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Revelation of Nano Topology in Čech Rough Closure Spaces

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Abstract: The purpose of this paper is to derive Nano topology in terms of Čech rough closure operators. In addition to this, we also establish the continuous functions on Čech rough closure space and its properties. Finally from these, we evolve a Čech nano topological space that satisfies the topological axioms on infinite universe.

1. Introduction

The aim of this paper is to drive a Nano Topology using Čech rough closure space. To achieve our purpose we make use of the following concepts. (a) Čech closure space: The seed of Čech closure space was first shown by E. Čech in 1966 [1, 2]. Henceforth many other researchers [8, 9] set their minds in this theory and developed it to a new hight. (b) Rough set Theory: Pawlak.Z [6] derived and gave shape to Rough set theory in terms of approximation using equivalence relation known as indiscernibility relation. (c) Nano Topology: Lellis Thivagar and Carmel Richard [4] further enhanced the rough set theory into a topology, called Nano Topology, which has at most five elements in it and he [5] also extended this theory into a multi granular nano topology. Hence using the above concepts we establish a new topology called Čech Nano Topology in term of Čech rough closure operators. Also we proceed to derive the related properties of Čech rough continuous function on the Čech rough closure space. To strengthen the theory suitable examples are sited.

2. Preliminaries

We shall recall here some definitions and concepts that are basics for the proceedings of this paper.

Definition 2.1 ([1]). *A function $\mathcal{C} : P(X) \longrightarrow P(X)$, where $P(X)$ is a power set of a set X, is called a Čech closure operator for X provided the following conditions are satisfied:*

(1). $\mathcal{C}(\emptyset) = \emptyset$

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(2). $A \subset \mathcal{C}(A)$ for each $A \subset X$

(3). $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$ for each $A, B \subset X$.

Then the pair $(X, \mathcal{C})$, where X is a non-empty set and $\mathcal{C}$ is a Čech closure operator for X, is called a Čech closure space. If $(X, \mathcal{C})$ is a Čech closure space and $A \subset X$, then $\mathcal{C}(A)$ is called the closure of A in $(X, \mathcal{C})$. The Čech closure space $(X, \mathcal{C})$ is said to be Kuratowski(topological) space, if $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$ for each $A \subset X$.

Definition 2.2 ([6]). *Let U be a non empty set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements that belong to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

(1). The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup\limits_{x \in U} \{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.

(2). The upper approximation of X with respects to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup\limits_{x \in U} \{R(x) : R(x) \cap X \neq \emptyset\}$.

(3). The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.3 ([6]). *If (U, R) is an approximation space and $X \subseteq U$ is said to be a rough set (inexact) with respect to R if $B_R(X) = U_R(X) - L_R(X) \neq \emptyset$, that is, $U_R(X) \neq L_R(X)$. Otherwise the set X is said to be crisp (exact) with respect to R.*

Definition 2.4 ([7]). *If (U, R) is an approximation space and $X, Y \subseteq U$. Then the set X is said to be rough subset of Y with respect to R if $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$. Note that every subset X of a rough set Y is a rough subset of Y.*

3. Čech Rough Closure Spaces

In this section we introduce Čech rough closure and interior operators on a rough set X in the approximation space (U, R) and also establish their relations.

Definition 3.1. *Let $P(X)$ be the power set of a rough set X in the approximation space (U, R). A function $C : P(X) \to P(X)$ is called a Čech rough closure (simply, rough closure) operator for X if it satisfy the following conditions:*

(1). $C(\emptyset) = \emptyset$

(2). $A \subset C(A)$ for each $A \subset X$

(3). $C(A \cup B) = C(A) \cup C(B)$ for each $A, B \subset X$.

Then the rough set X together with the Čech rough closure operator C is called a Čech rough closure space (simply, rough closure space) and it is denoted by (X, C). If (X, C) is a rough closure space and $A \subset X$, then $C(A)$ is called the rough closure of A in (X, C). A subset A of X is said to be rough closed in (X, C), if $C(A) = A$ and is said to be rough open if its complement is rough closed, that is $C(X - A) = X - A$. Also a subset A of X is said to be rough clopen in (X, C) if A is both rough open and rough closed in (X, C). Let (X, C) be a rough closure space, then its associated rough topology on X, denoted by τ_C is defined by $\tau_C = \{G \subset X : C(X - G) = X - G\}$.

Definition 3.2. *A rough closure operator C is said to be finer than a rough closure operator C_1 on the same rough set X (or C_1 is coarser than C) if $C(A) \subset C_1(A)$ for each $A \subset X$ and it is denoted as $C > C_1$.*

Example 3.3. *Let $U = \{a, b, c, d\}$ with $U|R = \{\{a\}, \{b\}, \{c, d\}\}$. Clearly $X = \{a, b, c\} \subseteq U$ is a rough set on U with respect to R. Define $C : P(X) \to P(X)$ by $C(\emptyset) = \emptyset$, $C(\{a\}) = \{a, b\}$, $C(\{b\}) = C(\{c\}) = C(\{b, c\}) = \{b, c\}$, and $C(\{a, b\}) = C(\{a, c\}) = C(X) = X$. Then the operator C is the rough closure operator and (X, C) is a rough closure space and the collection $\tau_C = \{\emptyset, X, \{a\}\}$ is the set of all rough open sets in (X, C). Note that (X, C) is not a rough topological space.*

Remark 3.4. *Let $P(X)$ be the power set of a rough set X in the approximation space (U, R). If a function $C : P(X) \to P(X)$ defined by $C(A) = A$ for all $A \subseteq X$, then clearly C is a finest rough closure operator and a rough topological closure operator on X. It gives rough discrete topology τ_C on X. Also if we define a function $C_1 : P(X) \to P(X)$ by*

$$C_1(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ X & \text{otherwise} \end{cases}$$

Then clearly C_1 is a coarsest rough closure operator and rough topological closure operator on X. Also it gives rough indiscrete topology τ_{C_1} on X.

