Int. J. Math. And Appl., 6(1–A)(2018), 187–198
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

On the Properties of k-Fibonacci-Like Sequence

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Abstract: In this article, we introduce a new generalization of Fibonacci sequence and we call it as k-Fibonacci-Like sequence. After that we obtain some fundamental properties for k-Fibonacci-Like sequence and also we present some relations among k-Fibonacci-Like sequence, k-Fibonacci sequence and k-Lucas sequence by some algebraic methods.

MSC: 11B37, 11B39.

Keywords: Fibonacci Sequence, Generalized Fibonacci Sequence, k-Fibonacci Sequence and k-Lucas Sequence.

1. Introduction

Fibonacci sequence is the most prominent examples of recursive sequence. It is famous for possessing wonderful and amazing properties; though some are simple and known, others find broad scope in research work. The most well-known Fibonacci sequence is given by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \text{ and } F_0 = 0,\ F_1 = 1 \tag{1}$$

The Fibonacci sequence has been studied extensively and generalized in many ways. However, in the present article, we are most interested in the generalizations of the Fibonacci sequence. Several authors made possible generalizations of the Fibonacci numbers, included the real numbers (and sometimes the complex numbers) in their domain. The two most intriguing generalizations of Fibonacci sequence are k-Fibonacci sequence $\langle F_{k,n}\rangle$ [1] and k-Lucas sequence $\langle L_{k,n}\rangle$ [2]. Chong et al. [3] introduced the generalized Fibonacci sequence $\langle U_n\rangle$ and obtained some identities for it and the authors defined generalized Fibonacci sequence $\langle U_n\rangle$ as

$$U_{n+2} = pU_{n+1} + qU_n,\ n \geq 0 \text{ and } U_0 = 0,\ U_1 = 1 \tag{2}$$

where $p, q \in \mathbb{Z}^+$. Panwar et al. [4] presented some properties of generalized Fibonacci sequence and is given by the following equation:

$$F_k = pF_{k-1} + qF_{k-2},\ k \geq 2 \text{ and } F_0 = a,\ F_1 = b \tag{3}$$

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where p, q, a, b are positive integers. In [5] the authors introduced another generalized Fibonacci sequence and they named it as Fibonacci-Like sequence and is given by

$$H_n = 2H_{n-1} + H_{n-2},\ n \geq 2 \text{ and } H_0 = 2,\ H_1 = 1 \tag{4}$$

Some new generalizations are introduced for Fibonacci and Lucas sequences by Bilgici [6] and these generalizations are defined respectively by

$$f_n = 2af_{n-1} + \left(b^2 - a\right) f_{n-2},\ f_0 = 0,\ f_1 = 1 \tag{5}$$

$$l_n = 2al_{n-1} + \left(b^2 - a\right) l_{n-2},\ l_0 = 2,\ l_1 = 2a \tag{6}$$

where $a,\ b$ are non-zero real numbers. $\langle q_n \rangle$ is another generalization of Fibonacci numbers introduced by Edson and Yayenie [7], which is delineated by

$$q_0 = 0,\ q_1 = 1,\ q_n = \begin{cases} aq_{n-1} + q_{n-2} & if\ n\ is\ even \\ bq_{n-1} + q_{n-2} & if\ n\ is\ odd \end{cases} \qquad (n \geq 2) \tag{7}$$

Some other generalizations are introduced by the authors in [8–11].

2. Preliminary Notes

Definition 2.1 ([1]). *For $k \in \mathbb{R}$, the k-Fibonacci sequence $\langle F_{k,n} \rangle$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1},\ n \geq 1 \text{ and } F_{k,0} = 0,\ F_{k,1} = 1 \tag{8}$$

Definition 2.2 ([2]). *For $k \in \mathbb{R}$, the k-Lucas sequence $\langle L_{k,n} \rangle$ is defined recurrently by*

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1},\ n \geq 1 \text{ and } L_{k,0} = 2,\ L_{k,1} = k \tag{9}$$

Definition 2.3. *For $k \in \mathbb{R}$, the k-Fibonacci-Like sequence say $\langle V_{k,n} \rangle$ is defined by the following equation:*

$$V_{k,n+1} = kV_{k,n} + V_{k,n-1},\ n \geq 1 \text{ and } V_{k,0} = 2m,\ V_{k,1} = p + mk \tag{10}$$

where m and p are positive integers.

