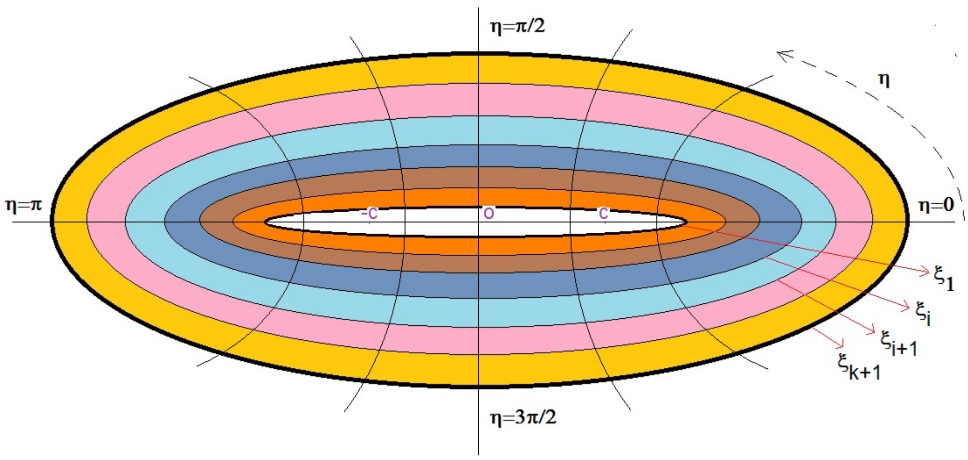

Figure 1. **Schematic representation of a k -layer elliptic annulus**

2.1. Heat Conduction

The differential equation governing transient temperature distribution $T_i(\xi, \eta,\, t)$ with internal heat source for a multilayer elliptic annulus along with the conditions can be defined as

$$h^2\left[T_i(\xi,\eta,t)_{,\xi\xi} + T_i(\xi,\eta,t)_{,\eta\eta}\right] + (1/\lambda_i) g_i(\xi,\eta) = (1/\kappa_i)\, T_i(\xi,\eta,t)_{,t} \tag{2}$$

with the initial condition as

$$T_i(\xi, \eta, z, 0) = p_i(\xi, \eta) \tag{3}$$

Moreover, the other boundary conditions as

1. Inner surface of the first layer (i =1 for $t > 0$)

$$\alpha_1 T_1(\xi,\eta,t)_{,\xi} - h_0 T_1(\xi,\eta,t)\big|_{\xi=\xi_1} = f_1(t),\ h_0 \geq 0, \tag{4}$$

2. Outer surface of the k^{th} layer ($i= k$ for $t > 0$)

$$\alpha_k T_k(\xi,\eta,t)_{,\xi} + h_k T_k(\xi,\eta,t)|_{\xi=\xi_{k+1}} = f_2(t),\ h_k \geq 0, \tag{5}$$

3. Interface of the i^{th} layer (i=2,...,k -1 for $t > 0$)

$$\begin{aligned}\alpha_i T_i(\xi,\eta,t)_{,\xi} + h_i[T_i(\xi,\eta,t) - T_{i+1}(\xi,\eta,t)]|_{\xi=\xi_{i+1}} = 0\\ \alpha_{i+1} T_{i+1}(\xi,\eta,t)_{,\xi} + h_i[T_i(\xi,\eta,t) - T_{i+1}(\xi,\eta,t)]|_{\xi=\xi_{i+1}} = 0\end{aligned} \tag{6}$$

in which the function $T_i(\xi,\eta,t)$ represents the temperature at every instance and at all points of the elliptic annulus under the influence of boundary conditions, prime (,) in equations denotes differentiation with respect to the variable specified in the subscript, $f_1(t)$ and $f_2(t)$ represents sectional heat supply at $\xi = \xi_1$ and $\xi = \xi_{k+1}$, respectively. Here λ_i denotes the thermal conductivity, the heat capacity per unit volume is given as $(\rho C)_i$ with ρ_i for density and C_i as specific heat respectively for the i^{th} layer, respectively. The physical significance of the interface boundary conditions [i.e. equation (6)] follows as that the finite value of the layer coefficient $h_i > 0,\ i = 1, 2, ..(k-1)$ represents a discontinuity of temperature at the corresponding interface.

2.2. Associated Thermal Stresses

The differential equation governing the Airy stress function $\chi(\xi,\eta,t)$ is given as

$$h^2\nabla^2\, h^2\nabla^2\, \chi_i + h^2\nabla^2 T_i = 0 \tag{7}$$

The stress distribution for each layer in the structure can be written in terms of Airy stress function as

$$\begin{aligned}\sigma^{(i)}_{\xi\xi} &= h^2\chi^{(i)}_{,\eta\eta} + \tfrac{c^2h^4}{2}\sinh 2\xi\ \chi^{(i)}_{,\xi} - \tfrac{c^2h^4}{2}\sin 2\eta\, \chi^{(i)}_{,\eta}\\ \sigma^{(i)}_{\eta\eta} &= h^2\chi^{(i)}_{,\xi\xi} - \tfrac{c^2h^4}{2}\sinh 2\xi\ \chi^{(i)}_{,\xi} + \tfrac{c^2h^4}{2}\sin 2\eta\ \chi^{(i)}_{,\eta}\\ \sigma^{(i)}_{\xi\eta} &= -h^2\chi^{(i)}_{,\xi\eta} + \tfrac{c^2h^4}{2}\sin 2\eta\ \chi^{(i)}_{,\xi} + \tfrac{c^2h^4}{2}\sinh 2\xi\, \chi^{(i)}_{,\eta}\end{aligned} \tag{8}$$

The following equation expresses the boundary conditions on the traction free surfaces for the thermal stress problem without external force