Remark 3.5. *Let (X, C) be a rough closure space and $A, B \subset X$. Then the following statements are true:*

(1). If $A \subset B$, then $C(A) \subset C(B)$.

(2). $C(A \cap B) \subset C(A) \cap C(B)$.

Proof.

(1). Clearly we have, $C(A) \subset C(A) \cup C(B) = C(B)$.

(2). Since $A \cap B \subset A$ and $A \cap B \subset B$, then $C(A \cap B) \subset C(A) \cap C(B)$. □

Theorem 3.6. *Let (X, C) be a rough closure space and $A, B \subset X$. Then the collection of all rough closed sets of a rough closure space (X, C) is closed under finite unions and arbitrary intersections.*

Proof. By the definition of C and for finite number n, we have $C(\bigcup_1^n A_i) = C(A_1) \cup C(A_2) \cup ... \cup C(A_n) = \bigcup_1^n A_i$. Since $\bigcap_1^\infty A_i \subset A_i$ for each $i = 1, 2, ...$, then $C(\bigcap_1^\infty A_i) \subset C(A_i) = A_i$ for each $i = 1, 2, ...$, and implies that $C(\bigcap_1^\infty A_i) \subset \bigcap_1^\infty A_i$. By the definition of C, we have $C(\bigcap_1^\infty A_i) = \bigcap_1^\infty A_i$. □

Remark 3.7. *In a rough closure space (X, C), $C(A)$ need not be a rough closed. It is clear from the following example.*

Example 3.8. *Let $U = \{a, b, c, d\}$ with $U|R = \{\{a\}, \{b\}, \{c, d\}\}$. Clearly $X = \{a, b, c\} \subseteq U$ is a rough set on U with respect to R. Define $C : P(X) \to P(X)$ by $C(\emptyset) = \emptyset$, $C(\{a\}) = \{a, b\}$, $C(\{b\}) = C(\{c\}) = C(\{b, c\}) = \{b, c\}$, and $C(\{a, b\}) = C(\{a, c\}) = C(X) = X$. Then the operator C is rough closure operator and (X, C) is a rough closure space. The collection $\tau_C = \{\emptyset, X, \{a\}\}$ is the set of all rough open sets in (X, C) in which $C(\{a\}) = \{a, b\}$ is not rough closed.*

Definition 3.9. *Let (X, C) be a rough closure space on X. A function $Int : P(X) \to P(X)$ is defined by $Int(A) = X - C(X - A)$ and is called a Čech rough interior (simply, rough interior) operator of A in (X, C).*

Example 3.10. *Let $U = \{a, b, c, d\}$ with $U|R = \{\{a\}, \{b\}, \{c, d\}\}$. Clearly $X = \{a, b, c\} \subseteq U$ is a rough set on U with respect to R. Define $C : P(X) \to P(X)$ by $C(\emptyset) = \emptyset$, $C(\{a\}) = \{a, b\}$, $C(\{b\}) = C(\{c\}) = C(\{b, c\}) = \{b, c\}$, and $C(\{a, b\}) = C(\{a, c\}) = C(X) = X$. Then (X, C) is a rough closure space. Here $Int(\{a, c\}) = X - C(\{b\}) = \{a\}$.*

Definition 3.11. *A rough neighbourhood of a subset A of a rough closure space (X, C) is defined as any subset O of X containing A in its rough interior. Thus G is a rough neighbourhood of A if and only if $A \subset X - C(X - G)$. By a rough neighbourhood of a point x of (X, C), we mean a rough neighbourhood of $\{x\}$, that is, $G \subset X$ is said to be a rough neighbourhood of a point x if $x \in Int(G)$. Rough neighbourhood system of a set $G \subset X$(resp. a point $x \in X$) in the rough closure space (X, C) is the collection of all rough neighbourhoods of the set G(resp.the point x).*

Example 3.12. *Let $U = \{a, b, c, d, e\}$ with $U|R = \{\{a, b, c, d, e\}\}$. Then $X = \{a, b, c, d\} \subseteq U$ is a rough set on U with respect to R. Define a map $C : P(X) \to P(X)$ as $C(\{a\}) = \{a\}$, $C(\{b\}) = C(\{c\}) = \{b, c\}$, and $C(\{d\}) = \{a, d\}$. Then for any $A \subseteq X$, define*

$$C(A) = \begin{cases} \emptyset & \text{if } A=\emptyset \\ \cup\{C(\{x\}) : x \in A\} & \text{otherwise} \end{cases}$$

Clearly (X, C) is a rough closure space. Then rough neighbourhood system of $\{d\}$ is the collection $\{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ and the rough neighbourhood system of the set $\{a, d\}$ is $\{\{a, d\}, \{a, b, d\}, \{a, c, d\}, X\}$.

