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Best Proximity Point Theorems for Some Well Known Mappings in Complete $b-$Metric spaces

J. Beny[1,*] and J. Maria Joseph

1 P.G and Research Department of Mathematics, Holy Cross College(Autonomous), Tiruchirappalli, Tami Nadu, India.

2 P.G and Research Department of Mathematics, St.Joseph's College(Autonomous), Tiruchirappalli, Tami Nadu, India.

Abstract: In this paper, we prove best proximity point theorems for types of cyclic $b-$contraction mappings in the setting of complete $b-$ metric spaces which generalize some results in the current literature.

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1. Introduction and Preliminaries

Fixed point Theory plays vital role in Mathematical analysis. Best approximations and best proximity points are considered as an extension of fixed point theory. In 1922, Stefan Banach has come up with beautiful theorem known as banach contraction theorem. This theorem laid foundation for all fixed point theorems. Eldred and Veeramani [1] proved existence and convergence of best proximity points in 2006. Then, many authors presented best proximity point results for different types of mappings [2–8]. In this section, we provide some basic definitions.

Definition 1.1. *Let A and B be nonempty subsets of a metric space (M, d). An element x^* in A is said to be a best proximity point of a mapping $T : A \to B$ if $d(x^*, Tx^*) = d(A, B)$.*

Definition 1.2. *Let (M, d) and (N, ρ) be two metric spaces. A mapping $S : M \to N$ is said to be a non-self-Lipschitzian mapping if there exists a constant $L \geq 0$ such that $\rho(S_x S_y) \leq Ld(x, y)$ for all $x, y \in M$.*

Definition 1.3. *Let (M, d) and (N, ρ) be two metric spaces. A Lipschitzian mapping $S : M \to N$ with the Lipschitz constant $L < 1$ is said be a non-self-contractive mapping.*

Definition 1.4. *Let A and B be nonempty subsets of a metric space (M, d). A mapping $T : A \to B$ is said to be*

(1). a non-self-Kannan mapping (see [9] for the self-mapping case) if there exists a constantk $\in [0, 1/2)$ such that $d(T_x, T_y) \leq k(d(x, T_x) + d(y, T_y))$ for all $x, y \in A$.

(2). a non-self-Chatterjea mapping (see [9] for the self-mapping case) if there exists a constantk $\in [0, 1/2)$ such that $d(T_x, T_y) \leq k(d(x, T_x) + d(y, T_y))$ for all $x, y \in A$.

* E-mail: benykutty@gmail.com

Definition 1.5. *Let A and B be a nonempty subsets of a metric space (M,d). A mapping $T : A \to B$ is said to be a generalized non self- Kannan and chatterjea if there exist nonnegative constants k_1, k_2 , k_3 such that $k_1 + 2k_2 + 2k_3 < 1$ and $d(T_x, T_y) \leq k_1(d(x,y) + k_2(d(x,T_x + d(y,T_y)) + k_3(d(x,T_y + d(y,T_x))$ for all $x, y \in A$. It is obvious that (4) is in a generalized form of (3)and (2)*

Definition 1.6. *Let A and B be a nonempty subsets of a metric space (M,d) and let $S : B \to A.A$ mapping $T : A \to B$ is said to be*

(1). a non-self-Kannan mapping (see [9] for the self-mapping case) if there exists a constant$k \in [0,1/2)$ such that $d(T_x, T_y) \leq k(d(x,T_x) + d(y,T_y))$ for all $x, y \in A$.

(2). a non-self-Chatterjea mapping with respect to the mapping S if there exists a constant$k \in [0,1/2)$ such that $d(T_x, T_y) \leq k(d(x,T_x) + d(y,T_y))$ for all $x, y \in A$.