1. Inner surface of the first layer (i =1 for $t > 0$)

$$\sigma^{(1)}_{\xi\xi} = \sigma^{(1)}_{\xi\eta} = 0 \tag{9}$$

2. Outer surface of the k^{th} layer ($i= k$ for $t > 0$)

$$\sigma^{(k)}_{\xi\xi} = \sigma^{(k)}_{\xi\eta} = 0 \tag{10}$$

3. Interface of the i^{th} layer (i=2,...,k for $t > 0$)

$$\sigma^{(i)}_{\xi\xi} = \sigma^{(i)}_{\xi\eta} = 0 \tag{11}$$

In the end, analysing the thermal stress problem for the multilayer elliptic annulus, which is a multiply-connected region, by the stress function method give a general solution for the differential equation of equation (7). In this paper, therefore, we derived the integral conditions necessary for the assurance of the single-valuedness of the rotation and displacements in the multiply-connected region in terms of Airy's stress function as follows

$$\int_{0\ |\ \xi=\xi_1,\,\xi_{i+1},\,\xi_{k+1}}^{2\pi} [h^2\nabla^2\chi + T]_{,\xi}\ d\eta = 0 \tag{12}$$

The equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

3. Solution to the Problem

To solve fundamental differential equation (2) using the theory on integral transformation, firstly we introduce a new integral transform of order n and m over the variable ξ and η as

$$\bar{f}_i(q_{2n,.m}) = \beta_i \int_0^{2\pi} \int_{\xi_i}^{\xi_{i+1}} f_i(\xi,\eta)\,(\cosh 2\xi - \cos 2\eta) \times \psi_{i,2n}\;(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) d\xi d\eta \tag{13}$$

in which $\psi_{i,2n}(\xi, q_{2n,m})$ is the kernel of the transform and $\bar{f}_i(q_{2n,.m})$ is Sturm-Liouville transform for the composite region (refer Appendix A). The inversion theorem is given by

$$f_i(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{i=1}^{k} \frac{\bar{f}_i(\xi,\eta)\,\psi_{i,2n}(\xi,q)\,ce_{2n}(\eta,q)}{\pi\int_{\xi_i}^{\xi_{i+1}} \psi_{i,2n}^2(\xi,q)(\cosh 2\xi - \Theta_{2n,m})\,d\xi} \tag{14}$$

In this article, we have introduced a new integral transform which has given in equation (13). Moreover, its inversion theorem is given in equation (14). The detailed mathematical calculation is given in the appendix section. On applying the new Sturm-Linville transform defined in equation (13) on equation (2), we obtain

$$\frac{1}{\kappa_i}\frac{\partial \bar{T}_i}{\partial t} = \frac{\bar{g}_i(q_{2n,m})}{\lambda_i\,\beta_i} - \left(\frac{4\,q_{2n,m}\,\bar{T}_i}{c^2\,\alpha_i} + F(q_{2n,m},t)\right) \tag{15}$$

in which

$$F(q_{2n,m},t) = \frac{4\pi A_0^{(2n)}}{c^2\,\alpha_i}\,\{\psi_{k,n,m}(\xi_{k+1})f_2(t) - \psi_{1,n,m}(\xi_1)f_1(t)\}$$

and

$$\bar{g}_i(q_{2n,m}) = \beta_i \int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}} g_i(\xi,\eta)(\cosh 2\xi - \cos 2\eta) \times \psi_{i,2n}(\xi,q_{2n,m}) ce_{2n}(\eta,q_{2n,m}) d\xi d\eta$$

After some mathematical simplification, we get the expression

$$\frac{\partial \bar{T}_i}{\partial t} = \mu_i \bar{g}_i(q_{2n,m}) - (\alpha_{2n,m}^2 \bar{T}_i + k_i F(q_{2n,m},t)) \tag{16}$$

in which $\mu_i = \frac{\kappa_i}{\lambda_i\,\beta_i}$ and $\alpha_{2n,m}^2 = \frac{\kappa_i}{\alpha_i}\frac{4\,q_{2n,m}}{c^2}$. Now, applying the Laplace transform to the Eq. (16), one obtains

$$\overline{\overline{T_i}}(q_{2n,m},s) = \;[p_i(\xi,\eta) + \mu_i \bar{g}_i(q_{2n,m}) - k_i\,\bar{F}\,(q_{2n,m},s)]/(s+\alpha_{2n,m}^2) \tag{17}$$

in which $\overline{\overline{T_i}}(q_{2n,m},s)$ and $\bar{F}(q_{2n,m},s)$ are the transformed function of $\bar{T}_i(q_{2n,m},t)$ and $F(q_{2n,m},t)$ respectively, s is transformed Laplace parameter. Then accomplishing Laplace inversion theorems on equation (17), we obtain

$$\bar{T}_i(q_{2n,m},t) = \exp[-\alpha_{2n,m}^2 t]\,[p_i(\xi,\eta) + \mu_i \bar{g}_i(q_{2n,m})] - k_i \int_0^t \exp[\alpha_{2n,m}^2 \tau]\,\bar{F}\,(q_{2n,m},t-\tau) d\tau \tag{18}$$

Finally, applying the inversion theorem defined in equation (14), on Eq. (18), one obtains

$$\begin{aligned} T_i(\xi,\eta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \psi_{i,2n}\,(\xi,q_{2n,m}) ce_{2n}(\eta,q_{2n,m}) \sum_{i=1}^{k} \Bigg\{ & \exp[-\alpha_{2n,m}^2 t] \times \exp[-\alpha_{2n,m}^2 t]\,[p_i(\xi,\eta) + \mu_i \bar{g}_i(q_{2n,m})] \\ & - k_i \int_0^t \exp[\alpha_{2n,m}^2 \tau]\,\bar{F}\,(q_{2n,m},t-\tau) d\tau \Bigg\} / C_{2n,m} \end{aligned} \tag{19}$$

where

$$C_{2n,m} = \pi \int_{\xi_i}^{\xi_{i+1}} \psi_{i,2n}^2(\xi, q_{2n,m})(\cosh 2\xi - \Theta_{2n,m})\, d\xi$$

and

$$\Theta_{2n,m} = \frac{1}{\pi}\int_0^{2\pi} \cos 2\eta \, ce_{2n}^2(\eta, q_{2n,m})\, d\eta$$
$$= A_0^{(2n)} A_2^{(2n)} + \sum_{r=0}^{\infty} A_{2r}^{(2n)} A_{2r+2}^{(2n)}$$