Then by [12], all the above three recurrence relations have the common characteristic equation $y^2 - ky - 1 = 0$, with two distinct roots r and s. Note that r and s are

$$r = \frac{k + \sqrt{k^2 + 4}}{2} \quad \text{and} \quad s = \frac{k - \sqrt{k^2 + 4}}{2} \tag{11}$$

Also we examine easily r and s have the following properties:

a) $rs = -1$, $r + s = k$ and $r - s = \sqrt{k^2 + 4}$

b) $r^2 - 1 = kr$ and $s^2 - 1 = ks$

c) $r^2 + 1 = kr + 2 = (r - s)\, r$ and $s^2 + 1 = ks + 2 = -(r - s)\, s$

3. Main Results

3.1. Some Fundamental Results of k-Fibonacci-Like Sequence $\langle V_{k,n}\rangle$

Theorem 3.1 (Binet's Formula for $\langle V_{k,n}\rangle$). *For $n \in \mathbb{Z}^+$, the n^{th} term for $\langle V_{k,n}\rangle$ is given by the following equation:*

$$V_{k,n} = p\frac{r^n - s^n}{r-s} + m\left(r^n + s^n\right) \tag{12}$$

where r and s are given in the equation (11).

Proof. The general form of k-Fibonacci-Like sequence (10) may be expressed in the form:

$$V_{k,n} = Ar^n + Bs^n \tag{13}$$

where A and B are constants that can be determined by the initial conditions of recurrence relation (10). So put the values $n = 0$ and $n = 1$ in equation (13), we have

$$A + B = 2m \quad \text{and} \quad Ar + Bs = p + mk$$

After solving the above system of equations for A and B, we achieve

$$A = \frac{p + mk - 2ms}{r-s} \quad \text{and} \quad B = \frac{2mr - p - mk}{r-s} \tag{14}$$

Thus,

$$\begin{aligned}
V_{k,n} &= \frac{1}{r-s}\Big[\left(p + mk - 2ms\right)r^n + \left(2mr - p - mk\right)s^n\Big] \\
&= p\frac{r^n - s^n}{r-s} + \frac{1}{r-s}\left(mkr^n - 2msr^n - mks^n + 2mrs^n\right) \\
&= p\frac{r^n - s^n}{r-s} + \frac{1}{r-s}\left(mkr^n + 2mr^{n-1} - mks^n - 2mrs^{n-1}\right) \\
&= p\frac{r^n - s^n}{r-s} + \frac{1}{r-s}\left[mr^{n-1}\left(rk+2\right) - ms^{n-1}\left(sk+2\right)\right] \\
&= p\frac{r^n - s^n}{r-s} + \frac{1}{r-s}\Big[mr^n\left(r-s\right) + ms^n\left(r-s\right)\Big] \\
&= p\frac{r^n - s^n}{r-s} + m\left(r^n + s^n\right)
\end{aligned}$$

Hence the result. □

Theorem 3.2 (General form for Negative k-Fibonacci-Like Numbers $\langle V_{k,-n}\rangle$). *For $n \in \mathbb{Z}_0$, the n^{th} term for $\langle V_{k,-n}\rangle$ is given by*

$$V_{k,-n} = (-1)^n\left[m\left(r^n + s^n\right) - p\frac{r^n - s^n}{r-s}\right] \tag{15}$$

Proof. From equation (13), we have

$$\begin{aligned}
V_{k,-n} &= \left(Ar^{-n} + Bs^{-n}\right) \\
&= \frac{A}{r^n} + \frac{B}{s^n}
\end{aligned}$$

$$= (-1)^{-n} \left(As^n + Br^n \right)$$

After the use of value of A and B from the equation (14), we achieve

$$\begin{aligned}
V_{k,-n} &= \frac{(-1)^n}{r-s} \left[(p + mk - 2ms)\, s^n + (2mr - p - mk)\, r^n \right] \\
&= (-1)^n \left[-p\frac{r^n - s^n}{r-s} + \frac{2mr^{n+1} - mkr^n - 2ms^{n+1} + mks^n}{r-s} \right] \\
&= (-1)^n \left[-p\frac{r^n - s^n}{r-s} + \frac{mr^n\,(2r-k) - ms^n\,(2s-k)}{r-s} \right] \\
&= (-1)^n \left[-p\frac{r^n - s^n}{r-s} + \frac{mr^n\,(r-s) + ms^n\,(r-s)}{r-s} \right] \\
&= (-1)^n \left[-p\frac{r^n - s^n}{r-s} + m\,(r^n + s^n) \right] \\
&= (-1)^n \left[m\,(r^n + s^n) - p\frac{r^n - s^n}{r-s} \right]
\end{aligned}$$

Hence the result. □

Theorem 3.3 (Generating Function for $\langle V_{k,n} \rangle$)**.**

$$\sum_{n=0}^{\infty} V_{k,n} y^n = \frac{2m + (p - mk)\, y}{1 - ky - y^2} \tag{16}$$