Definition 3.13. *A collection $\mathcal{B}$ of subsets of (X, C_R) is called a local base of the rough neighbourhood system of a set $A \subset X$ (resp. a point x) if and only if each $B \in \mathcal{B}$ is a rough neighbourhood of A (resp. of x) and every rough neighborhood of A (resp. of x) contains a $B' \in \mathcal{B}$.*

Example 3.14. *In example 3.12, the collection $\mathcal{B} = \{\{a, d\}, \{a, b, d\}\}$ is a local base of the rough neighbourhood system of the set $\{a, d\}$.*

Theorem 3.15. *Let (X, C) be a rough closure space and $A, B \subset X$. Then the collection of all rough open sets of a rough closure space (X, C) is closed under arbitrary unions and finite intersections.*

Proof. Proof is trivial by using de Morgan formula and $Int(A) = X - C(X - A)$. □

Remark 3.16. *In a rough closure space (X, C), $\emptyset$ and X are rough closed as well as rough open sets.*

Theorem 3.17. *Let (X, C) be a rough closure space. Then the following are true:*

(1). $Int(\emptyset) = \emptyset$ and $Int(X) = X$.

(2). $Int(A) \subset A$ for each $A \subset X$.

(3). $Int(A) \subset Int(B)$ for each $A \subset B \subset X$.

(4). $Int(A \cap B) = Int(A) \cap Int(B)$ for each $A, B \subset X$.

(5). $Int(A \cup B) \supset Int(A) \cup Int(B)$ for each $A, B \subset X$.

(6). $Int(A) = A$ if and only if A is a rough open set.

Proof. Here we prove part (4) only. In a similar manner, by using the definitions 3.9 and 3.11, we can prove (1), (2), (3), (5) and (6). Since $A \cap B \subset A$, and $A \cap B \subset B$, $Int(A \cap B) \subset Int(A) \cap Int(B)$. On the other hand, let $x \in Int(A) \cap Int(B)$, then A and B are two rough neighbourhood of x so that their intersection is also a rough neighbourhood of x. Hence $x \in Int(A \cap B)$. Thus, we have $Int(A \cap B) = Int(A) \cap Int(B)$ for each $A, B \subset X$. □

Remark 3.18. *Let (X, C) be a rough closure space. Then a subset G of X is a rough neighbourhood of a subset A of X if and only if G is a rough neighbourhood of each point of A. A subset G of X is rough open if and only if it is a rough neighbourhood of all of its points.*

Proof. The proof is obvious from definition 3.11 and theorem 3.17 (6). □

Proposition 3.19. *Let (X, C) be a rough closure space and $A \subset X$. Then $x \in C(A)$ if and only if every rough neighbourhood G of x in (X, C) such that $G \cap A \neq \emptyset$.*

Proof. Suppose a rough neighbourhood G of x such that $G \cap A = \emptyset$, then $x \in Int(G) \subset Int(X - A) = X - C(A)$, which shows that $x \notin C(A)$. Conversely, if $x \notin C(A)$, then $X - A$ is a rough neighbourhood of x such that $(X - A) \cap A = \emptyset$. □

Theorem 3.20. *Let (X, C) be a rough closure space. Then the following statements are equivalent:*

(1). The rough closure of each subset of X is rough closed in X, that is, $C(C(A)) = C(A)$ for each $A \subset X$

(2). The rough interior of each subset of X is rough open in X, that is, $Int(Int(A)) = Int(A)$ for each $A \subset X$

(3). For each $x \in X$, then the collection of all rough neighbourhoods of x is a local base at x

(4). For each $x \in X$ and if O is a rough neighbourhood of x, then there exists a rough neighbourhood G of x such that O is a rough neighbourhood of each point of G.

Proof. (1)⇒(2): Assume that $C(A)$ is rough closed for each $A \subset X$. Then $C(X - A)$ is rough closed and $Int(A) = X - C(X - A)$ is rough open for each $A \subset X$. Thus rough interior of every subset of X is rough open.

(2)⇒(3): Assume that the rough interior of each subset of X is rough open and let G be a rough neighbourhood of $x \in X$. Then $Int(G)$ is rough open and by Remark 3.18, $Int(G)$ is a rough neighbourhood of all of its points as well as x. Since $Int(G) \subset G$, this leads the part (3) by Definition 3.13.

(3)⇒(4): Since every rough open set is a rough neighbourhood of all of its points, this implies the part (4).

(4)⇒(1): Assume $x \in C(C(A))$ for any $A \subset X$. By Remark 3.19, it is sufficient to show that every rough neighbourhood O of x such that $O \cap A \neq \emptyset$. By hypothesis, there exists a rough neighbourhood G of x such that O is a rough neighbourhood of each point of G. Since $x \in C(C(A))$, by Remark 3.19, we have $G \cap C(A) \neq \emptyset$, and therefore we can choose $y \in G \cap C(A)$. Since $y \in C(A)$, then O is a rough neighbourhood of y and O is a rough neighbourhood of each $z \in G$. Then, we get $O \cap A \neq \emptyset$. □

4. Continuity in Rough Closure Spaces

In this section we introduce continuous functions in rough closure spaces and establish their properties. Through out this section, the rough closure spaces (X, C_1) and (Y, C_2) represents simply as X and Y respectively.

