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Inflection Point Control by Using Rational Cubic Spline

M. Dube[1] and Jagriti Sharma[2,*]

1 Department of Mathematics and Computer Science, Rani Durgavati University, Jabalpur, India.

2 Department of Mathematics, Nachiketa College of Computer Science, Commerce and Advanced Technology, Jabalpur, India.

Abstract: Consider the rational spline of the form cubic / quadratic with two shape parameters. The shape control of rational interpolation such as value point control, inflection point control and convexity control at a point is studied. The main aim of the paper is to control the shape of interpolating curve. Also, the numerical examples are given to validate the proposed interpolation technique.

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1. Introduction

In Computer Aided Geometric Design (CAGD) the construction of curves and surfaces is an important part. The spline interpolation play a crucial role to design the curves and surfaces. In the solution of many interpolation problems which occur in industrial designing and manufacturing, the given data must preserve some shape properties such as positivity, monotonicity and convexity. Many mathematician, computer scientist and engineers (cf. [3, 5–8]) are working on the shape preserving methods. The Rational splines have very important role in computer graphics, because they provide an exact representation for quadratic curves such as circle and ellipse. Now a days, the rational splines with parameters have received more attention in the area of shape preservation. Shape control became an even more important task in constructing curves and surface such as convexity control, monotonicity control and positivity control.

Butt and Brodlie [1] presented shape preservation of the curve by interval subdivision technique. Fahr and Kalley [6] considered a monotone rational B-Spline of degree one to preserve the shape of monotone data. Beside the shape control of curve many researcher give the attention on local control of interpolating curve. Duan et al. [4] have discussed the point control of the interpolating function of the form cubic/quadratic with two parameters and Duan at al.[2] have presented the local control of interpolating function of the form cubic/ linear.

Karim and Kong [8] discussed the local control of interpolating function by using rational cubic spline (cubic/quadratic) with three parameters and discussed the bounded properties of the rational cubic interpolants and shape controls of the rational interpolants. Also, they constructed the value control, inflection point control and convexity control at a point by using the rational cubic spline.

* *E-mail: jagriti.sharma87@gmail.com*

The rational spline of the form cubic/quadratic with two shape parameters is considered, which is an extension of idea given by Karim and Kong [7]. By using the above mentioned idea, the condition of inflexion point control is presented by suitable value of the parameters. We establish the error bound value control by applying the rational spline of the form cubic/quadratic with two shape parameters. Also, we present necessary and sufficient condition for inflexion point control and convexity control of the curve.

The paper is organised as follows: Section 2 discusses interpolatory conditions. Section 3 presents error bound. Section 4 describes value control of the rational spline interpolant. Section 5 establishes inflection point and convexity control of the curves. The last Section concludes the paper.

2. Interpolation

Let $\{(x_i, f_i, d_i); i = 0, 1, ..., n\}$ be a given set of points $a = x_0 < x_1 < ... < x_n = b$ is the knot spacing, where f_i and d_i are the values of the interpolating function $f(x)$ and its first derivatives at the knots $x_i, i = 0, 1, 2, ..., n-1$ respectively.
Consider $p_i(x)$ as cubic polynomial and $p_i(x)$ as quadratic polynomial, then C^1 continuous piecewise rational cubic spline with quadratic denominator is defined as follows:

$$S(x) = \frac{p_i(x)}{q_i(x)}; \quad i = 0, 1, ..., n-1; \quad x \in [x_i, x_{i+1}]$$

$$S(x) = \frac{(1-\theta)^2 v_i A_i + \theta^2(1-\theta)B_i + \theta(1-\theta)^2 C_i + \theta^2 w_i D_i}{(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^3 w_i} \tag{1}$$

where

$$\theta(x) = \frac{x - x_i}{h_i}, \qquad h_i = x_{i+1} - x_i \tag{2}$$

and $A_i = f_i$, $B_i = f_{i+1} - w_i h_i d_{i+1}$, $C_i = f_i + v_i h_i d_i$, $D_i = f_{i+1}$. Which satisfies the following interpolatery conditions

$$S(x_i) = f_i; \quad S(x_{i+1}) = f_{i+1}$$
$$S'(x_i) = d_i; \quad S^{'}(x_{i+1}) = d_{i+1}$$

