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Strong Edge Coloring of Some Classes of Unicyclic Graphs

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Abstract: Let G be an undirected simple graph. A strong edge coloring of a graph G is a function $f : E \rightarrow \{1, 2, \ldots, k\}$ such that $f(e_1) \neq f(e_2)$ whenever e_1 and e_2 lie within distance 2 from each other. In other words, no two edges lie on a path of length 3 receive same colors. The smallest number of colors essential for strong edge coloring of a graph G is entitled as strong chromatic index and is represented by $\chi'_s(G)$. In this paper, we investigate strong chromatic index of some classes of unicyclic graphs.

MSC: 05C15, 05C40.

Keywords: Strong edge coloring, strong chromatic index, unicyclic graph.

1. Introduction

The concept of strong edge coloring was first introduced by Fouquet and Jolivet [3] for cubic planar graphs. Strong edge coloring of graphs discovers its application in the area of frequency assignment. The problem of frequency assignment arises when different radio transmitters function in the same geographical region interfere with one another when assigned the same or closely related frequency channels. This situation is common in a wide variety of real world applications related to mobile or general wireless networks and can be modelled as a problem in strong edge coloring. The problem of finding $\chi'_s(G)$ of a graph is NP -complete [4]. One of the open problems proposed by Erdos and Nesetril [2] is as follows. If G is a simple graph with maximum degree Δ, then $\chi'_s(G) \leq (5\Delta^2 - 2\Delta + 1)/4$ if Δ is odd and $\chi'_s(G) \leq 5\Delta/4$ if Δ is even. This conjecture is true for $\Delta \leq 3$. They asked if there is any $\in > 0$ such that, for every such G , $\chi'_s(G) \leq (2- \in)\Delta^2$. Chung et al [1] studied extensively the upper bound of strong chromatic index. They showed that the upper bounds are exactly the numbers of edges in $2K_2$ free graphs. Recently strong edge coloring was studied by several authors as well.

2. Characterization of Unicyclic Graphs

Definition 2.1. *A unicyclic graph is a connected graph that contains precisely one cycle.*

Lemma 2.2. *For any unicyclic graph G with cycle C_3, $\chi'_s(G) = |E(G)|$ if and only if every vertex not in C_3 is a leaf.*

Proof. Since G contains the cycle C_3, the three edges of C_3 should be assigned with three distinct colors for strong edge coloring. Assume that $\chi'_s(G) = |E(G)| \Leftrightarrow$ The minimum number of colors required for strong edge coloring of G is equal

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to the number of edges of the graph $G \Leftrightarrow$ The distance $d(e_i, e_j) < 2$ for any two edges e_i and e_j, $(i \neq j)$ of $G \Leftrightarrow$ Since G is unicyclic and the distance $d(e_i, e_j) < 2$ for any two edges e_i and e_j, $(i \neq j)$ of G, implying that the edges incident at the cycle vertices other than the cycle edges must be the pendent edges $\Leftrightarrow$ Vertices which are not in C_3 must be the leaf vertices. Hence the proof. □

Lemma 2.3. *Let G be a unicyclic graph with cycle C_4 and $\Delta = 3$. Then $5 \leq \chi_s'(G) \leq 6$.*

Proof. Assume the contrary that $\chi_s'(G) < 5$. Let $C_4 = (u_1, u_2, u_3, u_4, u_1)$. For strong edge colouring of C_4, the edges of C_4 should be colored as $c(u_1u_2) = c_1$, $c(u_2u_3) = c_2$, $c(u_3u_4) = c_3$ and $c(u_4u_1) = c_4$. It is given that $\Delta = 3$. Let e_1 and e_2 be the edges incident at u_1 and u_2 respectively as shown in Figure 1. The edges e_1 and e_2 lie within distance two from each other implying that they should be assigned with two new colors c_5 and c_6. Similarly, if e_1 and e_2 are the edges incident at u_1 and u_3 respectively. Then the edges e_1 and e_2 can be assigned with a new color c_5 as the distance between e_1 and e_2 is exactly 2. This is a contradiction to our assumption that $\chi_s'(G) < 5$. It is obvious that $\chi_s'(G)$ cannot be greater than 6 as $\Delta = 3$. Thus $5 \leq \chi_s'(G) \leq 6$.

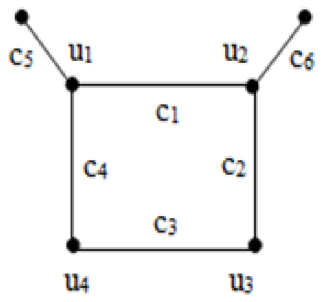

Figure 1. **Unicyclic graph with $\chi_s'(G) = 6$**

□

Lemma 2.4. *Let G be a unicyclic graph with cycle C_5 and $\Delta = 3$. Then $\chi_s'(G) = 5$.*

Proof. Let $V(C_5) = \{v_i | 1 \leq i \leq 5\}$ and $E(C_5) = \{v_iv_{i+1} | 1 \leq i \leq 4\} \bigcup \{v_5v_1\}$. The cycle C_5 requires 5 colors for strong edge coloring. Attaching a pendent edge at v_i, $1 \leq i \leq 5$, will not increase the number of colors as the pendent edge incident at the cycle vertex acquire color from the cycle edge that is two distance away from it. Thus $\chi_s'(G) = 5$. □

Theorem 2.5. *Let G be a unicyclic graph with cycle C_n, $n \geq 6$ and $\Delta = 3$. Then $\chi_s'(G) \leq 6$.*

Proof. Let $V(C_n) = \{v_i \,|1 \leq i \leq n\}$ and $E(C_n) = \{e_i = v_iv_{i+1} \,|1 \leq i \leq n-1\} \bigcup \{e_n = v_nv_1\}$. The following cases arise.