Proof. Since

$$\begin{aligned}
\sum_{n=0}^{\infty} V_{k,n} y^n &= \sum_{n=0}^{\infty} \left(Ar^n + Bs^n \right) y^n \\
&= A \sum_{n=0}^{\infty} r^n y^n + B \sum_{n=0}^{\infty} s^n y^n \\
&= \frac{A}{1 - ry} + \frac{B}{1 - sy} \\
&= \frac{(A+B) - (As + Br)\, y}{1 - ky - y^2}
\end{aligned}$$

Since $A + B = 2m$ and $As + Br = -(p - mk)$ and then

$$\sum_{n=0}^{\infty} V_{k,n} y^n = \frac{2m + (p - mk)\, y}{1 - ky - y^2}$$

as required. □

Theorem 3.4. *Let $\langle V_{k,n} \rangle$ denote the k-Fibonacci-Like sequence defined in equation (10), we have the following results:*

$$\sum_{i=1}^{n} V_{k,i} = \frac{V_{k,n+1} + V_{k,n} - V_{k,1} - V_{k,0}}{k} \tag{17}$$

$$\sum_{i=1}^{n} V_{k,2i} = \frac{V_{k,2n+1} - V_{k,1}}{k} \tag{18}$$

$$\sum_{i=1}^{n} V_{k,2i-1} = \frac{V_{k,2n} - V_{k,0}}{k} \tag{19}$$

$$\sum_{i=1}^{\infty} \frac{V_{k,i}}{t^i} = \frac{V_{k,0} + tV_{k,1}}{t^2 - tk - 1}, \quad \forall\, t \in \mathbb{R} \quad \text{and} \quad t > r \tag{20}$$

Proof (17). Since $V_{k,n+1} = kV_{k,n} + V_{k,n-1}$ and then

$$\begin{aligned}
\sum_{i=1}^{n} V_{k,i+1} &= k\sum_{i=1}^{n} V_{k,i} + \sum_{i=1}^{n} V_{k,i-1} \\
\sum_{i=2}^{n+2} V_{k,i} &= k\sum_{i=1}^{n} V_{k,i} + \sum_{i=0}^{n-1} V_{k,i} \\
\sum_{i=1}^{n} V_{k,i} + V_{k,n+1} - V_{k,1} &= k\sum_{i=1}^{n} V_{k,i} + \sum_{i=1}^{n} V_{k,i} - V_{k,n} + V_{k,0} \\
\sum_{i=1}^{n} V_{k,i} &= \frac{V_{k,n+1} + V_{k,n} - V_{k,1} - V_{k,0}}{k}
\end{aligned}$$

Hence the result. □

Proof (18). Since $V_{k,2n} = kV_{k,2n-1} + V_{k,2n-2}$, we have

$$\begin{aligned}
\sum_{i=1}^{n} V_{k,2i} &= k\sum_{i=1}^{n} V_{k,2i-1} + \sum_{i=1}^{n} V_{k,2i-2} \\
\sum_{i=1}^{n} V_{k,2i} &= k\sum_{i=1}^{n} (V_{k,2i-1} + V_{k,2i} - V_{k,2i}) + \sum_{i=1}^{n} V_{k,2i-2} \\
\sum_{i=1}^{n} V_{k,2i} &= k\sum_{i=1}^{2n} V_{k,i} - k\sum_{i=1}^{n} V_{k,2i} + \sum_{i=1}^{n} V_{k,2i} - V_{k,2n} + V_{k,0} \\
k\sum_{i=1}^{n} V_{k,2i} &= k\sum_{i=1}^{2n} V_{k,i} - V_{k,2n} + V_{k,0}
\end{aligned}$$

By using equation (17), we have

$$\begin{aligned}
k\sum_{i=1}^{n} V_{k,2i} &= V_{k,2n+1} + V_{k,2n} - V_{k,1} - V_{k,0} - V_{k,2n} + V_{k,0} \\
\sum_{i=1}^{n} V_{k,2i} &= \frac{V_{k,2n+1} - V_{k,1}}{k}
\end{aligned}$$

Hence the proof. □

Proof (19).

$$\begin{aligned}
\sum_{i=1}^{n} V_{k,2i-1} &= k\sum_{i=1}^{n} (V_{k,2i-1} + V_{k,2i} - V_{k,2i}) \\
\sum_{i=1}^{n} V_{k,2i-1} &= k\sum_{i=1}^{2n} V_{k,i} - \sum_{i=1}^{n} V_{k,2i}
\end{aligned}$$