Now assume Airy's stress function which satisfies condition (7) as,

$$\chi_i(\xi,\eta,t) = h^2 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \times \sum_{i=1}^{k} \left\{\exp[-\alpha_{2n,m}^2 t]\ \mu_i \bar{g}_i(q_{2n,m})\ [A_{2n,m}\sin 2\xi - \cos 2\xi]\right\}/C_{2n,m} \quad (20)$$

in which $A_{2n,m}$is the arbitrary constant that can be determined finally by using condition (11).

$$A_{2n,m} = \frac{ce_{2\eta\, \alpha(\xi)} - \cot 2\xi\, \beta(\eta)}{ce_{2\eta\, \gamma(\xi)} - \beta(\eta)} \quad (21)$$

in which

$$\alpha(\xi) = \cot 2\xi \ \sinh 2\xi \ \psi(\xi)/\psi(\xi)_{,\xi} - 2\sinh 2\xi$$
$$\beta(\eta) = \sinh 2\eta \, ce(\eta, q_{2n,m})_{,\eta} - 2ce(\eta, q_{2n,m})_{,\eta\eta}/c^2 h^2$$
$$\gamma(\xi) = 2\cot 2\xi \ \sinh 2\xi \ - \sinh 2\xi \, \psi(\xi)/\psi(\xi)_{,\xi}$$

Substituting $A_{2n,m}$ in equations (20), we get

$$\chi_i(\xi,\eta,t) = h^2 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m})}{C_{2n,m}[\beta(\eta) - ce_{2n}(\eta, q_{2n,m})\gamma(\xi)]} \times \sum_{i=1}^{k} \left\{\exp[-\alpha_{2n,m}^2 t]\ \mu_i \bar{g}_i(q_{2n,m})\ [-\sin 2\xi\alpha(\xi) + \cos 2\xi\gamma(\xi)]\right\} \quad (22)$$

The resulting equations of stresses can be obtained by substituting the Eq. (22) in Eq. (8) and Eq. (12) has been satisfied by substituting the value of Airy's stress function from Eq. (22). The equations of stresses are rather lengthy. Consequently the same have been omitted here for the sake of brevity, but have been considered during graphical discussion using MATHEMATICA software.

4. Numerical Results, Discussion and Remarks

We introduce the following dimensionless values

$$\left.\begin{aligned} &\bar{\xi}_i = \xi_i/b,\ \bar{a} = a/b,\ \bar{b} = b/b,\ e = c/b,\ \tau = \kappa\, t/b^2, \\ &T = T/T_0,\ \bar{\sigma}_{mn} = \sigma_{mn}/E_i\, \alpha_i\, T_0,\ (i = 1,2,3;\ m,\, n = \xi, \eta) \end{aligned}\right\} \quad (23)$$

For the sake of simplicity of numerical calculations, we consider a three-layered composite elliptic annulus plate. The numerical computations have been carried out for Aluminium, Tin and copper metal with initial temperature and for ($t > 0$) the temperature raised to finite value. The physical parameters are considered as $\bar{\xi}_1 = a =$ 0.1 m, $\bar{\xi}_4 = b = 1$ m, $\bar{\xi}_2 = 0.4$ m, $\bar{\xi}_3$=0.7m, reference temperature T_0 as 150 ^{0}C. The mechanical material properties are considered as specific heat capacity at constant pressure C_{v1}=0.900 J/g^{0}C, C_{v2}=0.21 J/g^{0}C, C_{v3}=0.385 J/g^{0}C, Modulus of Elasticity, $E_1 =$ 69 GPa, $E_2 = 47$ GPa and $E_3 = 117$ GPa; Thermal expansion coefficient, $\alpha_1 = 23.0\times10^{-6}/^0$C, $\alpha_2 = 24.8\times10^{-6}\ /^0$C, $\alpha_3 = 16.5\times10^{-6}\ /^0$C; Thermal conductivity $\lambda_1 = 204.2$ Wm^{-1} K^{-1}, $\lambda_2 = 66$ Wm^{-1} K^{-1} and$\lambda_3 =$ 386 Wm^{-1} K^{-1}; $f_1(t)$=20^{0}C,

$f_2(t)$=30^{0}C. Substituting the dimensionless value of equation (23) in the equation of temperature distribution (19) and in its stress components, we obtain the expressions for the temperature and stresses for our numerical discussion. In order to examine the influence of heating on the plate, we performed the numerical calculation for all variables, and numerical calculations are depicted in the following figures with the help of MATHEMATICA software.

Figs. 2-10 illustrates the numerical results of temperature distribution, stresses of the elliptical plate due to internal heat generation within the solid. A similar trend was observed for Fig. 2 as well as Fig. 4. Fig. 2 and Fig. 4 shows the temperature distribution along $\bar{\xi}$- direction for different time and η, respectively. In both the figures, temperature increases gradually towards the outer end of each layer due to the combined effect of sectional heat supply and internal heat energy. As shown in Fig. 3, initially the temperature approaches to a maximum whereas attains minimum as time increases. Fig. 5, shows that the temperature approaches to a minimum at both extreme ends i.e. at $\eta = 0$ and $\eta = \pi$ due to more compressive force, whereas due to a tensile force, the temperature is high at centre i.e. at $\eta = \pi/2$, which gives an overall bell-shaped curve for all three layers of different materials.