Definition 4.1. *Let (X, C_1) and (Y, C_2) be two rough closure spaces on rough sets X and Y respectively and a function $f : X \to Y$ is said to be rough continuous at a point $x \in X$ if $f(x) \in C_2(f(A))$ for $A \subseteq X$ and $x \in C_1(A)$. Let (X, C_1) and (Y, C_2) be two rough closure spaces on rough sets X and Y respectively and a function $f : X \to Y$ is said to be rough continuous on X if it is rough continuous at each point of X, or equivalently, if $f(C_1(A)) \subseteq C_2(f(A))$ for each $A \subseteq X$.*

Example 4.2. *Let $U_1 = \{a, b, c, d\}$ with $U_1|R_1 = \{\{a\}, \{b\}, \{c, d\}\}$. Clearly $X = \{a, b, c\} \subseteq U_1$ is a rough set on U_1 with respect to R_1. Define $C_1 : P(X) \to P(X)$ by $C_1(\emptyset) = \emptyset$, $C_1(\{a\}) = \{a\}$, $C_1(\{b\}) = C_1(\{c\}) = C_1(\{b, c\}) = \{b, c\}$, and $C_1(\{a, b\}) = C_1(\{a, c\}) = C_1(X) = X$. Then C_1 is the rough closure operator and (X, C_1) is a rough closure space.*

Let $U_2 = \{a, b, c\}$ with $U_2|R_2 = \{\{a\}, \{b, c\}\}$. Clearly $Y = \{a, b\} \subseteq U_2$ is a rough set on U_2 with respect to R_2. Define $C_2 : P(Y) \to P(Y)$ by $C_2(\emptyset) = \emptyset$, $C_2(\{a\}) = \{a\}$, $C_2(\{b\}) = \{b\}$, and $C_2(Y) = Y$. Then C_2 is the rough closure operator and (Y, C_2) is a rough closure space. Define $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = b$. Clearly, f is rough continuous on the rough closure space (X, C_1).

Theorem 4.3. *Let (X, C_1) and (Y, C_2) be two rough closure spaces on rough sets X and Y respectively and a function $f : X \to Y$ is said to be rough continuous on X if and only if $C_1(f^{-1}(B)) \subseteq f^{-1}(C_2(B))$ for every $B \subseteq Y$.*

Proof. Assume $f : X \to Y$ is a rough continuous on X and take $A = f^{-1}(B)$ for every $B \subseteq Y$. Then $f(C_1(A)) \subseteq C_2(B)$ and implies $C_1(A) \subseteq f^{-1}(C_2(B))$. Therefore, $C_1(f^{-1}(B)) \subseteq f^{-1}(C_2(B))$ for every $B \subseteq Y$. Conversely, assume $C_1(f^{-1}(B)) \subseteq f^{-1}(C_2(B))$, for every $B \subseteq Y$. If we take $B = f(A)$ and $A_1 = f^{-1}(B) \supseteq A$ for $A \subseteq X$. Then we have $C_1(A_1) \subseteq f^{-1}(C_2(B))$ and implies that $f(C_1(A_1)) \subseteq C_2(B) = C_2(f(A))$. Since $A \subseteq A_1$, $f(C_1(A)) \subseteq f(C_1(A_1)) \subseteq C_2(f(A))$ for every $A \subseteq X$. $\square$

Theorem 4.4. *Let (X, C_1) and (Y, C_2) be two rough closure spaces on rough sets X and Y respectively and a function $f : X \to Y$ is rough continuous on X if and only if the inverse image of each rough neighbourhood of $f(x)$ be a rough neighbourhood of x.*

Proof. Suppose $O = f^{-1}(G)$ is not a rough neighbourhood of x, where G is a rough neighbourhood of $f(x)$. By Definition 3.11, $x \in C_1(X - O)$ and by Definition 4.1, $f(x) \in C_2(f(X - O)) \subset C_2(Y - G)$. This leads to G is not a rough neighbourhood of $f(x)$. Conversely, suppose f is not a rough continuous, then for $x \in X$ and $A \subset X$, $f(x) \notin C_2(f(A))$. This implies $B = Y - f(A)$ is a rough neighbourhood of $f(x)$ and by hypothesis, $f^{-1}(B)$ is a rough neighbourhood of x and $f^{-1}(B) \cap X = \emptyset$. By proposition 3.19, it is clear that $x \notin C_1(A)$. $\square$

Theorem 4.5. *Let (X, C_1) and (Y, C_2) be two rough closure spaces on rough sets X and Y respectively. If a function $f : X \to Y$ is rough continuous on X then the inverse image of every rough open(resp. closed) set in Y is rough open (resp. closed) set in X.*

Proof. Assume $f : X \to Y$ is a rough continuous on X and let $B \subseteq Y$ be a rough closed set in Y. Then $C_1(f^{-1}(B)) \subseteq f^{-1}(C_2(B)) = f^{-1}(B)$ and implies $C_1(f^{-1}(B)) = f^{-1}(B)$. Thus the inverse image of every rough closed set in Y is rough closed set in X. For the proof of the inverse image of every rough open set in Y is rough open set in X, we can use Definition 3.9. $\square$