Bu using equations (18) and (19), we get the desired result. □

Proof (20). If we consider the equation (13), we get

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{V_{k,i}}{t^i} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{Ar^i + Bs^i}{t^i} \\
&= \lim_{n\to\infty} \left(A\sum_{i=1}^{n} \frac{r^i}{t^i} + B\sum_{i=1}^{n} \frac{s^i}{t^i} \right) \\
&= \lim_{n\to\infty} \left\{ A\frac{\frac{r}{t}\left[\left(\frac{r}{t}\right)^n - 1\right]}{\frac{r}{t} - 1} + B\frac{\frac{s}{t}\left[\left(\frac{s}{t}\right)^n - 1\right]}{\frac{s}{t} - 1} \right\} \\
&= A\frac{-\frac{r}{t}}{\frac{r}{t} - 1} + B\frac{-\frac{s}{t}}{\frac{s}{t} - 1}
\end{aligned}$$

$$
\begin{aligned}
&= A\frac{r}{t-r} + B\frac{s}{t-s} \\
&= \frac{t\left(Ar+Bs\right) - \left(Ars+Brs\right)}{\left(t-r\right)\left(t-s\right)} \\
&= \frac{t\left(Ar+Bs\right) - \left(rs\right)\left(A+B\right)}{t^2 - t\left(r+s\right) + rs} \\
&= \frac{V_{k,0} + tV_{k,1}}{t^2 - tk - 1}
\end{aligned}
$$

Hence the result. □

Lemma 3.1. *For $n \in \mathbb{Z}_0$, the following result holds:*

$$\frac{2mV_{k,n+1} - \left(p+mk\right)V_{k,n}}{m^2k^2 + 4m^2 - p^2} = \frac{r^n - s^n}{r-s} \tag{21}$$

Proof. If we consider equation (13), we have

$$
\begin{aligned}
2mV_{k,n+1} - \left(p+mk\right)V_{k,n} &= 2m\left(Ar^{n+1} + Bs^{n+1}\right) - \left(p+mk\right)\left(Ar^n + Bs^n\right) \\
&= Ar^n\left(2mr - p - mk\right) - Bs^n\left(p + mk - 2ms\right)
\end{aligned}
$$

By using equation (14), we get

$$
\begin{aligned}
2mV_{k,n+1} - \left(p+mk\right)V_{k,n} &= ABr^n\left(r-s\right) - ABs^n\left(r-s\right) \\
&= AB\left(r-s\right)\left(r^n - s^n\right)
\end{aligned}
$$

After the use of value $AB = \dfrac{m^2k^2 + 4m^2 - p^2}{\left(r-s\right)^2}$, the proof is clearly seen. □

Theorem 3.5 (Catalan's Identity for $\langle V_{k,n}\rangle$)**.** *For $l, n \in \mathbb{Z}_0$, we get*

$$V_{k,n+l}V_{k,n-l} - V_{k,n}^2 = \frac{\left(-1\right)^{n-l}}{m^2k^2 + 4m^2 - p^2}\left[2mV_{k,l+1} - \left(p+mk\right)V_{k,l}\right]^2, \quad 0 \le l \le n \tag{22}$$

Proof. If we consider equation (13), we have

$$
\begin{aligned}
V_{k,n+l}V_{k,n-l} - V_{k,n}^2 &= \left(Ar^{n+l} + Bs^{n+l}\right)\left(Ar^{n-l} + Bs^{n-l}\right) - \left(Ar^n + Bs^n\right)^2 \\
&= AB\left(r^{n+l}s^{n-l} + r^{n-l}s^{n+l} - 2r^ns^n\right) \\
&= AB\left(rs\right)^{n-l}\left[\left(r^{2l} + s^{2l}\right) - 2\left(rs\right)^l\right] \\
&= \left(-1\right)^{n-l}\left(m^2k^2 + 4m^2 - p^2\right)\left(\frac{r^l - s^l}{r-s}\right)^2 \\
&= \frac{\left(-1\right)^{n-l}}{m^2k^2 + 4m^2 - p^2}\left[2mV_{k,l+1} - \left(p+mk\right)V_{k,l}\right]^2
\end{aligned}
$$

This completes the proof of the theorem. □

Theorem 3.6 (Cassini's Identity for $\langle V_{k,n}\rangle$)**.** *For $n \in \mathbb{N}$, we have*

$$V_{k,n+1}V_{k,n-1} - V_{k,n}^2 = \left(-1\right)^{n-1}\left(m^2k^2 + 4m^2 - p^2\right) \tag{23}$$

We prove the equation (23) by two ways as *Proof (1)* and *Proof (2)*.