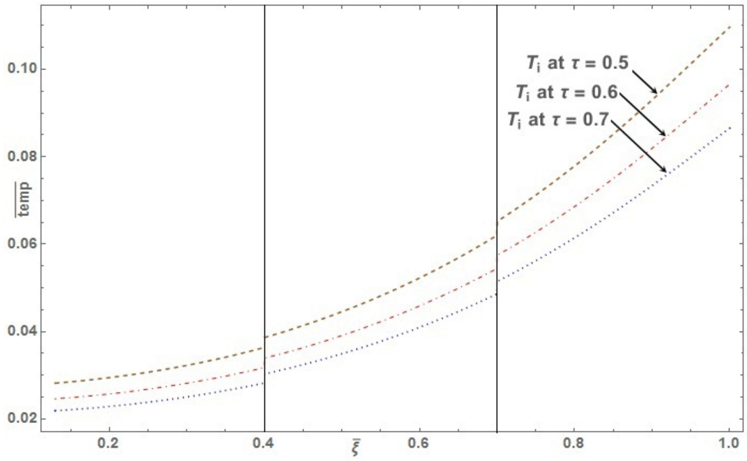

Figure 2. **Temperature distribution along $\bar{\xi}$ -direction for different τ**

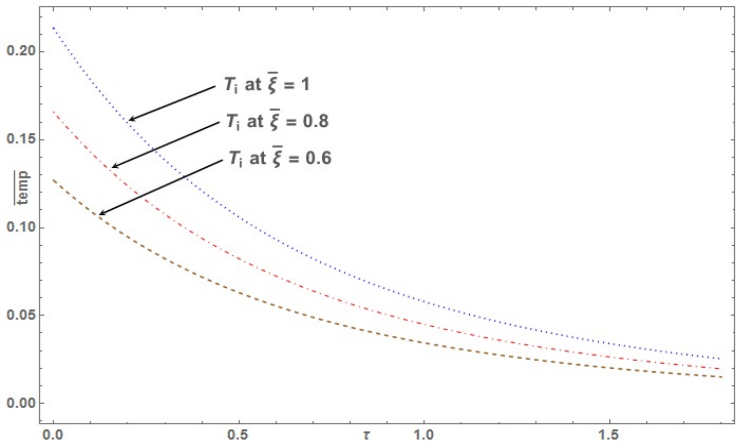

Figure 3. **Temperature distribution along τ for different $\bar{\xi}$**

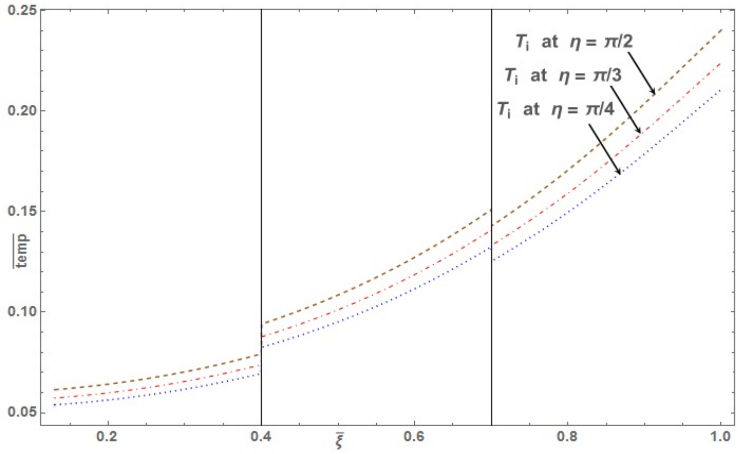

Figure 4. **Temperature distribution along $\bar{\xi}$-direction for different η**