Example 4.6. *The converse of the theorem 4.5, need not be true. Let $U = \{a, b, c, d\}$ with $U|R_1 = \{\{a\}, \{b\}, \{c, d\}\}$. Clearly $X = \{a, b, d\} \subseteq U$ is a rough set on U with respect to R_1. Define $C_1 : P(X) \to P(X)$ by $C_1(\emptyset) = \emptyset$, $C_1(\{a\}) = \{a\}$, $C_1(\{b\}) = \{b, d\}$, $C_1(\{d\}) = C_1(\{a, d\}) = \{a, d\}$, and $C_1(\{a, b\}) = C_1(\{b, d\}) = C_1(X) = X$. Then C_1 is the rough closure operator on X. Let $U = \{a, b, c, d\}$ with $U|R_2 = \{\{a\}, \{b, c, d\}\}$. Clearly $Y = \{a, b, c\} \subseteq U$ is a rough set on U with respect to R_2. Define $C_2 : P(Y) \to P(Y)$ by $C_2(\emptyset) = \emptyset$, $C_2(\{a\}) = \{a, b\}$, $C_2(\{b\}) = C_2(\{b, c\}) = \{b, c\}$, $C_2(\{c\}) = \{c\}$, and $C_2(\{a, b\}) = C_2(\{a, c\}) = C_2(Y) = Y$. Then C_2 is the rough closure operator on Y. Define $f : X \to Y$ by $f(a) = c$, $f(b) = a$ and $f(d) = c$. Here the inverse image of every rough open (resp. closed) set in Y is rough open (resp. closed) set in X but f is not rough continuous on (X, C_1), since $f(C_1(\{b\})) = \{a, c\} \not\subseteq C_2(f(\{b\})) = \{a, b\}$.*

Theorem 4.7. *Let (X, C_1), (Y, C_2) and (Z, C_3) be three rough closure spaces on rough sets X, Y and Z respectively and if functions $f : X \to Y$ and $g : Y \to Z$ are rough continuous on X and Y respectively, then $g \circ f : X \to Z$ is a rough continuous function.*

Proof. Let $A \subseteq X$. Then $g \circ f(C_1(A)) = g(f(C_1(A))) \subseteq g(C_2(f(A))) \subseteq C_3(g(f(A))) = C_3(g \circ f(A))$. $\square$

5.   Nano Topology in Čech Rough Closure Space

In this section we derive nano topology in terms of Čech closure operators on rough set X in the approximation space $(\mathcal{U}, R)$ and investigate its properties.

Definition 5.1. *Let (X, C) be a rough closure space and $A \subseteq X$. Then $int(A)$ is defined as the union of all rough open sets of (X, C) contained in A, that is $int(A) = \cup\{G \subseteq X : G \in \tau_C \text{ and } G \subseteq A\}$ and $cl(A)$ is defined as the intersection of all rough closed sets of (X, C) containing A in (X, C), that is $cl(A) = \cap\{F \subseteq X : F \in \tau_C' \text{ and } A \subseteq F\}$ and $Bd(A) = cl(A) - int(A)$ is called the boundary of A in (X, C).*

Example 5.2. *Let $U = \{a, b, c, d\}$ with $U|R_1 = \{\{a\}, \{c\}, \{b, d\}\}$. Clearly $X = \{a, b, c\} \subseteq U$ is a rough set on U with respect to R_1. Define $C : P(X) \to P(X)$ by $C(\emptyset) = \emptyset$, $C(\{a\}) = \{a, b\}$, $C(\{b\}) = \{a, b\}$, $C(\{c\}) = \{a, c\}$, $C(\{a, b\}) = \{a, b\}$, and $C(\{a, c\}) = C(\{b, c\}) = C(X) = X$. Then C is the rough closure operator on X and $C(C(\{c\})) = \{X\} \neq C(\{c\})$. Even though the collection $\tau_C = \{\emptyset, X, \{c\}\}$, the set of all rough open sets in (X, C) forms a topology yet $cl(\{c\}) = X \neq C(\{c\})$.*

Theorem 5.3. *If (X, C) is a rough closure space and $A, B \subseteq X$, then*

(1). $int(A) \subseteq A \subseteq cl(A)$.

(2). $int(\emptyset) = cl(\emptyset) = \emptyset$ and $int(X) = cl(X) = X$.

(3). $cl(A \cup B) = cl(A) \cup cl(B)$.

(4). $cl(A \cap B) \subseteq cl(A) \cap cl(B)$.

(5). $int(A \cup B) \supseteq int(A) \cup int(B)$.

(6). $int(A \cap B) = int(A) \cap int(B)$.

(7). $int(A) \subseteq int(B)$ and $cl(A) \subseteq cl(B)$ whenever $A \subseteq B$.

(8). $cl(A^c) = [int(A)]^c$ and $int(A^c) = [cl(A)]^c$.

(9). $cl(cl(A)) = cl(A)$.

(10). $int(int(A)) = int(A)$.