Case (1): $n \equiv 0(mod\ 3)$. In this case $\chi_s'(C_n) = 3$. Also, n can be even or odd. If n is even, then attaching edges to the cycle vertices will increase the number of colors by at most 2. Thus $\chi_s'(G) \leq 5$. If n is odd, then attaching edges to the cycle vertices will increase the number of colors by at most 3. Consequently, $\chi_s'(G) \leq 6$.

Case (2): $n \equiv 1(mod\ 3)$. In this case $\chi_s'(C_n) = 4$. Also, n can be even or odd. In both the cases, attaching edges to the cycle vertices will increase the number of colors by at most 2. Thus $\chi_s'(G) \leq 6$.

Case (3): $n \equiv 2(mod\ 3)$. In this case $\chi_s'(C_n) = 4$. Also, n can be even or odd. In both the cases, attaching edges to the cycle vertices will increase the number of colors by at most 2. Thus $\chi_s'(G) \leq 6$.

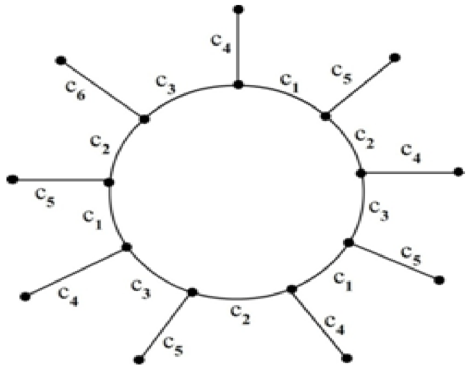

Figure 2. **Strong edge coloring of unicyclic graph with C_9 and $\Delta = 3$**

From the above cases, it is clear that $\chi'_s(G) \leq 6$. See Figure 2. Let $U_{n,r}$ be a unicyclic graph of order n obtained from the two vertex disjoint graphs C_r and P_{n-r} by adding an edge that joins a vertex of C_r to an end vertex of P_{n-r}. □

Theorem 2.6. *Let G be a unicyclic graph $U_{n,r}$. For $n > r > 5$, $4 \leq \chi'_s(G) \leq 5$.*

Proof. Consider the following cases.

Case (1): $r \equiv 0(mod\ 3)$. In this case, $\chi'_s(C_r) = 3$. Further, the edge joining C_r and P_{n-r} should be assigned with a new color and the edges of the path P_{n-r} acquire the colors that are already assigned to the cycle edges. Thus $\chi'_s(G) = 4$.

Case (2): $r \equiv 1(mod\ 3)$. Then the cycle C_r requires 4 colors for strong edge coloring. The edges of the path P_{n-r} should acquire colors from the edges of the cycle C_r. The edge joining C_r and P_{n-r} can be colored according as $r \equiv 0(mod\ 4)$ or $r \not\equiv 0(mod\ 4)$. If $r \equiv 0(mod\ 4)$, then the edge joining C_r and P_{n-r} should be assigned with a new color. Thus $\chi'_s(G) = 5$. Otherwise, the edge joining C_r and P_{n-r} as well as the edges of the path P_{n-r} can acquire colors from the colors that are assigned to the edges of the cycle. In this case $\chi'_s(G) = 4$.

Case (3): $r \equiv 2(mod\ 3)$. In this case, the cycle C_r requires 4 colors for strong edge coloring. The edge joining C_r and P_{n-r} can be colored according as $r \equiv 0(mod\ 4)$ or $r \not\equiv 0(mod\ 4)$. If $r \equiv 0(mod\ 4)$, then the edge joining C_r and P_{n-r} should be assigned with a new color. The edges of the path P_{n-r} should acquire colors from the colors already assigned to the edges of the cycle C_r. Thus $\chi'_s(G) = 5$. If $r \not\equiv 0(mod\ 4)$, then the edge joining C_r and P_{n-r} as well as the edges of P_{n-r} can acquire colors from the colors that are already assigned to the edges of the cycle as given in Figure 3 In this case $\chi'_s(G) = 4$. Thus, from the above discussion $4 \leq \chi'_s(G) \leq 5$.

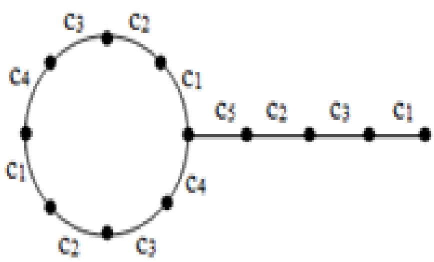

Figure 3. **Strong edge coloring of $U_{12,8}$**

□

3. Conclusion

In this paper, we characterized unicyclic graphs. It would be interesting to study the strong chromatic index for cycle related graphs and interconnection networks.

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