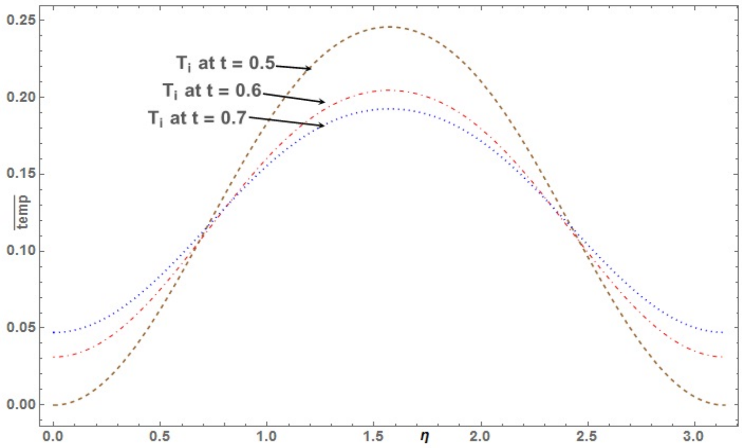

Figure 5. **Temperature distribution along η -direction for different τ**

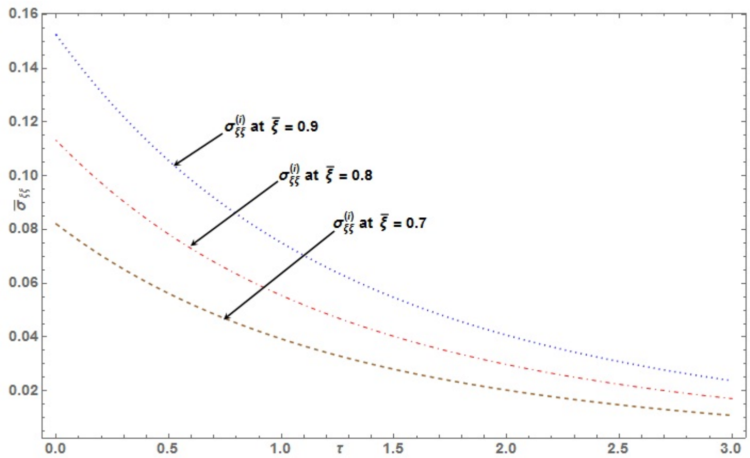

Figure 6. **Axial stress along τ for different $\bar{\xi}$**

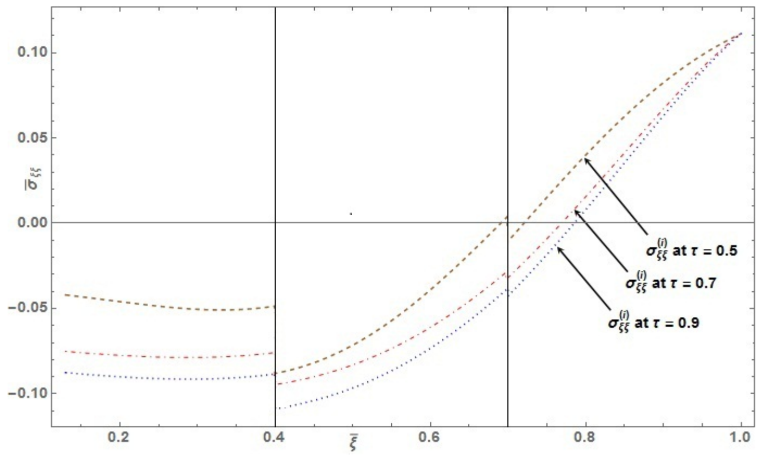

Figure 7. **Axial stress along $\bar{\xi}$- direction for different τ**

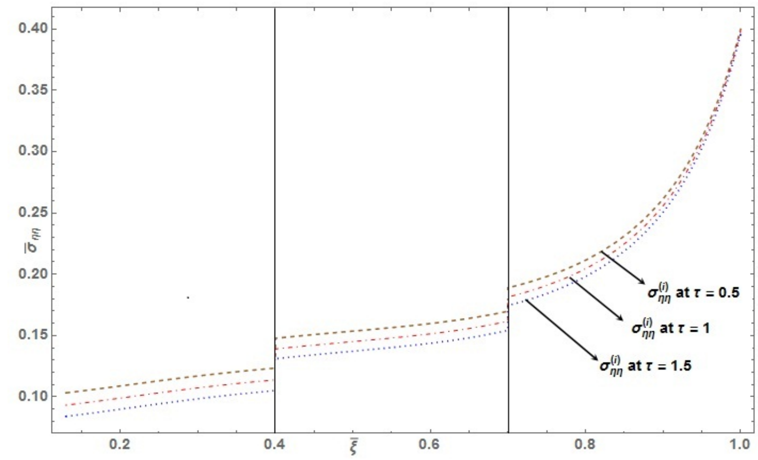

Figure 8. **Tangential stress along $\bar{\xi}$- direction for different τ**

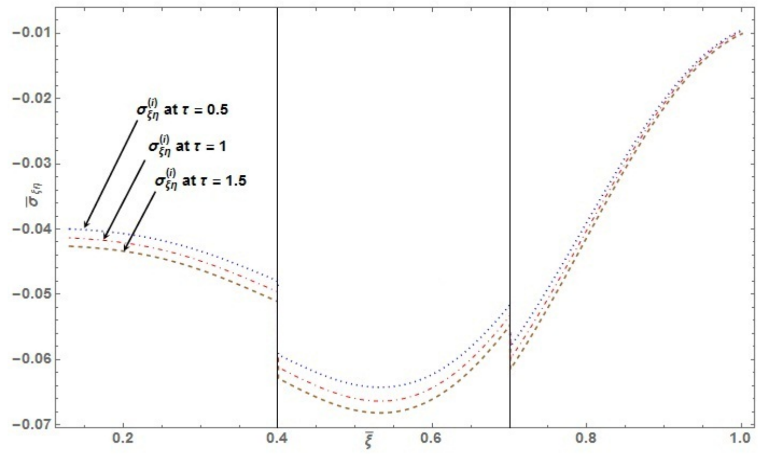

Figure 9. **Shear stress along $\bar{\xi}$- direction for different τ**

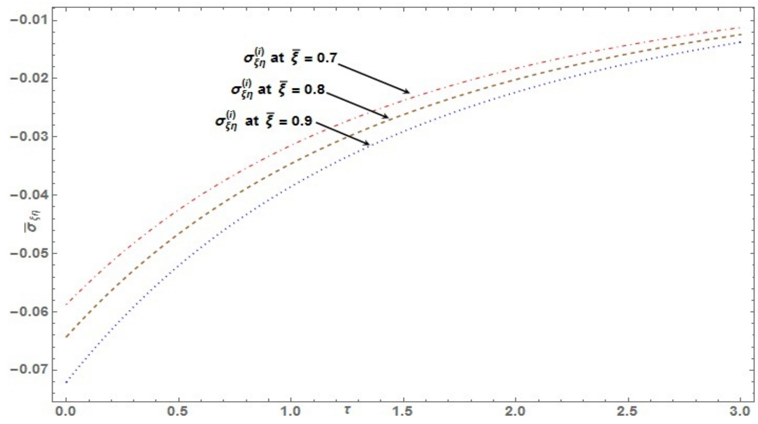

Figure 10. **Shear stress along τ for different $\bar{\xi}$**

Fig. 6 illustrates the axial stress decreases with time. Initially, the axial stress attains maximum expansion due to the accumulation of thermal energy dissipated by sectional heat supply and internal heat energy, which further decreases of time. Fig. 7 indicated that the axial stress along the $\bar{\xi}$- direction is initially on the negative side due to more compressive stress occurring at the inner edge which goes on increasing along $\bar{\xi}$- direction and attains a maximum at the outer edge. Fig. 8 depicts tangential stress along radial direction attaining minimum at inner and maximum at the outer edge. In Fig. 9, shear stress is of negative magnitude, initially, it attains minimum which goes on decreasing at the middle core and suddenly increases towards the outer region. Fig. 10 observed the minimum tangential stress with a negative magnitude which increases gradually with time.

5. Conclusion

The proposed analytical closed-form solution of transient thermal stress problem in an elliptic annulus region was discussed with multiple layers in the radial direction in presence of the time-independent internal heat source in each layer. The solution is investigated using integral transform method establishing Sturm-Liouville integral transform which is analogous to Vodicka's approach considering series expansion function in terms of an eigenfunction to solve the heat conduction partial differential equation in elliptical coordinates. The temperature and thermal stresses for the three-layer elliptic region have been computed numerically and exhibited graphically. The following results were obtained during our research

(1). The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions.