Proof.   Here we shall prove part (3),(4) and (9) only. The remaining parts are obvious. Part (3): Let $x \in cl(A \cup B) = \cap\{F \subseteq X : F \in \tau_C' \text{ and } A \cup B \subseteq F\}$ iff $x \in F$ and $A \cup B \subseteq F$ for all $F \subseteq \tau_C'$ iff $x \in F$ and $A \subseteq F$ for all $F \in \tau_C'$ or $x \in F$ and $B \subseteq F$ for all $F \subseteq \tau_C'$ iff $x \in \cap\{F \subseteq X : F \in \tau_C' \text{ and } A \subseteq F\}$ or $x \in \cap\{F \subseteq X : F \in \tau_C' \text{ and } B \subseteq F\}$ iff $x \in cl(A) \cup cl(B)$. Part (4): Let $x \notin cl(A) \cap cl(B)$, then $x \notin cl(A)$ or $x \notin cl(B)$, then $x \notin \cap\{F \subseteq X : F \in \tau_C' \text{ and } A \subseteq F\}$ or $x \notin \cap\{F \subseteq X : F \in \tau_C' \text{ and } B \subseteq F\}$, then there exists a $F_i \in \tau_C'$ such that $x \notin F_i$ and $A \subseteq F_i$ or there exists a $F_j \in \tau_C'$ such that $x \notin F_j$ implies $x \notin F_i \cap F_j$ such that $F_m = F_i \cap F_j \in \tau_C'$ and $x \notin \cap\{F_m \subseteq X : F_m \in \tau_C' \text{ and } A \cap B \subseteq F_m\}$. Therefore $x \notin cl(A \cap B)$. Part (9): By part (1), we have $cl(A) \subseteq cl(cl(A))$. On the other hand, if $x \in cl(cl(A)) = \cap\{F \subseteq X : F \in \tau_C' \text{ and } cl(A) \subseteq F\}$, then $x \in F$ and $cl(A) \subseteq F$ for all $F \subseteq \tau_C'$ implies $x \in F$ and $A \subseteq F$ for all $F \subseteq \tau_C'$ and $x \in \cap\{F \subseteq X : F \in \tau_C' \text{ and } A \subseteq F\}$. Then we have $x \in cl(A)$. □

Remark 5.4. *Let (X, C) be a rough closure space. For $A \subseteq X$, $\tau_{NC}(A) = \{X, \emptyset, int(A), cl(A), Bd(A)\}$. We note that X and $\emptyset$ are in $\tau_{NC}(A)$. Since $int(A) \subseteq cl(A)$, then $int(A) \cup cl(A) = cl(A) \in \tau_{NC}(A)$, $cl(A) \cup Bd(A) = cl(A) \in \tau_{NC}(A)$ and $int(A) \cup Bd(A) = cl(A) \in \tau_{NC}(A)$. Also, $int(A) \cap cl(A) = int(A) \in \tau_{NC}(A)$, $cl(A) \cap Bd(A) = Bd(A) \in \tau_{NC}(A)$ and $int(A) \cap Bd(A) = \emptyset \in \tau_{NC}(A)$.*

Definition 5.5. *Let (X, C) be a rough closure space, $A \subseteq X$, then the collection $\tau_{NC}(A) = \{X, \emptyset, int(A), cl(A), Bd(A)\}$ satisfies the following axioms:*

(1). X and $\emptyset$ are in $\tau_{NC}(A)$.

(2). The union of the elements of any sub-collection of $\tau_{NC}(A)$ is in $\tau_{NC}(A)$.

(3). The intersection of the elements of any sub-collection of $\tau_{NC}(A)$ is in $\tau_{NC}(A)$.

That is, $\tau_{NC}(A)$ forms a topology on rough closure space (X, C) called the nano Čech topology on X w.r.t A. We call $(X, \tau_{NC}(A))$ as the nano Čech topological space. The elements of $\tau_{NC}(A)$ are called nano Čech-open sets and a set is said to be nano Čech-closed, if its complement is nano Čech-open.

Example 5.6. *Let $\mathbb{R}^+$ be the positive real line and ρ be an equivalence relation on $\mathbb{R}^+$ defined by the partition $\mathbb{R}^+|\rho = \{(0,1], (1,2], (2,3], ...\}$, where $(x, x+1] = \{y \in \mathbb{R}^+ : x < y \leq x+1\}$. Then the set of all natural numbers $\mathbb{N} \subset \mathbb{R}^+$ is a rough set on $\mathbb{R}^+$, since $B_\rho(\mathbb{N}) = U_\rho(\mathbb{N}) - L_\rho(\mathbb{N}) = \mathbb{R}^+ \neq \emptyset$. Define $C : P(\mathbb{N}) \to P(\mathbb{N})$ by*

$$C(\{n\}) = \begin{cases} \{2, 4, 6, ...\} & \text{if } n \text{ is an even +ve integer} \\ \{1, 3, 5, ...\} & \text{if } n \text{ is an odd +ve integer} \end{cases}$$

and for $A \subseteq \mathbb{N}$,

$$C(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \bigcup \{C(\{n\}) : n \in A\} & \text{otherwise} \end{cases}$$

Then C is a rough closure operator and $(\mathbb{N}, C)$ is a rough closure space and the collection $\tau_C = \{\emptyset, \mathbb{N}, \{1, 3, 5, ...\}, \{2, 4, 6, ...\}\}$ is the collection of all rough clopen sets in $(\mathbb{N}, C)$. Let $A = \{1, 3, 5, ...\} \subset \mathbb{N}$, then $cl(A) = \{1, 3, 5, ...\}$, $int(A) = \{1, 3, 5, ...\}$ and $Bd(A) = \emptyset$. Therefore the nano Čech topology is $\tau_{NC}(A) = \{\mathbb{N}, \emptyset, \{1, 3, 5, ...\}\}$.

Proposition 5.7. *If $\tau_{NC}(A)$ is the nano Čech topology on (X, C) with respect to A, then the set $B = \{X, int(A), Bd(A)\}$ is the basis for $\tau_{NC}(A)$.*

Proof.

(1). Clearly, $\bigcup_{A \in B} A = X$.