3. Error Bound

In order to solve the bounded property of the interpolating function, the interpolating function $S(x)$ in (1) can be rewritten as follows:

$$S(x) = \xi_0(\theta, v_i, w_i) f_i + \xi_1(\theta, v_i, w_i) f_{i+1} + \xi_2(\theta, v_i, w_i) d_i h_i + \xi_3(\theta, v_i, w_i) d_{i+1} h_i \tag{3}$$

where

$$\begin{aligned}
\xi_0 &= \frac{(1-\theta)^2 v_i + \theta(1-\theta)^2}{(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2 w_i} \\
\xi_1 &= \frac{\theta^2(1-\theta) + \theta^2 w_i}{(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2 w_i} \\
\xi_2 &= \frac{\theta(1-\theta)^2 v_i}{(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2 w_i} \\
\xi_3 &= \frac{-\theta^2(1-\theta) w_i}{(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2 w_i}
\end{aligned}$$

Here $\xi(\theta, v_i, w_i)$ are interpolating bases with $\theta \in (0, 1)$. The interpolation bases have the following properties:

(a). $\xi_0(\theta, v_i, w_i) + \xi_1(\theta, v_i, w_i) = 1$

(b). $\xi_2(\theta, v_i, w_i) - \xi_3(\theta, v_i, w_i) = \frac{\theta(1-\theta)[(1-\theta)v_i+\theta w_i]}{(1-\theta)^2 v_i+\theta^2(1-\theta)+\theta(1-\theta)^2+\theta^2 w_i}$

(c). $\xi_0(\theta, v_i, w_i) \geq 0,\ \ \xi_1(\theta, v_i, w_i) \geq 0,\ \xi_2(\theta, v_i, w_i) \geq 0,\ \ \xi_3(\theta, v_i, w_i) \leq 0$

Here we easily see that no matter what the value of positive parameters v_i and w_i might be, the interpolating function defined in (1) is bounded on the interval [a, b].

Theorem 3.1. *Let $S(x)$ be the interpolation function defined by (1) on $[x_i, x_{i+1}]$ and denotes*

$$F = \max_{[j=i,i+1]} |f_j|, \quad D = \max_{x\in[x_i,x_{i+1}]} |f^{'}(x)| \tag{4}$$

satisfy the following condition:

$$|S(x)| \leq F + h_i D \tag{5}$$

Proof. From equation (3) following expression can be written as follows:

$$|S(x)| \ \leq \ |\xi_0(\theta, v_i, w_i) f_i| + |\xi_1(\theta, v_i, w_i) f_{i+1}| + |\xi_2(\theta, v_i, w_i) h_i d_i| + |\xi_3(\theta, v_i, w_i) h_i d_{i+1}| \tag{6}$$

The above inequality may be written as

$$|S(x)| \ \leq \ F + \frac{\theta(1-\theta)[(1-\theta)v_i+\theta w_i]}{(1-\theta)^2 v_i+\theta^2(1-\theta)+\theta(1-\theta)^2+\theta^2 w_i} h_i D$$

Let

$$\begin{aligned}
r(\theta) &= \frac{\theta(1-\theta)^2 v_i + \theta^2(1-\theta) w_i}{(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2 w_i} \\
&= \frac{\theta(1-\theta)[(1-\theta) v_i + \theta w_i]}{(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2 w_i} \\
&\leq \frac{\theta(1-\theta)[(1-\theta) v_i + \theta w_i]}{(1-\theta)^2 v_i + \theta^2 w_i} \\
&= \frac{\theta(1-\theta)(1-\theta) v_i}{(1-\theta)^2 v_i + \theta^2 w_i} + \frac{\theta(1-\theta)\theta w_i}{(1-\theta)^2 v_i + \theta^2 w_i} \\
&\leq \frac{\theta(1-\theta)(1-\theta) v_i}{(1-\theta)^2 v_i} + \frac{\theta(1-\theta)\theta w_i}{\theta^2 w_i} \\
&= \theta + 1 - \theta \\
&= 1
\end{aligned}$$