Proof (1). If we put $l = 1$ in equation (22), we get the result. □

Proof (2). To prove the required result, we use Cramer's rule. Consider a 2×2 linear system of equations

$$\left\{\begin{array}{l} V_{k,n}x_1 + V_{k,n-1}x_2 = V_{k,n+1} \\ V_{k,n+1}x_1 + V_{k,n}x_2 = V_{k,n+2} \end{array}\right\} \tag{24}$$

Certainly $V_{k,n}^2 - V_{k,n+1}V_{k,n-1} \neq 0$ for $n \geq 1$. Let $D = V_{k,n}^2 - V_{k,n+1}V_{k,n-1}$ then by the concept of Cramer's rule, we achieve

$$x_1 = (D)^{-1} \begin{vmatrix} V_{k,n+1} & V_{k,n-1} \\ V_{k,n+2} & V_{k,n+1} \end{vmatrix} \quad \text{and} \quad x_2 = (D)^{-1} \begin{vmatrix} V_{k,n} & V_{k,n+1} \\ V_{k,n+1} & V_{k,n+2} \end{vmatrix}$$

By virtue of the recurrence relation (10), $x_1 = k$ and $x_2 = 1$ is the unique solution of the system (24). Therefore

$$1 = (D)^{-1} \begin{vmatrix} V_{k,n} & V_{k,n+1} \\ V_{k,n+1} & V_{k,n+2} \end{vmatrix}$$

$$V_{k,n+2}V_{k,n} - V_{k,n+1}^2 = -\left(V_{k,n+1}V_{k,n-1} - V_{k,n}^2\right)$$

Let $U_{k,n} = V_{k,n+1}V_{k,n-1} - V_{k,n}^2$ and $U_{k,n+1} = V_{k,n+2}V_{k,n} - V_{k,n+1}^2$. Then clearly $U_{k,n+1} = -U_{k,n}$ is a first order homogeneous recurrence relation. Thus $U_{k,n} = (-1)^{n+1}\left(m^2k^2 + 4m^2 - p^2\right)$ is its general solution. Therefore

$$V_{k,n+1}V_{k,n-1} - V_{k,n}^2 = (-1)^{n+1}\left(m^2k^2 + 4m^2 - p^2\right)$$

$$V_{k,n+1}V_{k,n-1} - V_{k,n}^2 = (-1)^{n-1}\left(m^2k^2 + 4m^2 - p^2\right) \quad \because \ (-1)^{n+1} = (-1)^{n-1}$$

Hence the result. □

Corollary 3.7.

$$\sum_{i=1}^{n} V_{k,i}^2 = \frac{V_{k,n+1}V_{k,n} - V_{k,0}V_{k,1}}{k} \tag{25}$$

Proof. Since

$$\begin{aligned} V_{k,i}^2 &= \left(\frac{V_{k,i+1} - V_{k,i-1}}{k}\right)^2 \\ \sum_{i=1}^{n} V_{k,i}^2 &= \sum_{i=1}^{n}\left(\frac{V_{k,i+1} - V_{k,i-1}}{k}\right)^2 \\ &= \frac{1}{k^2}\left(\sum_{i=1}^{n} V_{k,i+1}^2 + \sum_{i=1}^{n} V_{k,i-1}^2 - 2\sum_{i=1}^{n} V_{k,i-1}V_{k,i+1}\right) \\ &= \frac{1}{k^2}\left\{\sum_{i=1}^{n} V_{k,i+1}^2 + \sum_{i=1}^{n} V_{k,i-1}^2 - 2\sum_{i=1}^{n}\left[V_{k,i}^2 - (-1)^i\left(m^2k^2 + 4m^2 - p^2\right)\right]\right\} \\ &= \frac{1}{k^2}\left[\left(\sum_{i=1}^{n} V_{k,i}^2 + V_{k,n+1}^2 - V_{k,1}^2\right) + \left(\sum_{i=1}^{n} V_{k,i}^2 + V_{k,n}^2 + V_{k,0}^2\right) - 2\sum_{i=1}^{n} V_{k,i}^2 + 2\left(m^2k^2 + 4m^2 - p^2\right)\sum_{i=1}^{n}(-1)^i\right] \end{aligned}$$

Since $\sum_{i=1}^{n}(-1)^i = \dfrac{(-1)^n - 1}{2}$ and then

$$\sum_{i=1}^{n} V_{k,i}^2 = \frac{1}{k^2}\left[V_{k,n+1}^2 - V_{k,n}^2 - V_{k,1}^2 + V_{k,0}^2 + 2\left(m^2k^2 + 4m^2 - p^2\right)\frac{(-1)^n - 1}{2}\right]$$

$$= \frac{1}{k^2}\Big[V_{k,n+1}^2 - V_{k,n+1}V_{k,n-1} - (-1)^n\left(m^2k^2+4m^2-p^2\right) - V_{k,1}^2 + V_{k,0}^2 + (-1)^n\left(m^2k^2+4m^2-p^2\right) - \left(m^2k^2+4m^2-p^2\right)\Big]$$