(2). Consider X and $int(A)$ from B. Let $W = int(A)$. Since $X \cap int(A) = int(A)$, $W \subset X \cap int(A)$ and every $x \in X \cap int(A) \in W$. If we consider X and $Bd(A)$ from B, taking $W = Bd(A)$, $W \subset X \cap Bd(A)$ and every $X \in X \cap Bd(A) \in W$, since $X \cap Bd(A) = Bd(A)$. And when we consider $int(A)$ and $Bd(A)$, $int(A) \cap Bd(A) = \emptyset$. Thus B is a basis for $\tau_{NC}(A)$. $\square$

Remark 5.8. *Let (X, C) be a rough closure space and $A \subseteq X$.*

(1). If $int(A) = \emptyset$ and $cl(A) = X$, then $\tau_{NC}(A) = \{X, \emptyset\}$, the indiscrete nano Čech topology on (X, C).

(2). If $int(A) = cl(A) = A$, then the nano Čech topology, $\tau_{NC}(A) = \{X, \emptyset, int(A)\}$.

(3). If $int(A) = \emptyset$ and $cl(A) \neq X$, then $\tau_{NC}(A) = \{X, \emptyset, cl(A)\}$.

(4). If $int(A) \neq \emptyset$ and $cl(A) = X$, then $\tau_{NC}(A) = \{X, \emptyset, int(A), Bd(A)\}$.

(5). If $int(A) \neq cl(A)$ where $int(A) \neq \emptyset$ and $cl(A) \neq X$, then $\tau_{NC}(A) = \{X, \emptyset, int(A), cl(A), Bd(A)\}$ is the discrete nano Čech topology on (X, C).

Definition 5.9. *Let $(X, \tau_{NC}(A))$ be a nano Čech topological space and $S \subseteq X$. Then $\tau_{NC}\text{-}int(S)$ is defined as the union of all nano Čech-open sets contained in S, that is $\tau_{NC}\text{-}int(S)$ is the largest nano Čech-open subsets of S. The $\tau_{NC}\text{-}cl(S)$ is defined as the intersection of all nano Čech-closed sets containing S, that is $\tau_{NC}\text{-}cl(S)$ is the smallest nano Čech-closed supersets of S.*

Example 5.10. *From the Example 3.12, we have $\tau_C = \{X, \emptyset, \{b, c, d\}, \{b, c\}, \{a, d\}, \{d\}\}$ and $\tau_C' = \{X, \emptyset, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{a, b, c\} \subset X$, then $cl(A) = \{a, b, c\}$, $int(A) = \{b, c\}$ and $Bd(A) = \{a\}$. Therefore the nano Čech topology is $\tau_{NC}(A) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. If $S = \{a, b, d\} \subset X$, then $\tau_{NC}\text{-}int(S) = \{a\}$ and $\tau_{NC}\text{-}cl(S) = X$.*

Theorem 5.11. *If $(X, \tau_{NC}(A))$ is a nano Čech topological space, $x \in \tau_{NC}\text{-}cl(S)$ if and only if $G \cap S \neq \emptyset$ for every nano Čech-open set G containing x, where $S \subseteq X$.*

Proof. If $x \in \tau_{NC}\text{-}cl(S)$ and G is a nano Čech-open set containing x, then $X - G$ is nano Čech-closed. If $G \cap S = \emptyset$, then $S \subseteq X - G$. That is, $X - G$ is a nano Čech-closed set containing A. Therefore $\tau_{NC}\text{-}cl(S) \subseteq X - G$ which is a contradiction, since $x \in \tau_{NC}\text{-}cl(S)$ and $x \notin X - G$. Hence $G \cap S \neq \emptyset$ for every nano Čech-open set G containing x. Conversely, if $G \cap S \neq \emptyset$ for every nano Čech-open set G containing x and if $x \notin \tau_{NC}\text{-}cl(S)$, then $x \in X - \tau_{NC}\text{-}cl(S)$ which is a nano Čech-open and hence $(X - \tau_{NC}\text{-}cl(S)) \cap S \neq \emptyset$. But $S \subseteq \tau_{NC}\text{-}cl(S)$ and hence $X - \tau_{NC}\text{-}cl(S) \subseteq X - S$ which implies $[X - \tau_{NC}\text{-}cl(S)] \cap S \subseteq (X - S) \cap S$ and therefore $(X - S) \cap S \neq \emptyset$ which is a contradiction. Hence $x \in \tau_{NC}\text{-}cl(S)$. $\square$

Theorem 5.12. *Let $(X, \tau_{NC}(A))$ be a nano Čech topological space and $S, T \subseteq X$, then*

(1). $\tau_{NC}\text{-}int(S) \subseteq S \subseteq \tau_{NC}\text{-}cl(S)$.

(2). $\tau_{NC}\text{-}int(\emptyset) = \tau_{NC}\text{-}cl(\emptyset) = \emptyset$ and $\tau_{NC}\text{-}int(X) = \tau_{NC}\text{-}cl(X) = X$.

(3). $\tau_{NC}\text{-}cl(S \cup T) = \tau_{NC}\text{-}cl(S) \cup \tau_{NC}\text{-}cl(T)$.

(4). $\tau_{NC}\text{-}cl(S \cap T) \subseteq \tau_{NC}\text{-}cl(S) \cap \tau_{NC}\text{-}cl(T)$.

(5). $\tau_{NC}\text{-}int(S \cup T) \supseteq \tau_{NC}\text{-}int(S) \cup \tau_{NC}\text{-}int(T)$.

