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Common Fixed Point Theorems of $(g-\alpha_{s^p},\psi,\varphi)$ Contraction Mappings with Application

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Abstract: In this paper, we use the concept of generalized $(g-\alpha_{s^p},\psi,\varphi)$ contractive mappings to prove some existence and uniqueness results for common fixed point in the setting of $b-$metric spaces. We deduce some results as corollaries, examples and give an application to verify the results obtained.

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1. Introduction

Fixed point theory is one of the most important topics in Mathematics, specially in analysis. Due to its application in various disciplines like engineering, computer science, biological sciences, economics etc., some authors took their interest in fixed point theory and its application. The Banach fixed point theorem [1] popularly known as Banach contraction mapping principle is a rewarding results in fixed point theory. For decades, the Banach contraction principle has been improved on different directions at different spaces by mathematicians.

In 1993, Czerwik [2] introduced firstly the concept of $b-$metric space and proved some fixed point theorems of contractive mappings in $b-$metric space. In the sequel, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in $b-$metric spaces (see [3]-[6]). On the other hand, more recently, Samet et al. [7] introduced the concept of $\alpha-$ admissible and $\alpha-\psi-$contractive mappings and presented fixed point theorems for them. In [8] and [9], Zoto et al. studied generalized α_{s^p} contractive mappings and $(\alpha-\psi,\phi)-$ contractions in $b-$metric$-$like space. Also, it should be noted that some authors have studied the fixed point theorems in the generalized metric space and $b-$metric space (see [10]-[12]). In particular, Ma et al. [13] studied the sufficient conditions for the existence of a unique common fixed point of generalized $\alpha_s-\psi-$Geraghty contractions in an α_s-complete partial $b-$metric space. In 2021, Hao and Guan [14] introduced a new class of generalized weakly contractive mappings in the framework of $b-$metric spaces.

Motivated and inspired by Theorems 26 in [13] and Theorem 10 in [14], in this paper, our purpose is to introduce the concept of generalized $(g-\alpha_{s^p},\psi,\varphi)$ contractive mapping and obtain a few common fixed point results in the framework of $b-$metric space. Furthermore, we provide an example that elaborated the useability of our result. Meanwhile, we present an application to the existence of solutions to an integral equation by means of one of our results.

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2. Preliminaries

Firstly, we recall some definitions and lemmas in $b-$metric space.

Definition 2.1 ([2]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0, +\infty)$ is said to be a $b-$metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:*

(i). $d(x, y) = 0$ if and only if $x = y$;

(ii). $d(x, y) = d(y, x)$;

(iii). $d(x, y) \leq s(d(x, z) + d(y, z))$.

In general, (X, d) is called a $b-$metric space with parameter $s \geq 1$.

Remark 2.2. *Obviously, every metric space is a $b-$metric space with $s = 1$. We can find several examples of $b-$metric spaces which are not metric spaces [15].*

Example 2.3 ([16]). *Let (X, ρ) be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then $d(x, y)$ is a $b-$metric space with $s = 2^{p-1}$.*

Definition 2.4 ([17]). *Let (X, d) be a $b-$metric space with parameter $s \geq 1$. Then a sequence $\{x_n\}$ in X is said to be:*

(i). $b-$convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$;

(ii). a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ when $n, m \to +\infty$.

As usual, a $b-$metric space is called complete if and only if each Cauchy sequence in this space is $b-$convergent.

Definition 2.5 ([18]). *Let f and g be two self-mappings on a nonempty set X. If $w = fx = gx$, for some $x \in X$, then x is said to be the coincidence point of f and g, where w is called the point of coincidence of f and g. Let $C(f, g)$ denote the set of all coincidence points of f and g.*

Definition 2.6 ([18]). *Let f and g be two self-mappings defined on a nonempty set X. Then f and g is said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$ for every $x \in C(f, g)$.*

We cite the following lemma to obtain our main results:

Lemma 2.7 ([16]). *Let (X, d) be a $b-$metric space with parameter $s \geq 1$. Assume that $\{x_n\}$ and $\{y_n\}$ are $b-$convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \to +\infty} d(x_n, y_n) \leq \limsup_{n \to +\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n \to +\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n \to +\infty} d(x_n, z) \leq \limsup_{n \to +\infty} d(x_n, z) \leq s d(x, z).$$

3. Main Results

In this section, we will establish some results for the existence of a common fixed point of generalized weakly contractive mappings in the setting of complete $b-$metric spaces. Furthermore, we also provide an example to support our result.

A function $f : X \to [0, +\infty)$, where (X, d) is a $b-$metric space, is called lower semicontinuous if, for all $x \in X$ and $\{x_n\}$ is $b-$convergent to x, we have

$$f(x) \leq \liminf_{n\to+\infty} f(x_n).$$

Let Ω denote the class of all mappings $\beta : \mathbb{R}^+ \to [0, \frac{1}{s})$. We denote Ψ the class of the functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(1). ψ is non-decreasing,

(2). ψ is continuous,

(3). $\psi(t) = 0$ if and only if $t = 0$.