Since $-V_{k,1}^2 + V_{k,0}^2 - \left(m^2k^2+4m^2-p^2\right) = -kV_{k,0}V_{k,1}$ and then

$$\begin{aligned}\sum_{i=1}^{n} V_{k,i}^2 &= \frac{1}{k^2}\Big[V_{k,n+1}\left(V_{k,n+1}-V_{k,n-1}\right) - kV_{k,0}V_{k,1}\Big]\\ &= \frac{V_{k,n+1}V_{k,n} - V_{k,0}V_{k,1}}{k}\end{aligned}$$

This completes the proof of the corollary. □

Theorem 3.8 (d' Ocagne's Identity for $\langle V_{k,n}\rangle$)**.** *For $l, n \in \mathbb{Z}_0$, we have*

$$V_{k,l}V_{k,n+1} - V_{k,l+1}V_{k,n} = (-1)^l\left[2mV_{k,n-l+1} - (p+mk)V_{k,n-l}\right], \quad 0 \le l \le n \tag{26}$$

Proof. To prove the equation (26), we shall use induction on l. Certainly the result is true for $l = 0$. Suppose that the result is true for all values j less than or equal $l-1$ and then

$$\begin{aligned}V_{k,l-2}V_{k,n+1} - V_{k,l-1}V_{k,n} &= (-1)^{l-2}\left[2mV_{k,n-l+3} - (p+mk)V_{k,n-l+2}\right] \text{ and}\\ V_{k,l-1}V_{k,n+1} - V_{k,l}V_{k,n} &= (-1)^{l-1}\left[2mV_{k,n-l+2} - (p+mk)V_{k,n-l+1}\right]\end{aligned}$$

Now we prove that the equation (26) is true for l and then

$$\begin{aligned}V_{k,l}V_{k,n+1} - V_{k,l+1}V_{k,n} &= V_{k,n+1}\left(kV_{l-1} + V_{k,l-2}\right) - V_{k,n}\left(kV_{k,l} + V_{k,l-1}\right)\\ &= k\left(V_{k,n+1}V_{k,l-1} - V_{k,n}V_{k,l}\right) + V_{k,n-1}V_{k,l-2} - V_{k,n}V_{k,l-1}\\ &= (-1)^{l-1}k\left[2mV_{k,n-l+2} - (p+mk)V_{k,n-l+1}\right] - (-1)^{l-1}\left[2mV_{k,n-l+3} - (p+mk)V_{k,n-l+2}\right]\\ &= (-1)^{l-1}\left[2m\left(kV_{k,n-l+2} - V_{k,n-l+3}\right) - (p+mk)\left(kV_{k,n-l+1} - V_{k,n-l+2}\right)\right]\\ &= (-1)^{l-1}\left[-2mV_{k,n-l+1} + (p+mk)V_{k,n-l}\right]\\ &= (-1)^{l}\left[2mV_{k,n-l+1} - (p+mk)V_{k,n-l}\right]\end{aligned}$$

as required. □

Theorem 3.9 (Generalized Identity for $\langle V_{k,n}\rangle$)**.** *For $l, n, q \in \mathbb{Z}^+$, we have the following result*

$$V_{k,q}V_{k,n} - V_{k,q-l}V_{k,n+l} = \frac{(-1)^{q-l+1}}{(p^2k^2+4m^2-p^2)}\left[2mV_{k,l+1} - (p+mk)V_{k,l}\right]\left[2mV_{k,n+l-q+1} - (p+mk)V_{k,n+l-q}\right] \tag{27}$$

where $1 \le l \le q$ and $1 \le n \le q$

Proof. If we consider the equation (13), we get

$$\begin{aligned}V_{k,q}V_{k,n} - V_{k,q-l}V_{k,n+l} &= \left(Ar^q + Bs^q\right)\left(Ar^n + Bs^n\right) - \left(Ar^{q-l} + Bs^{q-l}\right)\left(Ar^{n+l} + Bs^{n+l}\right)\\ &= AB\left(r^qs^n - r^{q-l}s^{n+l} + r^ns^q - r^{n+l}s^{q-l}\right)\\ &= AB\left[r^qs^n\left(1 - \frac{s^l}{r^l}\right) + r^ns^q\left(1 - \frac{r^l}{s^l}\right)\right]\end{aligned}$$

$$= AB\left[r^{q-l}s^n\left(r^l - s^l\right) - r^n s^{q-l}\left(r^l - s^l\right)\right]$$
$$= \frac{\left(m^2k^2 + 4m^2 - p^2\right)}{r-s}\frac{\left(r^l - s^l\right)}{r-s}\left(r^{q-l}s^n - r^n s^{q-l}\right)$$