Definition 1.7. *Let A and B be a nonempty subsets of a metric space (M,d) and let $S : B \to A.A$ mapping $T : A \to B$ is said to be generalized non self- Kannan and chatterjea with respect to the mapping S if there exist nonnegative constants k_1, k_2 , k_3 such that $k_1+2k_2+2k_3 < 1$ and $d(T_x, T_y) \leq k_1(d(x,y)+k_2(d(x,ST_x+d(y,ST_y))+k_3(d(x,ST_y+d(y,ST_x))$ for all $x, y \in A$. It is clear that (7) is in a generalized form of (5) and (6).*

Definition 1.8. *Let A and B be nonempty subsets of a metric space (M,d). Given $T : A \to B$ and $S : B \to A$ the pair (S,T) is said to form a weak K-cyclic contraction if there exists a nonnegative $k < 1/2$ such that $d(T_x, ST_x) \leq k[(d(x,T_x) + d(T_x,ST_x] + (1-2k)d(A,B)$ for all $x, y \in B$.*

2. Main Results

Theorem 2.1. *Let X be a complete b-metric space with $S \geq 1$. Let A and B be non empty closed subsets of X. Let $F : A \to B$ and $G : B \to A$ satisfy the following conditions.*

(1). G is a Lipschitzian mapping with Lipschitz constant $k \geq 1$.

(2). $d(Fa, Fb) \leq c_1 d(a,b) + c_2[d(a,GFa) + d(b,GFb)] + c_3[d(a,GFb) + d(b,GFa)] + c_4[\frac{d(a,b)}{1+d(b,GFa)}]$ for all $a \in A$ and $b \in B$ with $c_1, c_2, c_3, c_4 \geq 0$, where $c_1 + 2c_2 + 2sc_3 + c_4 < \frac{1}{k}$.

The pair (F,G) forms a weak k-cyclic contraction. Then there exists elements $a \in A$ and $b \in B$ such that $d(a,Fa) = d(A,B)$, $d(b,Fb) = d(A,B)$, $d(a,b) = d(A,B)$. If a_0 is any point in A, $a_{2n+1} = Fa_{2n}$ and $a_{2n} = Ga_{2n-1}$, then the sequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to best proximity points of F and G.

Proof. Fix $a_0 \in A$. Define the sequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ by $a_{2n} = Ga_{2n-1}$, for all $n \geq 1$ and $a_{2n+1} = Fa_{2n}$, for all $n \geq 0$.

$$\begin{aligned}
d(a_{2n}, a_{2n+2}) &= d(Ga_{2n-1}, Ga_{2n+1}) \\
&\leq kd(a_{2n-1}, a_{2n+1}) = kd(Fa_{2n-2}, Fa_{2n}) \\
&\leq k[c_1 d(a_{2n-2}, a_{2n}) + c_2(d(a_{2n-2}, FGa_{2n-2}) + d(a_{2n}, FGa_{2n})) \\
&\quad + c_3(d(a_{2n-2}, FGa_{2n}) + d(a_{2n}, FGa_{2n-2})) + c_4\frac{d(a_{2n-2}, a_{2n})}{1 + d(a_{2n}, FGa_{2n-2})}] \\
&= k[c_1 d(a_{2n-2}, a_{2n}) + c_2(d(a_{2n-2}, a_{2n}) + d(a_{2n}, a_{2n+2})) \\
&\quad + c_3(d(a_{2n-2}, a_{2n+2}) + d(a_{2n}, a_{2n}) + c_4\frac{d(a_{2n-2}, a_{2n})}{1 + d(a_{2n}, a_{2n})}]
\end{aligned}$$

$$\leq k[c_1 d(a_{2n-2}, a_{2n}) + c_2(d(a_{2n-2}, a_{2n}) + d(a_{2n}, a_{2n+2}))$$

$$+ sc_3(d(a_{2n-2}, a_{2n}) + d(a_{2n}, a_{2n+2})) + c_4 d(a_{2n-2}, a_{2n})$$

$$= k[c_1 d(a_{2n-2}, a_{2n}) + c_2(d(a_{2n-2}, a_{2n}) + c_2 d(a_{2n}, a_{2n+2}))$$

$$+ sc_3 d(a_{2n-2}, a_{2n}) + sc_3 d(a_{2n}, a_{2n+2}) + c_4 d(a_{2n-2}, a_{2n})]$$

$$= k[(c_1 + c_2 + sc_3 + c_4) d(a_{2n-2}, a_{2n} + (c_1 + c_2) d(a_{2n}, a_{2n+2}]$$

$$d(a_{2n}, a_{2n+2}) \leq (\frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)}) d(a_{2n-2}, a_{2n})$$