Definition 3.1. *Let (X, d) be a $b-$metric space with parameter $s \geq 1$, and let $f, g : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be given mappings and $p \geq 1$ is an arbitrary constant. The mapping $f : X \to X$ is said to be $g - \alpha_{s^p}-$admissible if, for all $x, y \in X$, $\alpha(gx, gy) \geq s^p$ implies $\alpha(fx, fy) \geq s^p$.*

Definition 3.2. *Let (X, d) be a $b-$metric space with parameter $s \geq 1$, and let $f, g : X \to X$ two self-mappings. Assume that $\alpha : X \times X \to [0, +\infty)$ and $p \geq 1$ is a constant. A mapping f is called a generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mapping, if there exists $\psi \in \Psi, \beta \in \Omega$ and $L \geq 0, \frac{1}{s} + L < 1$ such that*

$$\psi(\alpha(gx, gy)[d(fx, fy) + \varphi(fx) + \varphi(fy)]) \leq \beta(\psi(M(x, y, d, f, g, \varphi)))\psi(M(x, y, d, f, g, \varphi)) + L\psi(N(x, y, d, f, g, \varphi)), \quad (1)$$

for all $x, y \in X$ with $\alpha(gx, gy) \geq s^p$ and $d(fx, fy) + \varphi(fx) + \varphi(fy) \neq 0$, where

$$\begin{aligned} M(x, y, d, f, g, \varphi) = \max \Big\{ & d(gx, gy) + \varphi(gx) + \varphi(gy), \frac{1}{2}\{d(fx, gx) + \varphi(fx) + \varphi(gx) + d(fy, gy) + \varphi(fy) + \varphi(gy)\}, \\ & \frac{1}{2s}\{d(fx, gy) + \varphi(fx) + \varphi(gy) + d(fy, gx) + \varphi(fy) + \varphi(gx)\} \Big\}, \\ \text{and} \quad N(x, y, d, f, g, \varphi) = \ & \frac{1}{2}\min\{d(fx, fy) + \varphi(fx) + \varphi(fy), d(fy, gy) + \varphi(fy) + \varphi(gy)\}, \end{aligned}$$

Remark 3.3.

(i). Note that, for $g = I$, the definition reduces to an α_{s^p}-admissible mapping in a $b-$metric space.

(ii). For $s = 1$, the definition reduces to the definition of an $\alpha-$admissible mapping in a metric space.

(iii). If take $\psi(t) = t$, the definition reduces to an $g - \alpha_{s^p}$-admissible mapping in a $b-$metric space.

Let (X, d) be a complete $b-$metric space with parameter $s \geq 1$ and $\alpha : X \times X \to [0, +\infty)$ be a function. Then

(H_{s^p}). If $\{x_n\}$ is a sequence in X such that $gx_n \to gx$ as $n \to +\infty$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ with $\alpha(gx_{n_k}, gx) \geq s^p$ for all $k \in N$;

(U_{s^p}). For all $u, v \in C(f, g)$, we have the condition of $\alpha(gu, gv) \geq s^p$ or $\alpha(gv, gu) \geq s^p$.

Theorem 3.4. *Let (X,d) be a complete $b-$metric space with parameter $s\geq 1$ and let $f,g:X\to X$ be given self-mappings satisfying g is injective and $f(X)\subset g(X)$ where $g(X)$ is closed. Suppose $\varphi:X\to[0,+\infty)$ is a lower semicontinuous function and $\alpha:X\times X\to[0,+\infty)$. If the following conditions are satisfied:*

(i). f is $g-\alpha_{s^p}-$ admissible mapping,

(ii). f is generalized $(g-\alpha_{s^p},\psi,\varphi)$ contractive mapping,

(iii). there is $x_0\in X$ with satisfying $\alpha(gx_0,fx_0)\geq s^p$,

(iv). properties (H_{s^p}) and (U_{s^p}) are satisfied,

(v). α has a transitive property type s^p, that is, for $x,y,z\in X$, $\alpha(x,y)\geq s^p$ and $\alpha(y,z)\geq s^p\Rightarrow\alpha(x,z)\geq s^p$.

Then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Proof. Let $x_0\in X$ such that $\alpha(gx_0,fx_0)\geq s^p$(using condition (iii)). Define the sequence $\{x_n\}$ and $\{y_n\}$ in X by $y_n=fx_n=gx_{n+1}$ for all $n\in N$. If $y_n=y_{n+1}$ for some $n\in\mathbb{N}$, then we have $y_n=y_{n+1}=fx_{n+1}=gx_{n+1}$ and we have nothing to prove. Without loss of generality, assume that $y_n\neq y_{n+1}$ for all $n\in N$. According to condition(i), we get

$$\begin{aligned}\alpha(gx_0,gx_1)&=\alpha(gx_0,fx_0)\geq s^p,\\ \alpha(gx_1,gx_2)&=\alpha(fx_0,fx_1)\geq s^p,\\ \alpha(gx_2,gx_3)&=\alpha(fx_1,fx_2)\geq s^p.\end{aligned}$$