By using equation (21), we have

$$V_{k,q}V_{k,n} - V_{k,q-l}V_{k,n+l} = \frac{\left[2mV_{k,l+1} - (p+mk)V_{k,l}\right]}{r-s}(rs)^{q-l+1}\left(r^{n+l-q} - s^{n+l-q}\right)$$
$$= \frac{(-1)^{q-l+1}}{(p^2k^2 + 4m^2 - p^2)}\left[2mV_{k,l+1} - (p+mk)V_{k,l}\right]\left[2mV_{k,n+l-q+1} - (p+mk)V_{k,n+l-q}\right]$$

This completes the proof of the theorem. □

Theorem 3.10 (Binomial Form of $\langle V_{k,n}\rangle$)**.** *For $n \in \mathbb{Z}_0$, the following result holds*

$$V_{k,2n} = \sum_{i=0}^{n}\binom{n}{i}k^i V_{k,i} \tag{28}$$

Proof. Again from the equation (13), we achieve

$$\begin{aligned} V_{k,2n} &= A\left(r^2\right)^n + B\left(s^2\right)^n \\ &= A\,(kr+1)^n + B\,(ks+1)^n \\ &= A\sum_{i=0}^{n}\binom{n}{i}(kr)^i + B\sum_{i=0}^{n}\binom{n}{i}(ks)^i \\ &= \sum_{i=0}^{n}\binom{n}{i}k^i\left(Ar^i + Bs^i\right) \\ &= \sum_{i=0}^{n}\binom{n}{i}k^i V_{k,i} \end{aligned}$$

Hence the result. □

3.2. Relation Properties

In this subsection we present the results which establishes the relation of k-Fibonacci-Like sequence $\langle V_{k,n}\rangle$ with k-Fibonacci sequence $\langle F_{k,n}\rangle$ and k-Lucas sequence $\langle L_{k,n}\rangle$.

Theorem 3.11. *For $n \in \mathbb{Z}_0$, the following result holds:*

$$2mV_{k,n+1} - (p+mk)V_{k,n} = \left(m^2k^2 + 4m^2 - p^2\right)F_{k,n} \tag{29}$$

Proof. In order to prove the required result, we will use induction on n. Clearly the result is true for $n = 0$. Suppose that the result is true for all values j less than or equal n and then

$$\begin{aligned} 2mV_{k,n+2} - (p+mk)V_{k,n+1} &= 2m\left(kV_{k,n+1} + V_{k,n}\right) - (p+mk)\left(kV_{k,n} + V_{k,n-1}\right) \\ &= k\left[2mV_{k,n+1} - (p+mk)V_{k,n}\right] + \left[2mV_{k,n} - (p+mk)V_{k,n-1}\right] \\ &= \left(m^2k^2 + 4m^2 - p^2\right)\left(kF_{k,n} + F_{k,n-1}\right) \\ &= \left(m^2k^2 + 4m^2 - p^2\right)F_{k,n+1} \end{aligned}$$

as required. □

Theorem 3.12. *For $n \in \mathbb{N}$, we have*

$$2mV_{k,n+2} - (p+mk)\,V_{k,n+1} + 2mV_{k,n} - (p+mk)\,V_{k,n-1} = \left(m^2k^2 + 4m^2 - p^2\right) L_{k,n} \tag{30}$$

Proof. In order to prove the equation (30), we use induction on n. Let $n = 1$, we get

$$\begin{aligned}
&2mV_{k,3} - (p+mk)\,V_{k,2} + 2mV_{k,1} - (p+mk)\,V_{k,0}\\
&= 2m\left(mk^3 + pk^2 + 3mk + p\right) - (p+mk)\left(mk^2 + pk + 2m\right) + 2m\,(p+mk) - (p+mk)\,2m\\
&= m^2k^3 + 4m^2k - p^2k\\
&= \left(m^2k^2 + 4m^2 - p^2\right) L_{k,1}\\
&= \text{R.H.S}
\end{aligned}$$

Assume that the result is true for all values j less than or equal $n-1$ and then

$$\begin{aligned}
&2mV_{k,n} - (p+mk)\,V_{k,n-1} + 2mV_{k,n-2} - (p+mk)\,V_{k,n-3} = \left(m^2k^2 + 4m^2 - p^2\right) L_{k,n-2} \quad \text{and}\\
&2mV_{k,n+1} - (p+mk)\,V_{k,n} + 2mV_{k,n-1} - (p+mk)\,V_{k,n-2} = \left(m^2k^2 + 4m^2 - p^2\right) L_{k,n-1}
\end{aligned}$$