□

4. Value Control of The Rational Spline Interpolant

The shape of the interpolating curve can be changed without changing the value of the function, the shape of the interpolant curve can be modified by selecting suitable parameters. If the value of interpolating function $S(x)$ at point x^* where $x^* \in [x_i, x_{i+1}]$ equal to real number M i.e $S(x^*) = M$ where M is any value between f_i and f_{i+1}, then this type of control is called value control of the interpolation. From equation (3) we have control equation for $x = x^*$ where$x^* \in [x_i, x_{i+1}]$ is

$$M = \xi_0(\theta^*, v_i, w_i) f_i + \xi_1(\theta^*, v_i, w_i) f_{i+1} + \xi_2(\theta^*, v_i, w_i) d_i h_i + \xi_3(\theta^*, v_i, w_i) d_{i+1} h_i \tag{7}$$

equation (7) is equivalent to the following

$$K_1 v_i + K_2 w_i + K_3 = 0 \tag{8}$$

where

$$\begin{aligned} K_1 &= (1-\theta^*)^2[M - f_i - h_i d_i \theta^*] \\ K_2 &= {\theta^*}^2[M - f_{i+1} + (1-\theta^*)h_i d_{i+1}] \\ K_3 &= \theta^*(1-\theta^*)[M - f_i(1-\theta^*) - f_{i+1}\theta^*] \end{aligned}$$

Here θ^* is known local coordinate. We see that parameters v_i and w_i are positive and satisfy the equation (8), therefore we have the following theorem on function value control.

Theorem 4.1. *Let $S(x)$ which is defined by (1) be the interpolation function over the interval $[x_i, x_{i+1}]$ and let $x^* \in [x_i, x_{i+1}]$, M is a real number, then the sufficient condition for the existence of the positive parameters v_i and w_i satisfy $S(x^*) = M$ is that $K_1 \times K_2 \times K_3 < 0$.*

Example 4.2. *We consider the interpolation interval [0,1] to test the value control by using the interpolation data from paper of Duan et al. [3]. If the value of one parameter is known, then by using condition which is given in equation (8), we can easily evaluate the value of another parameter which satisfies the condition of value control. In order to understand the concept of value control, we have many examples shown below.*

Let $v_i = 1$ and $w_i = 1$ interpolating function $S(x)$ in interval [0,1] be given by

$$S_1(x) = \frac{-2x^3 + 9x^2 - 4x + 3}{x^2 - x + 1} \tag{9}$$

whose value at 0.5 is 4 then equation (8) becomes $3v_i - 2w_i - 1 = 0$.

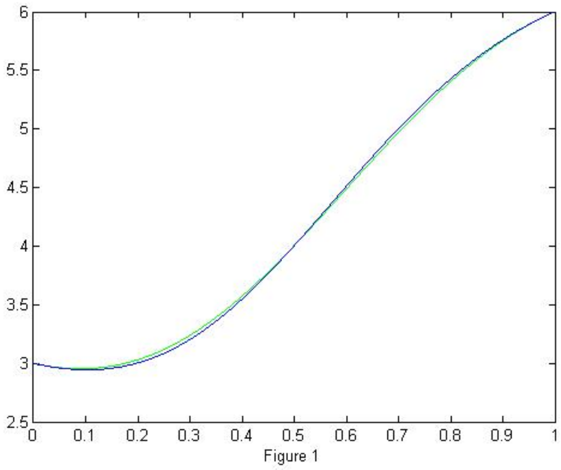

Figure 1. **The graph of $S_1(x)$ and $S_2(x)$**

It may verify that $S(0.5) = 4$. Other choice of parameters which satisfy (8) are given by $v_i = 1.5$, $w_i = 1.75$, then $S(x)$ becomes

$$S_2(x) = \frac{-x^3 + 14.5x^2 - 7.5x + 4.5}{2.25x^2 - 2x + 1.5} \tag{10}$$

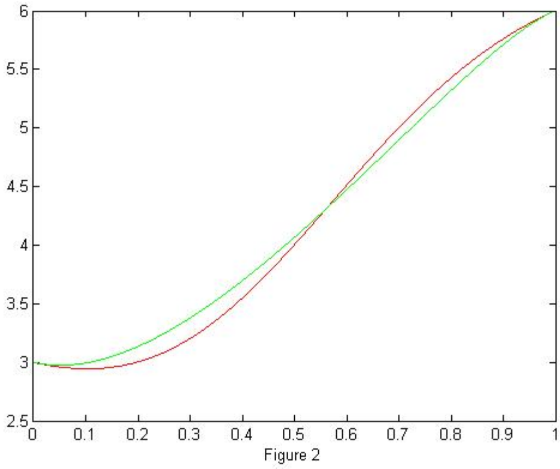

Figure 2. **The graph of interpolating curve $S_2(x)$(red) and interpolating curve(green)**