Therefore, continuing this process, we obtain $\alpha(gx_n,gx_{n+1})=\alpha(y_{n-1},y_n)\geq s^p$ for all $n\in\mathbb{N}$. Applying (1) with $x=x_n$ and $y=x_{n+1}$, we obtain

$$\begin{aligned}\psi(d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))&\leq\psi(s^p[d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1})])\\ &\leq\psi(\alpha(gx_n,gx_{n+1})[d(fx_n,fx_{n+1})+\varphi(fx_n)+\varphi(fx_{n+1})])\\ &\leq\beta(\psi(M(x_n,x_{n+1},d,f,g,\varphi))\psi(M(x_n,x_{n+1},d,f,g,\varphi))\\ &+L\psi(N(x_n,x_{n+1},d,f,g,\varphi)),\end{aligned}\tag{2}$$

where

$$\begin{aligned}M(x_n,x_{n+1},d,f,g,\varphi)=&\max\{d(gx_n,gx_{n+1})+\varphi(gx_n)+\varphi(gx_{n+1}),\frac{1}{2}\{d(fx_n,gx_n)+\varphi(fx_n)+\varphi(gx_n)+\\ &d(fx_{n+1},gx_{n+1})+\varphi(fx_{n+1})+\varphi(gx_{n+1})\},\frac{1}{2s}\{d(fx_n,gx_{n+1})+\varphi(fx_n)+\varphi(gx_{n+1})\\ &+d(fx_{n+1},gx_n)+\varphi(fx_{n+1})+\varphi(gx_n)\}\}\\ \leq&\max\{d(y_{n-1},y_n)+\varphi(y_{n-1})+\varphi(y_n),\\ &\frac{1}{2}\{d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1})+d(y_{n+1},y_n)+\varphi(y_{n+1})+\varphi(y_n)\},\\ &\frac{1}{2s}\{d(y_n,y_n)+\varphi(y_n)+\varphi(y_n)+d(y_{n+1},y_{n-1})+\varphi(y_{n+1})+\varphi(y_{n-1})\}\}\\ \leq&\max\{d(y_{n-1},y_n)+\varphi(y_{n-1})+\varphi(y_n),d(y_{n+1},y_n)+\varphi(y_{n+1})+\varphi(y_n)\}\end{aligned}\tag{3}$$

and

$$\begin{aligned}N(x_n,x_{n+1},d,f,g,\varphi)&=\frac{1}{2}\min\{d(fx_n,fx_{n+1})+\varphi(fx_n)+\varphi(fx_{n+1}),d(fx_{n+1},gx_{n+1})+\varphi(fx_{n+1})+\varphi(gx_{n+1})\}\\&=\frac{1}{2}\min\{d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}),d(y_{n+1},y_n)+\varphi(y_{n+1})+\varphi(y_n)\}\\&<d(y_{n+1},y_n)+\varphi(y_{n+1})+\varphi(y_n).\end{aligned}\tag{4}$$

If $d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1})>d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1})$, for some $n\in\mathbb{N}$, by means of (2), (3) and (4), we have

$$\begin{aligned}\psi(d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))&\le\frac{1}{s}\psi(M(x_n,x_{n+1},d,f,g,\varphi))+L\psi(N(x_n,x_{n+1},d,f,g,\varphi))\\&\le\frac{1}{s}\psi(d(y_{n+1},y_n)+\varphi(y_{n+1})+\varphi(y_n))+L\psi(d(y_{n+1},y_n)+\varphi(y_{n+1})+\varphi(y_n))\\&<\psi(d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1})),\end{aligned}$$

which is a contradiction. Hence, we have

$$d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1})\le d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}),\tag{5}$$

$$M(x_n,x_{n+1},d,f,g,\varphi)\le d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}),\tag{6}$$

$$\text{and}\quad N(x_n,x_{n+1},d,f,g,\varphi)<d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}).\tag{7}$$

From (5), we deduce that $\{d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1})\}$ is a non-increasing sequence and consequently there exists $r\ge 0$ such that

$$\lim_{n\to+\infty}(d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))=r.$$

Having in mind (2), (6) and (7), one can obtain

$$\begin{aligned}\psi(d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))&\le\beta(\psi(M(x_n,x_{n+1},d,f,g,\varphi))\psi(M(x_n,x_{n+1},d,f,g,\varphi))+L\psi(N(x_n,x_{n+1},d,f,g,\varphi))\\&\le\frac{1}{s}\psi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}))+L\psi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}))\\&<\psi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1})).\end{aligned}\tag{8}$$

We claim that $r=0$. On the contrary, assume that $r>0$. Taking limit as $n\to\infty$ in (8), we have

$$\begin{aligned}\psi(r)&=\lim_{n\to+\infty}\psi(d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))\\&\le\lim_{n\to+\infty}\beta(\psi(M(x_n,x_{n+1},d,f,g,\varphi))\psi(M(x_n,x_{n+1},d,f,g,\varphi))+L\lim_{n\to+\infty}\psi(N(x_n,x_{n+1},d,f,g,\varphi)))\\&\le\frac{1}{s}\lim_{n\to+\infty}\psi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}))+L\lim_{n\to+\infty}\psi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}))\\&<\lim_{n\to+\infty}\psi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}))\\&=\psi(r).\end{aligned}$$

The above holds unless $\lim_{n\to+\infty}(d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))=r=0$. It follows that $\lim_{n\to+\infty}d(y_n,y_{n+1})=0$ and $\lim_{n\to+\infty}\varphi(y_n)=0$.