Now we prove that the result is true for n and then

$$\begin{aligned}
&2mV_{k,n+2} - (p+mk)\,V_{k,n+1} + 2mV_{k,n} - (p+mk)\,V_{k,n-1}\\
&= 2m\left(kV_{k,n+1} + V_{k,n}\right) - (p+mk)\left(kV_{k,n} + V_{k,n-1}\right) + 2m\left(kV_{k,n-1} + V_{k,n-2}\right) - (p+mk)\left(kV_{k,n-2} + V_{k,n-3}\right)\\
&= k\big[2mV_{k,n+1} - (p+mk)\,V_{k,n} + 2mV_{k,n-1} - (p+mk)\,V_{k,n-2}\big] + \big[2mV_{k,n} - (p+mk)\,V_{k,n-1} + 2mV_{k,n-2}\\
&\qquad - (p+mk)\,V_{k,n-3}\big]\\
&= k\left(m^2k^2 + 4m^2 - p^2\right) L_{k,n-1} + \left(m^2k^2 + 4m^2 - p^2\right) L_{k,n-2}\\
&= \left(m^2k^2 + 4m^2 - p^2\right)\left(kL_{k,n-1} + L_{k,n-2}\right)\\
&= \left(m^2k^2 + 4m^2 - p^2\right) L_{k,n}
\end{aligned}$$

as required. □

Theorem 3.13. *For $n \in \mathbb{Z}_0$, the following result holds:*

$$V_{k,n} = pF_{k,n} + mL_{k,n}, \quad p,\ m \in \mathbb{Z}_0 \tag{31}$$

Proof. To prove the required result we use mathematical induction on n. Certainly the result is true for $n = 0$. Let us suppose that the result is true for all values j less than or equal n and then

$$\begin{aligned}
V_{k,n+1} &= kV_{k,n} + V_{k,n-1}\\
&= k\left(pF_{k,n} + mL_{k,n}\right) + \left(pF_{k,n-1} + mL_{k,n-1}\right)\\
&= p\left(kF_{k,n} + F_{k,n-1}\right) + m\left(kL_{k,n} + L_{k,n-1}\right)\\
&= pF_{k,n+1} + mL_{k,n+1}
\end{aligned}$$

as required. □

Theorem 3.14. *For $n \geq 0$ and $l \geq 1$ the following result holds:*

$$V_{k,n+1}V_{k,l} + V_{k,n}V_{k,l-1} = \left(m^2k^2 + 4m^2 + p^2\right) F_{k,n+l} + (2mp)\, L_{k,n+l} \tag{32}$$

Proof. In order to prove the equation (32), we use induction on l. For $l = 1$ and $n = 0$, we have

$$\begin{aligned} \text{L.H.S} &= V_{k,1}V_{k,1} + V_{k,0}V_{k,0} \\ &= (p + mk)^2 + 4m^2 \\ &= m^2k^2 + 4m^2 + p^2 + 2mpk \end{aligned}$$

and

$$\begin{aligned} \text{R.H.S} &= \left(m^2k^2 + 4m^2 + p^2\right) F_{k,1} + (2mp)\, L_{k,1} \\ &= m^2k^2 + 4m^2 + p^2 + 2mpk \end{aligned}$$

Let us suppose that the result is true for all values i less than or equal l. Now we prove that the result is true for $l + 1$ and then

$$\begin{aligned} V_{k,n+1}V_{k,l+1} + V_{k,n}V_{k,l} &= V_{k,n+1}\left(kV_{k,l} + V_{k,l-1}\right) + V_{k,n}\left(kV_{k,l-1} + V_{k,l-2}\right) \\ &= k\left(V_{k,n+1}V_{k,l} + V_{k,n}V_{k,l-1}\right) + \left(V_{k,n++1}V_{k,l-1} + V_{k,n}V_{k,l-2}\right) \\ &= k\Big[\left(m^2k^2 + 4m^2 + p^2\right) F_{k,n+l} + (2mp)\, L_{k,n+l}\Big] + \Big[\left(m^2k^2 + 4m^2 + p^2\right) F_{k,n+l-1} + (2mp)\, L_{k,n+l-1}\Big] \\ &= \left(m^2k^2 + 4m^2 + p^2\right)\left(kF_{k,n+l} + F_{k,n+l-1}\right) + (2mp)\left(kL_{k,n+l} + L_{k,n+l-1}\right) \\ &= \left(m^2k^2 + 4m^2 + p^2\right) F_{k,n+l+1} + (2mp)\, L_{k,n+l+1} \end{aligned}$$

as needed. □

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