Similarly,

$$d(a_{2n-2}, a_{2n}) \leq \frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)} d(a_{2n-4}, a_{2n-2})$$

Thus,

$$d(a_{2n-2}, a_{2n}) \leq \frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)} d(a_{2n-4}, a_{2n-2})$$

$$d(a_{2n}, a_{2n+2}) \leq (\frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)})^2 d(a_{2n-4}, a_{2n-2})$$

By induction,

$$d(a_{2n}, a_{2n+2}) \leq (\frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)})^n d(a_0, a_2)$$

Therefore, $\{a_{2n}\}$ is a Cauchy sequence is A and hence converges to some element $a \in A$. Now,

$$d(a_{2n+1}, a_{2n+3}) = d(Fa_{2n}, Fa_{2n+2})$$

$$\leq c_1 d(a_{2n}, a_{2n+2}) + c_2(d(a_{2n}, GFa_{2n}) + d(a_{2n+2}, GFa_{2n+2}))$$

$$+ c_3(d(a_{2n}, GFa_{2n+2}) + d(a_{2n+2}, GFa_{2n})) + c_4 \frac{d(a_{2n}, a_{2n+2})}{1 + d(a_{2n+2}, GFa_{2n})}$$

$$= c_1 d(Ga_{2n-1}, Ga_{2n+1}) + c_2(d(Ga_{2n-1}, Ga_{2n+1}) + d(Ga_{2n+1}, Ga_{2n+3}))$$

$$+ c_3(d(Ga_{2n-1}, Ga_{2n+3}) + d(Ga_{2n+1}, Ga_{2n+1})) + c_4 \frac{d(a_{2n}, a_{2n+2})}{1 + d(a_{2n+2}, GFa_{2n})}$$

$$\leq c_1 k d(a_{2n-1}, a_{2n+1}) + c_2 k(d(a_{2n-1}, a_{2n+1}) + d(a_{2n+1}, a_{2n+3}))$$

$$+ c_3 k(d(a_{2n-1}, a_{2n+3}) + (d(a_{2n+1}, a_{2n+1})) + c_4 d(a_{2n}, a_{2n+2})$$

$$\leq c_1 k d(a_{2n-1}, a_{2n+1}) + c_2 k d(a_{2n-1}, a_{2n+1}) + c_2 k d(a_{2n+1}, a_{2n+3})$$

$$+ c_3 ks(d(a_{2n-1}, a_{2n+})) + d(a_{2n+1}, a_{2n+3}))$$

$$d(a_{2n+1}, a_{2n+3}) \leq (\frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)} d(a_{2n-1}, a_{2n+1})$$

similarly

$$d(a_{2n-1}, a_{2n+1}) \leq (\frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)}) d(a_{2n-3}, a_{2n-1})$$

Thus,

$$d(a_{2n+1}, a_{2n+3}) \leq (\frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)}^2 d(a_{2n-3}, a_{2n-1})$$

By induction we obtain,

$$d(a_{2n+1}, a_{2n+3}) \leq (\frac{k(c_1 + c_2 + sc_3 + c_4)}{1 - k(c_2 + sc_3)}^n d(a_1, a_3)$$

$\{a_{2n+1}\}$ is a cauchy sequence is B and hence converges to some element $y \in B$.