(2). The maximum tensile stress shifting from central core to outer region may be due to heat, stress, concentration or available internal heat sources under considered temperature field.

(3). Finally, the maximum tensile stress occurring in the circular core on the major axis compared to elliptical central part indicates the distribution of weak heating. It might be due to insufficient penetration of heat through the elliptical inner surface.

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Appendix A: THE REQUIRED TRANSFORMATION

1. The transformation and its essential property

Consider a system of equations for composite region consisting of k-layers as

$$\begin{gathered}\alpha_i L[\psi_{i,2n}(\xi)ce_{2n}(\eta)] = -2\beta_i q_{2n,m}(\cosh 2\xi - \cos 2\eta)[\psi_{i,2n}(\xi)ce_{2n}(\eta)],\\ q_{2n,m} \geq 0,\ \xi_i \leq \xi \leq \xi_{i+1},\ i = 1,2,3...k,\ 0 \leq \eta \leq 2\pi\end{gathered} \tag{A1}$$

subjected to the interfacial and other boundary conditions under consideration

$$\left.\begin{aligned}&\alpha_1\,\psi'_{i,2n}(\xi_1)\ - h_0\,\psi_{i,2n}(\xi_1) = 0,\\ &\alpha_i\,\psi'_{i,2n}(\xi_{i+1}) + h_i[\psi_{i,2n}(\xi_{i+1}) - \psi_{i+1,2n}(\xi_{i+1})] = 0,\\ &\alpha_{i+1}\,\psi'_{i+1,2n}(\xi_{i+1}) + h_i[\psi_{i,2n}(\xi_{i+1}) - \psi_{i+1,2n}(\xi_{i+1})] = 0,\\ &\alpha_k\,\psi'_{k,2n}(\xi_{k+1})\ + h_k\,\psi_{k,2n}(\xi_{k+1}) = 0,\ \ i = 1,2,3...(k-1)\end{aligned}\right\} \tag{A2}$$

in which the prime ($'$) denotes differentiation with respect to the ξ variable, $\psi_{i,2n}(\xi)$ is the eigenfunction of the i^{th} layer, $q_{2n,m}$ is the eigenvalue of $[A_{i2n}Ce_{2n}(\xi, q_{2n,m}) + B_{i2n}Fey(\xi, q_{2n,m})]ce_{2n}(\eta) = 0$, $(\alpha_i\,, \beta_i)$ is the characteristics of the i^{th} layer, h_0 is the surface coefficients at $\xi = \xi_1$, h_i is the surface coefficients at $\xi = \xi_{i+1}$, h_k is the surface coefficients at $\xi = \xi_{k+1}$ and $L = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$. Furthermore, $ce_{2n}(\eta,\, q)$ is the ordinary Mathieu function of the first kind of order n [20], $Ce_{2n}(\xi,\, q)$ is a modified Mathieu function of the first kind of order n [20], and the recurrence relations for the Bessel functions Y_{2r}(are identical in form with those for J_{2r}) can be defined as

$$Fey_{2n}(\xi, q) = \frac{ce_{2n}(0,q)}{A_0^{(2n)}}\sum_{r=0}^{\infty} A_{2r}^{(2n)}\, Y_{2r}(2k'\,\sinh\xi)\ \begin{pmatrix}|\sinh\xi| > 1\\ R(\xi) > 0\end{pmatrix}\ [20],$$

with y in Fey signifies the Y-Bessel function and $q = k'^2 = \lambda c^2/4$.

The general solution of equation (A1) is of the form

$$\begin{aligned}\Phi_{i,2n,m}(\xi,\eta) &= \psi_{i,2n}(\xi)ce_{2n}(\eta)\\ &= [A_{i2n}Ce_{2n}(\xi, q_{2n,m}) + B_{i2n}Fey_{2n}(\xi, q_{2n,m})]\, ce_{2n}(\eta)\end{aligned} \tag{A3}$$

satisfying the following boundary conditions, we get a system of $2k$ simultaneous equations so that arbitrary constants A_{i2n} and B_{i2n} can be obtained. Also from this $2k$ system of equations, we get the frequency equation by eliminating, A_{i2n} and B_{i2n}. After substituting the values of A_{i2n} and B_{i2n}, we get the required solution of the Sturm-Liouville problem (A1) subjected to the boundary and interfacial conditions (A2)).

2. Orthogonal property of the eigenfunction $\psi_{i,2n,m}$

If $\Phi_{i,2n,m}$ and $\Phi_{i,2s,p}$ be the solutions of equation (A1) we have

$$\alpha_i L\,\Phi_{i,2n,m}(\xi,\eta) = -2\beta_i q_{2n,m}(\cosh 2\xi - \cos 2\eta)\Phi_{i,2n,m}(\xi,\eta) \tag{A4}$$

$$\alpha_i L\,\Phi_{i,2s,p}(\xi,\eta) = -2\beta_i q_{2s,p}(\cosh 2\xi - \cos 2\eta)\Phi_{i,2s,p}(\xi,\eta) \tag{A5}$$