Clearly, the resulting interpolating functions are similar to each other, that is no matter what the value of positive parameter might be as they satisfy equation (8). Figure 1 shows the graph of $S_1(x)$ and $S_2(x)$. Similarly, Figure 2 shows that the graph of interpolating curve $S_2(x)$ (red) and interpolating curve(green). Now if parameters v_i and w_i satisfy the condition (8), then to find the value of M we have following procedure. If value of interpolation function $S(x)$ at 0.5 is M i.e $S(0.5) = M$ then

$$S(0.5, v_i, w_i) = \frac{5v_i + 10w_i + 9}{2(v_i + w_i + 1)} \tag{11}$$

If we choose $w_i = 2.5$ then, we have

$$S(0.5, v_i) = \frac{5v_i + 34}{2v_i + 7}$$

Now if we require $S(0.5) = M$, then

$$M = \frac{5v_i + 34}{2(v_i + 7)} \tag{12}$$

from equation (12)

$$v_i = \frac{7M - 34}{5 - 2M} \tag{13}$$

When we choose $w_i = 2.5$ and $v_i > 0$, we have the following constraints for the value of M. Also, $2.5 < M < 4.8$ for $v_i > 0$. Now if we choose $v_i = 2$, then

$$\begin{aligned} S(0.5, w_i) &= \frac{19 + 10w_i}{6 + 2w_i} \\ M &= \frac{19 + 10w_i}{6 + 2w_i} \end{aligned} \tag{14}$$

from (14)

$$w_i = \frac{6M - 19}{10 - 2M}$$

Now, if we choose $v_i > 0$ and $w_i > 0$ then the value of M is lying between 3.1 and 5. We may obtain different values for the ranges of M by selecting different values of the parameters.

5. Inflection Point and Convexity Control of Curves

The convexity control of an interpolant curve depends on the second order derivative of the interpolating function. The second derivative of the rational cubic spline defined in (1) can be defined as follows

$$S''(x) = \frac{\sum_{i=0}^{5} \eta_i}{h_i[(1-\theta)^2 v_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2 w_i]^3}$$

where

$$\begin{aligned}
\eta_0 &= 2v_i^2(\Delta_i - d_i)(1-\theta)^3 \\
\eta_1 &= -2v_i^2 w_i(d_{i+1} - \Delta_i)(1-\theta)^2 \\
\eta_2 &= 2v_i^2 w_i(d_{i+1} - 3d_i + \Delta_i)\theta(1-\theta)^2 \\
\eta_3 &= 6v_i w_i^2 d_{i+1}\theta^2(1-\theta) \\
\eta_4 &= -6v_i w_i^2 \Delta_i d_i \theta^2 \\
\eta_5 &= 2w_i^2[(d_{i+1} - \Delta_i) + v_i(d_i + 2\Delta_i)]\theta^3
\end{aligned}$$

Let $R(\theta) = \sum\limits_{i=0}^{5} \eta_i$, then

$$\begin{aligned}
R(0) &= 2v_i^2(\Delta_i - d_i) \\
R(1) &= 2w_i^2[(d_{i+1} - \Delta_i) + v_i(d_i + 2\Delta_i)]
\end{aligned}$$

If $R(0)R(1) < 0$, the interpolation curve $S(x)$ must have an inflection point for $\theta \in (0,1)$. Therefore, we have the following theorem:

Theorem 5.1. *For the given interpolation data let $S(x)$ be the interpolation function defined by (1) in $[x_i, x_{i+1}]$. the sufficient and necessary condition for a point $x^*, s(x^*)$ to be an inflection point is that the positive parameter v_i, w_i must satisfy the condition*

$$R(\theta^*) = 0 \tag{15}$$

Similarly if the design require the convex point or concave point at x^* the the positive parameters must satisfy the condition