$$\begin{aligned}
d(a_{2n+2}, GFa) &= d(GFa_{2n}, GFa)\\
&\leq kd(Fa_{2n}, Fa)\\
&\leq k(c_1 d(a_{2n}, a) + c_2(d(a_{2n}, GFa_{2n}) + d(a, GFa)) + c_3(d(a_{2n}, GFa) + d(a, GFa_{2n})) + c_4 \frac{d(a_{2n}, a)}{1 + d(a, GFa_{2n})}\\
&= k(c_1 d(a_{2n}, a) + c_2(d(a_{2n}, a_{2n+2}) + d(a, GFa)) + c_3(d(a_{2n}, GFa) + d(a, a_{2n+2})) + c_4 \frac{d(a_{2n}, a)}{1 + d(a, a_{2n+2})}
\end{aligned}$$

Letting $n \to \infty$

$$\begin{aligned}
d(a, GFa) \leq kd(b, Fa) &\leq k(c_2 d(a, GFa) + c_3 d(a, GFa))\\
&\leq k(c_2 + c_3)d(a, GFa)
\end{aligned}$$

Notice that $0 \leq k(c_2 + c_3) < 1$. Its not hard verify that $d(b, F_a) = 0$ and then $b = F_a$. On the other hand, we also found that

$$\begin{aligned}
d(a_{2n+3}, FGb) &= d(FGa_{2n+1}, FGb)\\
&\leq c_1 d(Ga_{2n+1}, Gb) + c_2(d(Ga_{2n+1}, GFGa_{2n+1}) + d(Gb, GFGb))\\
&+ c_3(d(Ga_{2n+1}, GFGb) + d(Gb, GFGa_{2n+1})) + c_4 \frac{d(Ga_{2n+1}, Gb)}{1 + d(Gb, GFGa_{2n+1})}\\
&\leq c_1 kd(a_{2n+1}, b) + c_2 d(Ga_{2n+1}, Ga_{2n+3}) + c_2 kd(b, FGb)\\
&+ c_3 k(d(a_{2n+1}, FGb) + d(b, a_{2n+3})) + c_4 \frac{kd(a_{2n+1}, b)}{1 + kd(b, a_{2n+3})}\\
&\leq c_1 kd(a_{2n+1}, b) + c_2 k(d(a_{2n+3}, a_{2n+3})) + d(b, FGb))\\
&+ c_3 k(d(a_{2n+1}, FGb) + d(b, a_{2n+3})) + c_4 \frac{kd(a_{2n+1}, b)}{1 + kd(b, a_{2n+3})}\\
d(b, FGb) &\leq c_2 d(Gb, a) = 0
\end{aligned}$$

and hence $Gb = a$. Since the pair G, F forms a weak k-cyclic contraction, it follows that there exists $k \in [0, 1/2]$ such that

$$\begin{aligned}
d(a, b) &= d(Fa, GFa)\\
&\leq k(d(a, Fa) + d(Fa, GFa)) + (1 - 2k)d(A, B)\\
&\leq 2kd(a, b) + (1 - 2k)d(A, B)\\
(1 - 2k)d(a, b) &\leq (1 - 2k)d(A, B)\\
d(a, b) &= d(A, B)
\end{aligned}$$

Hence $d(A, B) = d(a, b)\{= d(a, Ga) = d(Ga, GFa) = d(b, Gb)\}$. This shows that a is a best proximity point of F and b is a best proximity point of G. □

Theorem 2.2. *Let A and B be non empty closed subsets of X. Let $F : A \to B$ and $G : B \to A$ satisfy the following contions.*

(1). G is a Lipschitzian mapping with Lipschitz constant $k \geq 1$.

(2). $d(Fa,Fb) \leq c_1 d(a,b) + c_2[d(a,GFa)+d(b,GFb)] + c_3[d(a,GFb)+d(b,GFa)]$ *for all* $a \in A$ *and* $b \in B$ *with* $c_1, c_2, c_3 \geq 0$, *where* $1 + s + sc_1 + 4s^2c_2 + 2s^2c_3 + 4s^3c_3 < \frac{1}{S}$.