Multiply equation (A4) by $\Phi_{i,2s,r}(\xi,\eta)$ and equation (A5) by $\Phi_{i,2n,m}(\xi,\eta)$, and then subtracting equation (A5) from equation (A4), leads to

$$\begin{aligned}&\alpha_i\{[\Phi_{i,2s,p}(\xi,\eta)\,\Phi'_{i,2n,m}(\xi,\eta) - \Phi_{i,2n,m}(\xi,\eta)\,\Phi'_{i,2s,p}(\xi,\eta)]'\\ &+[\Phi_{i,2s,p}(\xi,\eta)\,\Phi^{\circ}_{i,2n,m}(\xi,\eta) - \Phi_{i,2n,m}(\xi,\eta)\,\Phi^{\circ}_{i,2s,p}(\xi,\eta)]^{\circ}\}\\ &= -2\beta_i(q_{2n,m} - q_{2s,p})(\cosh 2\xi - \cos 2\eta)\Phi_{i,2n,m}(\xi,\eta)\Phi_{i,2s,p}(\xi,\eta)\end{aligned} \tag{A6}$$

in which the prime ($^\circ$) represents the differentiation with respect to the η variable. Further simplifying the equation (A6), we get

$$\begin{aligned}\alpha_i\left\{[X_i(\xi,\eta)]'+[Y_i(\xi,\eta)]^\circ\right\}&=-2\beta_i(q_{2n,m}-q_{2s,p})(\cosh 2\xi-\cos 2\eta)\\&\times\Phi_{i,2n,m}(\xi,\eta)\Phi_{i,2s,p}(\xi,\eta)\end{aligned}\tag{A7}$$

in which

$$X_i(\xi,\eta)=\begin{vmatrix}\Phi_{i,2n,m}(\xi,\eta) & \Phi'_{i,2n,m}(\xi,\eta)\\ \Phi_{i,2s,p}(\xi,\eta) & \Phi'_{i,2s,p}(\xi,\eta)\end{vmatrix},\ Y_i(\xi,\eta)=\begin{vmatrix}\Phi_{i,2n,m}(\xi,\eta) & \Phi^\circ_{i,2n,m}(\xi,\eta)\\ \Phi_{i,2s,p}(\xi,\eta) & \Phi^\circ_{i,2s,p}(\xi,\eta)\end{vmatrix}$$

Integrating equation (A7) with respect to the variable η within 0 to 2 π and with respect to the variable ξ within ξ_i to ξ_{i+1} $(i=1,2,3...k)$, one yield

$$\begin{aligned}&2\beta_i(q_{2s,p}-q_{2n,m})\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}(\cosh 2\xi-\cos 2\eta)\Phi_{i,2n,m}(\xi,\eta)\Phi_{i,2s,p}(\xi,\eta)\\&=-\alpha_i\left\{\int_0^{2\pi}\left(|X_i(\xi,\eta)|_{\xi_i}^{\xi_{i+1}}\right)d\eta+\int_{\xi_i}^{\xi_{i+1}}\left(|\,Y_i(\xi,\eta)|_0^{2\pi}\right)d\xi\right\}\end{aligned}\tag{A8}$$

Due to periodicity of the function, the second term on the Right Hand Side (R.H.S) of equation (A7) vanishes

$$\begin{aligned}&2\beta_i(q_{2s,p}-q_{2n,m})\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}(\cosh 2\xi-\cos 2\eta)\Phi_{i,2n,m}(\xi,\eta)\Phi_{i,2s,p}(\xi,\eta)\\&=\alpha_i\left\{\int_0^{2\pi}[X_i(\xi_i,\eta)-X_i(\xi_{i+1},\eta)]\,d\eta\right\}\end{aligned}\tag{A9}$$

According to the second and third boundary condition of equation (A2), one obtains

$$\phi_i(\xi_i)-\phi_{i-1}(\xi_i)=\frac{1}{h_{i-1}}\phi'_{i-1}(\xi_i),\phi'_i(\xi_i)=\phi'_{i-1}(\xi_i)\,\text{for all}\,n,m,s,p\tag{A10}$$

Then equation (A9) reduces to

$$\begin{aligned}&2\beta_i(q_{2n,m}-q_{2s,p})\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}(\cosh 2\xi-\cos 2\eta)\Phi_{i,2n,m}(\xi,\eta)\Phi_{i,2s,p}(\xi,\eta)\,d\xi\,d\eta\\&=\int_0^{2\pi}\alpha_i\left(|X_{i-1,2n,m}(\xi_i,\eta)\,|-|\,X_{i,2n,m}(\xi_{i+1},\eta)\,|\right)d\eta\end{aligned}\tag{A11}$$

$$\begin{aligned}&2\beta_i(q_{2s,p}-q_{2n,m})\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}(\cosh 2\xi-\cos 2\eta)\Phi_{i,2n,m}(\xi,\eta)\Phi_{i,2s,p}(\xi,\eta)\,d\xi\,d\eta\\&=\int_0^{2\pi}\alpha_i(D_{i,2n,m,2s,p}-D_{i+1,2n,m,2s,p})\,d\eta\end{aligned}\tag{A12}$$

in which

$$D_{i,2n,m,2s,p}=\begin{vmatrix}\Phi_{i-1,2n,m}(\xi_i,\eta) & \Phi'_{i-1,2n,m}(\xi_i,\eta)\\ \Phi_{i-1,2s,p}(\xi_i,\eta) & \Phi'_{i-1,2s,p}(\xi_i,\eta)\end{vmatrix}$$

and

$$D_{i+1,2n,m,2s,p}=\begin{vmatrix}\Phi_{i,2n,m}(\xi_{i+1},\eta) & \Phi^\circ_{i,2n,m}(\xi_{i+1},\eta)\\ \Phi_{i,2s,p}(\xi_{i+1},\eta) & \Phi^\circ_{i,2s,p}(\xi_{i+1},\eta)\end{vmatrix}.$$

Let us consider that $D_{1,2n,m,2s,p}=0$, $D_{i+1,2n,m,2s,p}=0$ according to the last equation of (A2), then we have

$$\sum_{i=1}^{k}\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}\beta_i(\cosh 2\xi-\cos 2\eta)\,\Phi_{i,2n,m}(\xi,\eta)\,\Phi_{i,2s,p}(\xi,\eta)\,d\xi\,d\eta=0,\ \ n\neq s,m\neq p\tag{A13}$$