Now we shall prove that $\{y_n\}$ is a Cauchy sequence in X. If $\{y_n\}$ is not Cauchy, that is, there exists $\varepsilon>0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ satisfying n_k is the smallest index for which $n_k>m_k>k$,

$$\varepsilon\le d(y_{m_k},y_{n_k}),\tag{9}$$

$$\text{and}\ \ d(y_{m_k},y_{n_k-1})<\varepsilon. \tag{10}$$

By the triangle inequality in $b-$metric space and (9), (10), we have

$$\varepsilon\leq d(y_{m_k},y_{n_k})\leq sd(y_{m_k},y_{n_k-1})+sd(y_{n_k-1},y_{n_k})<s\varepsilon+sd(y_{n_k-1},y_{n_k}).$$

By passing to limit as $k\to+\infty$ in the above inequality, we have $\varepsilon\leq\limsup\limits_{k\to+\infty}d(y_{m_k},y_{n_k})\leq s\varepsilon$. Also,

$$d(y_{m_k},y_{n_k})\leq sd(y_{m_k},y_{n_k-1})+sd(y_{n_k-1},y_{n_k}), \tag{11}$$

$$d(y_{m_k},y_{n_k})\leq sd(y_{m_k},y_{m_k-1})+sd(y_{m_k-1},y_{n_k}), \tag{12}$$

$$d(y_{m_k-1},y_{n_k})\leq sd(y_{m_k-1},y_{m_k})+sd(y_{m_k},y_{n_k}). \tag{13}$$

Taking account into (9), (10) and (11), we obtain

$$\frac{\varepsilon}{s}\leq\limsup_{k\to+\infty}d(y_{m_k},y_{n_k-1})\leq\varepsilon.$$

Using (9), (12) and (13), we get

$$\frac{\varepsilon}{s}\leq\limsup_{k\to+\infty}d(y_{m_k-1},y_{n_k})\leq s^2\varepsilon.$$

Similarly,

$$\begin{aligned}d(y_{m_k-1},y_{n_k-1})&\leq sd(y_{m_k-1},y_{m_k})+sd(y_{m_k},y_{n_k-1}),\\ d(y_{m_k},y_{n_k})&\leq sd(y_{m_k},y_{m_k-1})+s^2d(y_{m_k-1},y_{n_k-1})+s^2d(y_{n_k-1},y_{n_k}),\end{aligned}$$

so there is

$$\frac{\varepsilon}{s^2}\leq\limsup_{k\to+\infty}d(y_{m_k-1},y_{n_k-1})\leq s\varepsilon.$$

In view of the definition of $M(x,y,d,f,g,\varphi)$, we deduce

$$\begin{aligned}M(x_{m_k},x_{n_k},d,f,g,\varphi)=&\max\{d(gx_{m_k},gx_{n_k})+\varphi(gx_{m_k})+\varphi(gx_{n_k}),\\ &\frac{1}{2}\{d(fx_{m_k},gx_{m_k})+\varphi(fx_{m_k})+\varphi(gx_{m_k})+d(fx_{n_k},gx_{n_k})+\varphi(fx_{n_k})+\varphi(gx_{n_k})\},\\ &\frac{1}{2s}\{d(fx_{m_k},gx_{n_k})+\varphi(fx_{m_k})+\varphi(gx_{n_k})+d(fx_{n_k},gx_{m_k})+\varphi(fx_{n_k})+\varphi(gx_{m_k})\}\}\\ =&\max\{d(y_{m_k-1},y_{n_k-1})+\varphi(y_{m_k-1})+\varphi(y_{n_k-1}),\\ &\frac{1}{2}\{d(y_{m_k},y_{m_k-1})+\varphi(y_{m_k})+\varphi(y_{m_k-1})+d(y_{n_k},y_{n_k-1})+\varphi(y_{n_k})+\varphi(y_{n_k-1})\},\\ &\frac{1}{2s}\{d(y_{m_k},y_{n_k-1})+\varphi(y_{m_k})+\varphi(y_{n_k-1})+d(y_{n_k},y_{m_k-1})+\varphi(y_{n_k})+\varphi(y_{m_k-1})\}\}.\end{aligned} \tag{14}$$

Taking the upper limit as $k\to+\infty$ in (14), we obtain

$$\limsup_{k\to+\infty}M(x_{m_k},x_{n_k},d,f,g,\varphi)\leq\max\{s\varepsilon,0,\frac{\varepsilon+s^2\varepsilon}{2s}\}=s\varepsilon.$$

Also, we have

$$\begin{aligned} N(x_{m_k}, x_{n_k}, d, f, g, \varphi) &= \frac{1}{2}\min\{d(fx_{m_k}, fx_{n_k}) + \varphi(fx_{m_k}) + \varphi(fx_{n_k}), d(fx_{n_k}, gx_{n_k}) + \varphi(fx_{n_k}) + \varphi(gx_{n_k})\} \\ &= \frac{1}{2}\min\{d(y_{m_k}, y_{n_k}) + \varphi(y_{m_k}) + \varphi(y_{n_k}), d(y_{n_k}, y_{n_k-1}) + \varphi(y_{n_k}) + \varphi(y_{n_k-1})\}. \end{aligned}$$