$$R(\theta^*) > 0 \qquad (for\ convex) \tag{16}$$

$$R(\theta^*) < 0 \qquad (for\ concave) \tag{17}$$

Example 5.2. *Let $f(x)$ be the interpolated function for the interpolation data given in Duan et al. [3]. For $v_i = 1$ and $w_i = 1$, let us suppose that $S_1(x)$ is the interpolation function defined by (9) on the interpolation interval [0,1]. Now we see that control of the inflection point and convexity of interpolation curve.*

$$S_1(x) = \frac{-2x^3 + 9x^2 - 4x + 3}{x^2 - x + 1} \tag{18}$$

Second derivative is given by

$$S_1''(x) = \frac{10x^3 - 24x^2 - 6x + 10}{(x^2 - x + 1)^3} \tag{19}$$

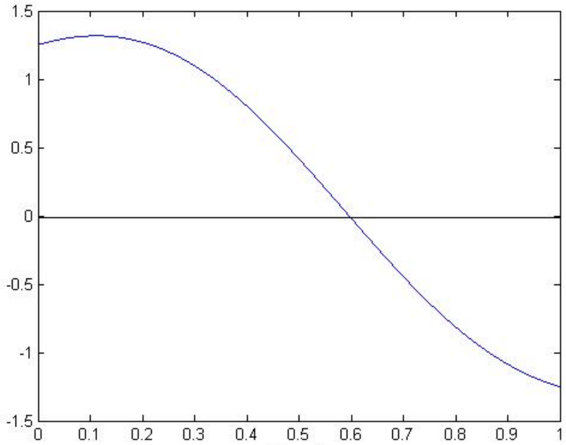

Figure 3. **Graph of** $S_1''(x)$

After simple calculation, we have $S''(0) = 10$ and $S''(0) = -10$, and the root of the equation $S''(0)$ in [0,1] is $x^* = 0.5967$. As $0 \leq x \leq x^*, S_1(x)$ is convex while as $x^* \leq x \leq 1, S_1(x)$ is concave. This implies that the point $(x^*, S_1(x^*))$ is the inflection point of interpolant curve $S_1(x)$. Suppose that if we require the inflection point at the point $x = 0.7$, then the parameters v_i, w_i must satisfy the condition $R(0.7, v_i, w_i) = 0$ for this purpose let us consider $w_i = 1$, then $R(0.7, v_i)$ becomes $R(0.7, v_i) = 2.3825v_i^2 - 3.626v_i - 0.686$. Hence, we get the positive root $v_i = 1.6924$ of the equation $R(0.7, v_i)$. After simplification, we have $S_4(x)$ which is the interpolating function given by

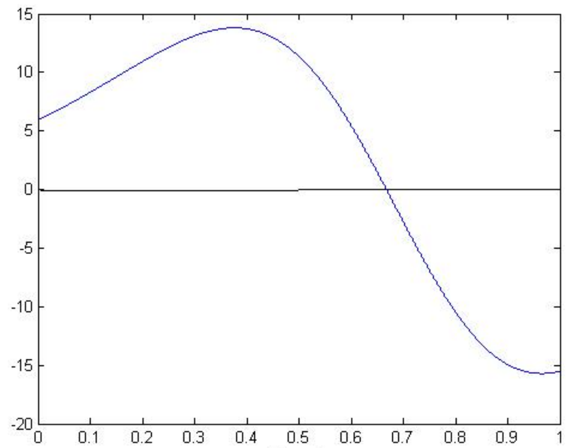

Figure 4. **Graph of** $S_3''(x)$

$$S_3(x) = \frac{-2.6924x^3 + 12.462x^2 - 8.8468x + 5.0772}{1.6924x^2 - 2.3848x + 1.6924}$$

The graph of the curve $S_1''(x)$ has been shown in Figure 3 and Figure 4 shows the graph of $S_3''(x)$. By referring Figure 4, it may be noted that $S_3''(x)$ has an inflection point at $x = 0.7$ and graph of $S_3''(x)$ is convex on [0, 0.7] and concave on [0.7,1].

6. Conclusion

The rational cubic spline with quadratic denominator has been considered. The spline interpolation has two free parameters v_i, w_i. By considering, rational spline of the form cubic/quadratic, we have studied the shape control of rational interpolation such as value point control, inflection point control and convexity control at a point. It is interesting to note that the shape of curve can be adjusted as we desire by simply altering the value of shape parameters without changing the data points. The numerical examples show that the capability of the method as well as it indicates that the scheme of interpolation works well.

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