If any function $f_i(\xi,\eta)$ is continuous and single-valued in the region $\xi_i\le\xi\le\xi_{i+1}$, $i=1,2,3,...k$, $0\le\eta\le 2\pi$ and satisfying the boundary and interfacial conditions of the eigenvalue problem, then we define the integral transform in the region as given in equation (13). Now at any point within the range [20]

$$f_i(\xi,\eta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\gamma_{2n}\,\psi_{i,2n}(\xi,q)ce_{2n}(\eta,q)\tag{A14}$$

where the constant γ_{2n} is to be determined, and for that multiply equation (A14) by $\psi_{i,2n}(\xi,q)\,ce_{2n}(\eta,q)\,(\cosh 2\xi-\cos 2\eta)$ and integrate with respect to the variable ξ from ξ_i to ξ_{i+1} and with respect to the variable η from 0 to 2π, one obtains

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi}\gamma_{2n}\ \psi_{i,2n}^2(\xi,q)\,ce_{2n}^2(\eta,q)(\cosh 2\xi-\cos 2\eta)d\eta\,d\xi$$
$$=\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi} f_i(\xi,\eta)\,\psi_{i,2n}(\xi,q)ce_{2n}(\eta,q)(\cosh 2\xi-\cos 2\eta)d\eta\,d\xi$$

On mathematical simplification, one obtains

$$\gamma_{2n}==\sum_{i=1}^{k}\frac{\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi} f_i(\xi,\eta)\,\psi_{i,2n}(\xi,q)ce_{2n}(\eta,q)(\cosh 2\xi-\cos 2\eta)d\eta\,d\xi}{\pi\int_{\xi_i}^{\xi_{i+1}}\psi_{i,2n}^2(\xi,q)\,(\cosh 2\xi-\Theta_{2n,m})\,d\xi} \tag{A15}$$

Hence the inversion formula of equation (13) is given by equation (14).

3. Property of the Transform

To solve the problem stated above, we consider the effect of the Sturm-Liouville transform on the expression

$$\alpha_i(f_i''(\xi,\eta)+f_i^{\circ\circ}(\xi,\eta))+2\beta_i(\cosh 2\xi-\cos 2\eta)f_i(\xi,\eta) \tag{A16}$$

Integrating by parts twice, one obtains

$$\sum_{i=1}^{k}\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi}\alpha_i(f_i''(\xi,\eta)+f_i^{\circ\circ}(\xi,\eta))[\psi_{i,2n}(\xi)ce_{2n}(\eta)]\,d\xi\,d\eta$$
$$=-2A_0^{(2n)}[\alpha_i\,\psi_{i,2n}(\xi)f_i'(\xi,\eta)-\alpha_i\,f_i(\xi,\eta)\,\psi_{i,2n}'(\xi)]_{\xi_i}^{\xi_{i+1}}$$
$$+\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi}\alpha_i f_i(\xi,\eta)\,[\psi_{i,2n}(\xi)ce_{2n}(\eta)]''\ d\eta\,d\xi$$
$$+\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi}\alpha_i f_i(\xi,\eta)[\psi_{i,2n}(\xi)ce_{2n}(\eta)]^{\circ\circ}d\eta\,d\xi$$

After some simplification, we get the expression

$$\sum_{i=1}^{k}\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi}\alpha_i\,(f_i''(\xi,\eta)+f_i^{\circ\circ}(\xi,\eta))[\psi_{i,2n}(\xi)ce_{2n}(\eta)]\,d\xi\,d\eta$$
$$=-2A_0^{(2n)}\left\{[\alpha_k\,\psi_{k,2n}(\xi)f_k'(\xi,\eta)-\alpha_k\,f_k(\xi,\eta)\,\psi_{k,2n}'(\xi)]_{\xi=\xi_{k+1}}\right.$$
$$+[\alpha_i\,\psi_{i,2n}(\xi)f_i'(\xi,\eta)-\alpha_i\,f_i(\xi,\eta)\psi_{i,2n}'(\xi)]_{\xi=\xi_{i+1}}$$
$$-[\alpha_{i+1}\,\psi_{i+1,2n}(\xi)f_i'(\xi,\eta)-\alpha_{i+1}\,f_{i+1}(\xi,\eta)\,\psi_{i+1,2n}'(\xi)]_{\xi=\xi_{i+1}}$$
$$\left.-\ [\alpha_1\,\psi_{1,2n}(\xi)f_1'(\xi,\eta)-\alpha_1\,f_1(\xi,\eta)\,\psi_{1,2n}'(\xi)]_{\xi=\xi_1}\right\}-2q_{2n,m}\bar{f}_i(q_{2n,m})$$

Using the boundary and interfacial conditions, one obtains

$$\begin{aligned}&\sum_{i=1}^{k}\int_{\xi_i}^{\xi_{i+1}}\int_0^{2\pi}\alpha_i(f_i''(\xi,\eta)+f_i^{\circ\circ}(\xi,\eta))[\psi_{i,2n}(\xi)ce_{2n}(\eta)]\,d\xi\,d\eta\\&=-2A_0^{(2n)}\{\psi_{k,2n}(\xi_{k+1})[\alpha_k\,f_k'(\xi_{k+1},\eta)+h_k\,f_k(\xi_{k+1},\eta)]\\&+\psi_{i,2n}(\xi_{i+1})[\alpha_i\,f_i'(\xi_{i+1},\eta)-h_i\,(f_{i+1}(\xi_{i+1},\eta)-f_i(\xi_{i+1},\eta))]\\&-\psi_{i+1,2n}(\xi_{i+1})[\alpha_{i+1}\,f_{i+1}'(\xi_{i+1},\eta)-h_i\,(f_{i+1}(\xi_{i+1},\eta)-f_i(\xi_{i+1},\eta))]\\&-\psi_{1,2n}(\xi_1)\,[\alpha_1\,f_1'(\xi_1,\eta)-h_0\,f_1(\xi_1,\eta)]\}-2q_{2n,m}\bar{f}_i(q_{2n,m})\end{aligned} \tag{A17}$$

Hence equation (A17) is the fundamental property of the Sturm-Liouville transform for the composite region defined in equation (13), which removes a group of terms quoted in equation (A16).