It follows that

$$\limsup_{k\to+\infty} N(x_{m_k}, x_{n_k}, d, f, g, \varphi) < s\varepsilon.$$

Using the transitivity property type s^p of α, we have $\alpha(gx_{m_k}, gx_{n_k}) \geq s^p$. As we have seen in (1) with $x = x_{m_k}$ and $y = x_{n_k}$, one can get

$$\begin{aligned} \psi(s\varepsilon) &\leq \psi(s^p\varepsilon) \\ &\leq \psi(\alpha(gx_{m_k}, gx_{n_k}) \limsup_{n\to+\infty}[d(y_{m_k}, y_{n_k}) + \varphi(y_{m_k}) + \varphi(y_{n_k})]) \\ &\leq \limsup_{n\to+\infty} \beta(\psi(M(x_{m_k}, x_{n_k}, d, f, g, \varphi)))\psi(M(x_{m_k}, x_{n_k}, d, f, g, \varphi)) + \limsup_{n\to+\infty} L\psi(N(x_{m_k}, x_{n_k}, d, f, g, \varphi)) \\ &\leq \frac{1}{s}\psi(s\varepsilon) + L\psi(s\varepsilon) \\ &< \psi(s\varepsilon), \end{aligned}$$

which is a contradiction. It follows that $\{y_n\}$ is a Cauchy sequence in X. The completeness of X ensures that there exists a $u \in X$ such that

$$\lim_{n\to+\infty} d(y_n, u) = \lim_{n\to+\infty} d(fx_n, u) = \lim_{n\to+\infty} d(gx_{n+1}, u) = \lim_{n,m\to+\infty} d(y_n, y_m) = 0. \tag{15}$$

Furthermore, we have $u \in g(X)$ since $g(X)$ is closed. It follows that one can choose a $z \in X$ such that $u = gz$, and one can write (15) as

$$\lim_{n\to+\infty} d(y_n, gz) = \lim_{n\to+\infty} d(fx_n, gz) = \lim_{n\to+\infty} d(gx_{n+1}, gz) = 0.$$

By virtue of the definition of φ, we get

$$\varphi(gz) = \varphi(u) \leq \liminf_{n\to+\infty} \varphi(y_n) = 0.$$

That is, $\varphi(gz) = \varphi(u) = 0$. The property H_{s^p} yields that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ so that $\alpha(y_{n_k-1}, gz) \geq s^p$ for all $k \in \mathbb{N}$. If $fz \neq gz$, taking $x = x_{n_k}$ and $y = z$ in contractive condition (1), we deduce that

$$\begin{aligned} \psi(d(fx_{n_k}, fz) + \varphi(fx_{n_k}) + \varphi(fz)) \leq &\psi(s^p[d(fx_{n_k}, fz) + \varphi(fx_{n_k}) + \varphi(fz)]) \\ \leq &\psi(\alpha(y_{n_k-1}, gz)[d(fx_{n_k}, fz) + \varphi(fx_{n_k}) + \varphi(fz)]) \\ \leq &\beta(\psi(M(x_{n_k}, z, d, f, g, \varphi)))\psi(M(x_{n_k}, z, d, f, g, \varphi)) + L\psi(N(x_{n_k}, z, d, f, g, \varphi)), \end{aligned} \tag{16}$$

where

$$\begin{aligned} M(x_{n_k}, z, d, f, g, \varphi) = \max\{&d(gx_{n_k}, gz) + \varphi(gx_{n_k}) + \varphi(gz), \\ &\frac{1}{2}\{d(fx_{n_k}, gx_{n_k}) + \varphi(fx_{n_k}) + \varphi(gx_{n_k}) + d(fz, gz) + \varphi(fz) + \varphi(gz)\}, \\ &\frac{1}{2s}\{d(fx_{n_k}, gz) + \varphi(fx_{n_k}) + \varphi(gz) + d(fz, gx_{n_k}) + \varphi(fz) + \varphi(gx_{n_k})\}\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{d(y_{n_k-1},gz)+\varphi(y_{n_k-1})+\varphi(gz),\\
&\quad \frac{1}{2}\{d(y_{n_k},y_{n_k-1})+\varphi(y_{n_k})+\varphi(y_{n_k-1})+d(fz,gz)+\varphi(fz)+\varphi(gz)\},\\
&\quad \frac{1}{2s}\{d(y_{n_k},gz)+\varphi(y_{n_k})+\varphi(gz)+d(fz,y_{n_k-1})+\varphi(fz)+\varphi(y_{n_k-1})\}\},\\
N(x_{n_k},z,d,f,g,\varphi) &= \frac{1}{2}\min\{d(fx_{n_k},fz)+\varphi(fx_{n_k})+\varphi(fz),d(fz,gz)+\varphi(fz)+\varphi(gz)\}\\
&= \frac{1}{2}\min\{d(y_{n_k},fz)+\varphi(y_{n_k})+\varphi(fz),d(fz,gz)+\varphi(fz)+\varphi(gz)\}.
\end{aligned}
$$

By simple calculation, we obtain

$$
\limsup_{k\to+\infty} M(x_{n_k},z,d,f,g,\varphi)\le d(fz,gz)+\varphi(fz), \tag{17}
$$

$$
\textit{and}\quad \limsup_{k\to+\infty} N(x_{n_k},z,d,f,g,\varphi)< d(fz,gz)+\varphi(fz). \tag{18}
$$

By taking the supper limit as $k\to+\infty$ in (16) and using (17) and (18), one can get

$$
\begin{aligned}
\psi(d(fz,gz)+\varphi(fz)) &\le \frac{1}{s}\psi(d(fz,gz)+\varphi(fz))+L\psi(d(fz,gz)+\varphi(fz))\\
&< \psi(d(fz,gz)+\varphi(fz)).
\end{aligned}
$$

Evidently, $d(fz,gz)+\varphi(fz)=0$, which implies that $fz=gz$ and $\varphi(fz)=0$.