(6). $\tau_{NC}\text{-}int(S \cap T) = \tau_{NC}\text{-}int(S) \cap \tau_{NC}\text{-}int(T)$.

(7). $\tau_{NC}\text{-}int(S) \subseteq \tau_{NC}\text{-}int(T)$ and $\tau_{NC}\text{-}cl(S) \subseteq \tau_{NC}\text{-}cl(T)$ whenever $S \subseteq T$.

(8). $\tau_{NC}\text{-}cl(S^c) = [\tau_{NC}\text{-}int(S)]^c$ and $\tau_{NC}\text{-}int(S^c) = [\tau_{NC}\text{-}cl(S)]^c$.

(9). $\tau_{NC}\text{-}cl(\tau_{NC}\text{-}cl(S)) = \tau_{NC}\text{-}cl(S)$.

(10). $\tau_{NC}\text{-}int(\tau_{NC}\text{-}int(S)) = \tau_{NC}\text{-}int(S)$.

Proof. The proof directly follows from Theorems 5.3 and 5.11. $\square$

Theorem 5.13. *If $(X, \tau_{NC}(A))$ is a nano Čech topological space and $S \subseteq X$, then*

(1). $\tau_{NC}\text{-}cl(S) = S$ if and only if S is nano Čech-closed set.

(2). $\tau_{NC}\text{-}int(S) = S$ if and only if S is nano Čech-open set.

Proof. Part (1): If S is nano Čech-closed, then τ_{NC}-$cl(S)$ is the smallest nano Čech-closed sets containing S and so τ_{NC}-$cl(S) = S$. On the other hand, if τ_{NC}-$cl(S) = S$, then S is the smallest nano Čech-closed sets containing itself and so A is nano Čech-closed.

Part (2): If S is nano Čech-open, $X - S$ is nano Čech-closed if and only if τ_{NC}-$cl(X - S) = X - S$ if and only if $X - cl(X - S) = S$ if and only if τ_{NC}-$int(S) = S$, by above theorem. $\qquad\square$

Theorem 5.14. *The operator τ_{NC}-cl is the Kuratowski closure operator.*

Proof. The proof follows from the parts (2), (3), (7) and (9) of Theorem 5.12. $\qquad\square$

Theorem 5.15. *If $(X, \tau_{NC}(A))$ is a nano Čech topological space, then*

(1). τ_{NC}-$cl(int(A)) = [Bd(A)]^c$.

(2). τ_{NC}-$cl(cl(A)) = X$.

(3). τ_{NC}-$cl(Bd(A)) = [int(A)]^c$.

Proof. The nano Čech-open sets in X are $X, \emptyset, int(A), cl(A)$ and $Bd(A)$ and hence the nano Čech-closed sets in X are $X, \emptyset, [int(A)]^c, [cl(A)]^c$ and $[Bd(A)]^c$, where $cl(A^c) = [int(A)]^c$, $int(A^c) = [cl(A)]^c$ and $[Bd(A)]^c = int(A) \cap int(A^c)$.

(1). Since $int(A) \subseteq cl(A), int(A) \cap [cl(A)]^c = \emptyset$. That is, $[int(A)]^c$ and $[cl(A)]^c$ cannot contain $int(A)$, unless $int(A) = \emptyset$, in which case, τ_{NC}-$cl(int(A)) = X = [Bd(A)]^c$. But $[Bd(A)]^c$ is a nano Čech-closed set containing $int(A)$. Thus X and $[Bd(A)]^c$ are the nano Čech-closed sets containing $int(A)$. Therefore τ_{NC}-$cl(int(A)) = [Bd(A)]^c$.

(2). If $cl(A) \subseteq [int(A)]^c$, then $int(A) \subseteq [cl(A)]^c \subseteq [int(A)]^c$. That is, $int(A) = \emptyset$. Hence $\tau_{NC}(A) = \{X, \emptyset, cl(A)\}$ and so X is the only nano Čech-closed set containing $cl(A)$. That is, τ_{NC}-$cl(cl(A)) = X$, if $cl(A) \subseteq [int(A)]^c$. If $cl(A) \subseteq [Bd(A)]^c$, then $Bd(A) \subseteq [cl(A)]^c \subseteq [Bd(A)]^c$ and hence $Bd(A) = \emptyset$. Therefore $cl(A) = int(A)$. Hence $\tau_{NC}(A) = \{X, \emptyset, cl(A)\}$ and so X is the only nano Čech-closed set containing $cl(A)$. Therefore τ_{NC}-$cl(cl(A)) = X$. Thus in both cases, τ_{NC}-$cl(cl(A)) = X$.

(3). Since $Bd(A) \subseteq cl(A), Bd(A) \cap [cl(A)]^c = \emptyset$. That is, $[Bd(A)]^c$ and $[cl(A)]^c$ cannot contain $Bd(A)$, unless $Bd(A) = \emptyset$. But $[int(A)]^c$ is a nano Čech-closed set containing $Bd(A)$. Therefore τ_{NC}-$cl(Bd(A)) = [int(A)]^c$. $\qquad\square$

6. Conclusion

Rough set theory is a vast area that has varied inventions, applications and interactions with many other branches of mathematical sciences. Deriving Nano topology from Čech closure space is one such interaction. To add strength and make our theory vivid we have also illustrated a few examples here. Further this concept can also be extended to rough compactness and rough connectedness. I hope the beauty of this work can pave way to many other research fields such as Fuzzy topology, bitopology, digital topology etc.

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