Now we claim that z is the unique coincidence point of f and g. If not, there exist $z,z'\in C(f,g)$ and $z\neq z'$, according to the property of U_{s^p}, we deduce that $\alpha(gz,gz')\ge s^p$, applying (1) with $x=z$ and $y=z'$, we obtain that

$$
\begin{aligned}
\psi(d(fz,fz')+\varphi(fz)+\varphi(fz')) &\le \psi(s^p[d(fz,fz')+\varphi(fz)+\varphi(fz')])\\
&\le \psi(\alpha(gz,gz')[d(fz,fz')+\varphi(fz)+\varphi(fz')])\\
&\le \beta(\psi(M(z,z',d,f,g,\varphi)))\psi(M(z,z',d,f,g,\varphi))+L\psi(N(z,z',d,f,g,\varphi)).
\end{aligned} \tag{19}
$$

Here,

$$
\begin{aligned}
M(z,z',d,f,g,\varphi) &= \max\Big\{d(gz,gz')+\varphi(gz)+\varphi(gz'),\frac{1}{2}\{d(fz,gz)+\varphi(fz)+\varphi(gz)+d(fz',gz')+\varphi(fz')+\varphi(gz')\},\\
&\quad \frac{1}{2s}\{d(fz,gz')+\varphi(fz)+\varphi(gz')+d(fz',gz)+\varphi(fz')+\varphi(gz)\}\Big\}\\
&\le d(gz,gz')+\varphi(gz'),\\
N(z,z',d,f,g,\varphi) &= \frac{1}{2}\min\{d(fz,fz')+\varphi(fz)+\varphi(fz'),d(fz',gz')+\varphi(fz')+\varphi(gz')\}\\
&< d(gz,gz')+\varphi(gz').
\end{aligned}
$$

It follows from (19) that

$$
\begin{aligned}
\psi(d(gz,gz')+\varphi(gz')) &\le \frac{1}{s}\psi(d(gz,gz')+\varphi(gz'))+L\psi(d(gz,gz')+\varphi(gz'))\\
&< \psi(d(gz,gz')+\varphi(gz')).
\end{aligned}
$$

Hence, we get that $d(gz,gz')+\varphi(gz')=0$, which implies that $gz=gz'$ and $\varphi(gz')=0$. Since g is a injective mapping, then $z=z'$, that is z is a unique coincidence point of f and g. Further, owing to the weak compatibility of f and g, it is easy to show that z is a unique common fixed point of f and g. This completes the proof. □

Example 3.5. *Let $X=[0,+\infty)$ and $d(x,y)=(x-y)^2$. Define mappings $f,g:X\to X$ by*

$$fx=\begin{cases}\frac{x^2}{8}, & x\in[0,1]\\ 2x, & x>1,\end{cases}\quad \text{and}\quad gx=\begin{cases}\frac{7x^2}{8}, & x\in[0,1]\\ \frac{7x}{4}, & x>1.\end{cases}$$

Define mapping $\alpha: g(X)\times g(X)\to[0,+\infty)$ by

$$\alpha(x,y)=\begin{cases}s, & x,y\in[0,\frac{7}{4}]\\ 0, & \text{otherwise.}\end{cases}$$

Define mappings $\psi:[0,+\infty)\to[0,+\infty)$ and $\varphi: X\to X$ with $\psi(t)=t,\varphi(x)=x^2, x\in[0,+\infty)$. Let β be a function on $[0,+\infty)$ defined by $\beta(t)=\frac{16}{49}+\frac{1}{49+t}$ for all $t\geq 0$, then $\beta\in\Omega$. It is clear that $f(X)\subset g(X)$. For $x,y\in X$ such that $\alpha(gx,gy)\geq s$, we can know that $gx,gy\in[0,\frac{7}{4}]$ and this implies that $x,y\in[0,1]$, so we obtain $fx,fy\in[0,\frac{7}{4}]$ by definitions and $\alpha(fx,fy)\geq s$. That is, f is $g-\alpha_s-$admissible mapping. For all $x,y\in[0,1]$, we have

$$\begin{aligned}\psi(\alpha(gx,gy)[d(fx,fy)+\varphi(fx)+\varphi(fy)]) =&2\cdot[(\frac{x^2}{8}-\frac{y^2}{8})^2+(\frac{x^2}{8})^2+(\frac{y^2}{8})^2]\\ =&4\cdot\frac{1}{64}(x^4+y^4-x^2y^2)\\ =&\frac{1}{16}(x^4+y^4-x^2y^2),\\ \beta(M(x,y,d,f,g,\varphi))\psi(M(x,y,d,f,g,\varphi))\geq&\frac{16}{49}[\psi(d(gx,gy)+\varphi(gx)+\varphi(gy))]\\ =&\frac{16}{49}[(\frac{7x^2}{8}-\frac{7y^2}{8})^2+(\frac{7x^2}{8})^2+(\frac{7y^2}{8})^2]\\ =&\frac{16}{49}\cdot\frac{49}{32}(x^4+y^4-x^2y^2)\\ =&\frac{1}{2}(x^4+y^4-x^2y^2).\end{aligned}$$

According to above inequalities, remark that

$$\begin{aligned}\psi(\alpha(gx,gy)[d(fx,fy)+\varphi(fx)+\varphi(fy)]) =&\frac{1}{16}(x^4+y^4-x^2y^2)\\ \leq&\frac{1}{2}(x^4+y^4-x^2y^2)\\ \leq&\beta(M(x,y,d,f,g,\varphi))\psi(M(x,y,d,f,g,\varphi))+L\psi(N(x,y,d,f,\varphi)).\end{aligned}$$

with $L\geq 0,\frac{1}{s}+L<1$. It follows that f is a generalized $(g-\alpha_{s^p},\psi,\varphi)$ contractive mapping and all conditions of Theorem 3.4 are satisfied with $s=2$. It is easy to obtain that 0 is the unique common fixed point of f and g.

If $\varphi=0$ in Theorem 3.4, we can get the following sequence:

Corollary 3.6. *Let (X,d) be a complete $b-$metric space with parameter $s\geq 1$ and let $f,g:X\to X$ be given self-mappings satisfying g is injective and $f(X)\subset g(X)$ where $g(X)$ is closed. Suppose $p\geq 1$ is a constant, and $\alpha: X\times X\to[0,+\infty)$ a given mapping. If the following conditions are satisfied:*

(i). f is $g-\alpha_{s^p}-$ admissible mapping,

(ii). for each $x,y\in X$

$$\psi(\alpha(gx,gy)d(fx,fy))\leq\beta(\psi(M_1(x,y)))\psi(M_1(x,y))+L\psi(N_1(x,y)),$$

where

$$M_1(x,y)=\max\{d(gx,gy),\frac{1}{2}\{d(fx,gx)+d(fy,gy)\},\frac{1}{2s}\{d(fx,gy)+d(fy,gx)\}\},$$
$$\text{and}\quad N_1(x,y)=\frac{1}{2}\min\{d(fx,fy),d(fy,gy)\},$$

(iii). there is $x_0\in X$ with satisfying $\alpha(gx_0,fx_0)\geq s^p$,

(iv). properties (H_{s^p}) and (U_{s^p}) are satisfied,

(v). α has a transitive property type s^p, that is, for $x,y,z\in X$, $\alpha(x,y)\geq s^p$ and $\alpha(y,z)\geq s^p\Rightarrow\alpha(x,z)\geq s^p$.

Then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

If we consider corresponding problem in the setting of metric space, that is, $s=1$ in Theorem 3.4, one can obtain that:

Corollary 3.7. *Let (X,d) be a complete $b-$metric space with parameter $s\geq 1$ and let $f,g:X\to X$ be given self-mappings satisfying g is injective and $f(X)\subset g(X)$ where $g(X)$ is closed. Suppose $p\geq 1$ is a constant, and $\alpha:X\times X\to[0,+\infty)$ a given mapping. If the following conditions are satisfied:*

(i). f is $g-\alpha_{s^p}-$ admissible mapping,

(ii). for each $x,y\in X$

$$\psi(\alpha(gx,gy)d(fx,fy))\leq\beta(\psi(M_2(x,y)))\psi(M_2(x,y))+L\psi(N(x,y)),$$

where

$$\begin{aligned}M_2(x,y,d,f,g,\varphi)=\max\Big\{&d(gx,gy)+\varphi(gx)+\varphi(gy),\frac{1}{2}\{d(fx,gx)+\varphi(fx)+\varphi(gx)+d(fy,gy)+\varphi(fy)+\varphi(gy)\},\\&\frac{1}{2}\{d(fx,gy)+\varphi(fx)+\varphi(gy)+d(fy,gx)+\varphi(fy)+\varphi(gx)\}\Big\},\end{aligned}$$

(iii). there is $x_0\in X$ with satisfying $\alpha(gx_0,fx_0)\geq s^p$,

(iv). properties (H_{s^p}) and (U_{s^p}) are satisfied,

(v). α has a transitive property type s^p, that is, for $x,y,z\in X$, $\alpha(x,y)\geq s^p$ and $\alpha(y,z)\geq s^p\Rightarrow\alpha(x,z)\geq s^p$.

Then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Theorem 3.8. *Let (X,d) be a complete $b-$metric space with parameter $s\geq 1$. Let $f:X\to X$ be a given mapping and $\varphi:X\to[0,+\infty)$ be a lower semicontinuous with $\varphi(t)=0$ for $t\in Fix(f)$. Suppose $p\geq 3$ is a constant. If the following conditions are satisfied:*

(i). f is α_{s^p}-admissible mapping,

(ii). there is $x_0\in X$ with satisfying $\alpha(x_0,fx_0)\geq s^p$,

(iii). if there exist $\psi \in \Psi$*, and* $\frac{1}{s} + L < 1$ *such that*

$$\psi(\alpha(x,y)d(fx,fy) + \varphi(fx) + \varphi(fy)) \leq \frac{1}{s}\psi(h(x,y,d,f,\varphi)) + L\psi(q(x,y,d,f,\varphi)),$$

where

$$\begin{aligned} h(x,y,d,f,\varphi) = \max\Big\{ & d(x,y) + \varphi(x) + \varphi(y), \frac{1}{2}\{d(x,fx) + \varphi(x) + \varphi(fx) + d(fy,y) + \varphi(fy) + \varphi(y)\}, \\ & \frac{1}{2s}\{d(fx,y) + \varphi(fx) + \varphi(y) + d(fy,x) + \varphi(fy) + \varphi(x)\}\Big\}, \\ q(x,y,d,f,\varphi) = {} & \frac{1}{2}\min\{d(x,y) + \varphi(x) + \varphi(y), d(fy,y) + \varphi(fy) + \varphi(y)\}. \end{aligned}$$

(iv). properties (H_{s^p}) *and* (U_{s^p}) *are satisfied when* $g = I$*.*

(v). α *has a transitive property type* s^p*, that is, for* $x, y, z \in X$*,* $\alpha(x,y) \geq s^p$ *and* $\alpha(y,z) \geq s^p \Rightarrow \alpha(x,z) \geq s^p$*.*

Then f *has a unique common fixed point.*

Proof. The proof is similar to that of Theorem 3.4, we omit it. □

4. Application

In this section, by using Theorem 3.8, we will show the existence of a solution to the integral equation:

$$x(t) = \int_0^T G(t,r,x(r))dr. \tag{20}$$

Let $X = C([0,T])$ be the set of real continuous functions defined on $[0,T]$. For $p \geq 1$, we define

$$d(x,y) = (\rho(x,y))^p = \sup_{t \in [0,T]} |x(t) - y(t)|^p \text{ for all } x, y \in X.$$

It is easy to prove that (X,d) is a complete $b-$metric space with $s = 2^{p-1}$. Consider the mapping $f : X \to X$ defined by

$$fx(t) = \int_0^T G(t,r,x(r))dr$$

and let $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function.

Theorem 4.1. *Consider equation (20) and suppose that*

(i). $G : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}^+$ *is continuous,*

(ii). there exists $x_0 \in X$ *such that* $\xi(x_0(t), fx_0(t)) \geq 0$ *for all* $t \in [0,T]$*.*

(iii). for all $t \in [0,T]$ *and* $x, y \in X$*,* $\xi(x(t), y(t)) \geq 0$ *implies* $\xi(fx(t), fy(t)) \geq 0$*.*

(iv). properties (H_{s^p}) *and* (U_{s^p}) *are satisfied when* $g = I$*,*

(v). there exists a continuous function $\gamma : [0,T] \times [0,T] \to R^+$ *such that*

$$\sup_{t \in [0,T]} \int_0^T \gamma(t,r)dr \leq 1.$$

(vi). there exists a constant $L\in(0,1)$ such that for $(t,r)\in[0,T]\times[0,T]$,

$$|G(t,r,x(r))-G(t,r,y(r))|\leq\sqrt[p]{\frac{1}{s^{p+1}}}\gamma(t,r)|x(r)-y(r)|.$$

Then the integral equation (20) has a unique solution $x\in X$.

Proof. Define $\alpha: X\times X\to[0,+\infty)$ by

$$\alpha(x,y)=\begin{cases}s^p, & \text{if } \xi(x(t),y(t))\geq 0,\\ 0, & otherwise.\end{cases}$$

It is easy to prove that f is α_{s^p}-admissible. For $x,y\in X$, by virtue of assumptions (i)-(vi), we have

$$\begin{aligned}s^pd(fx(t),fy(t))&=s^p\sup_{t\in[0,T]}|fx(t)-fy(t)|^p\\&=s^p\sup_{t\in[0,T]}|\int_0^T G(t,r,x(r))dr-\int_0^T G(t,r,y(r))dr|^p\\&\leq s^p\sup_{t\in[0,T]}(\int_0^T|G(t,r,x(r))-G(t,r,y(r))|dr)^p\\&\leq s^p\sup_{t\in[0,T]}(\int_0^T\sqrt[p]{\frac{1}{s^{p+1}}}\gamma(t,r)|x(r)-y(r)|dr)^p\\&\leq s^p\sup_{t\in[0,T]}(\int_0^T\sqrt[p]{\frac{1}{s^{p+1}}}\gamma(t,r)dr)^p\sup_{t\in[0,T]}|x(t)-y(t)|^p\\&\leq\frac{1}{s}\psi(h(x,y,d,f,\varphi)),\end{aligned}$$

which implies that

$$\psi(\alpha(x,y)d(fx,fy)+\varphi(fx)+\varphi(fy))\leq\frac{1}{s}\psi(h(x,y,d,f,\varphi))+L\psi(q(x,y,d,f,\varphi)).$$

Therefore, letting $\psi(t)=t$, and $\varphi(t)=0$, all the conditions of Theorem 3.8 are satisfied. As a result, the mapping f has a unique fixed point $x\in X$, which is a solution of the integral equation (20). □

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