The pair (F,G) *forms a weak k-cyclic contraction. Then there exists elements* $a \in A$ *and* $b \in B$ *such that*

$$d(a,F_a) = d(A,B)$$

$$d(b,F_b) = d(A,B)$$

$$d(a,b) = d(A,B).$$

If a_0 *is any point in* A, $a_{2n+1} = Fa_{2n}$ *and* $a_{2n} = Ga_{2n-1}$, *then the sequences* a_{2n+1} *converge to best proximity points of* F *and* G. *Further if* a^* *is another best proximity point of* F, *then* $d(a,a^*) \leq \frac{(s+s^2+2s^3(2C_2+C_3+SC_3))}{1-s^2(C_1+2S^2C_3)} d(A,B)$.

Proof. By previous theorem if we take $C_4 = 0$, we get a is the best proximity point of F and b is the best proximity point of G. We have to show that F has a unique best proximity point. It can be proved that $d(A,B) = d(a^*,Fa^*) = d(Fa^*,GFa^*)$

$$\begin{aligned}
d(a,a^*) &\leq s(d(a,Fa^*) + d(Fa^*,a^*)) \\
&= s(d(a,Fa^*)) + sd(Fa^*,a^*)) \\
&\leq s(s(d(a,Fa) + d(Fa,Fa^*))) + sd(Fa^*,a^*) \\
&\leq s^2 d(a,Fa) + s^2 d(Fa,Fa^*) + sd(Fa^*,a^*) \\
&= (s^2+s)d(A,B) + s^2 d(Fa,Fa^*) \\
&\leq (s^2+s)d(A,B) + s^2(c_1 d(a,a^*)) + c_2[d(a,GFa) + d(a^*,GFa^*)] + c_3(d(a,GFa^*) + d(a^*,GFa)) \\
&\leq (s^2+s)d(A,B) + s^2c_1 d(a,a^*) + s^2c_2[s(d(a,Fa) + d(Fa,GFa))] + s[d(a^*,Fa^*) + d(Fa^*,GFa^*)] \\
&+ s^2c_3[s(d(a,Fa^*)) + d(Fa^*,GFa^*)] + s[d(a^*,Fa) + d(Fa,GFa)] \\
&\leq (s^2+s)d(A,B) + s^2c_1 d(a,a^*) + s^3c_2(d(a,Fa) + d(Fa,GFa))] + s[d(a^*,Fa^*) + d(Fa^*,GFa^*)] \\
&+ s^3c_2(d(Fa,GFa)) + d(Fa^*,GFa^*)] + s[d(a^*,Fa) + d(Fa,GFa) + s^3c_2 d(a^*,Fa^*) \\
&+ s^3c_2(d(Fa^*,GFa^*) + s^2c_3[s^2s(d(a,a^*) + d(a^*,Fa^*))] + s^3c_3 d(Fa^*,GFa^*) \\
&+ s^2c_3 s[s(d(a^*,a) + d(a,Fa) + s^3c_3 d(Fa,GFa)] \\
&\leq (s^2+s)d(A,B) + s^2c_1 d(a,a^*) + s^3c_2(d(a,Fa) + s^3c_2(d(Fa,GFa)) + s^3c_2 d(a^*,Fa^*) \\
&+ s^3c_2(d(Fa^*,GFa^*) + s^4c_3(d(a,a^*) + s^4c_3(d(a^*,Fa^*) \\
&+ s^3c_3(d(Fa^*,GFa^*) + s^4c_3(d(a^*,a) + s^4c_3(d(a,Fa) + s^3c_3(d(Fa,GFa) \\
&\leq (s^2+s)d(A,B) + (s^2c_1 + s^4c_3 + s^3c_3)d(a,a^*) + (s^3c_2 + s^4c_3)d(a,Fa) + s^3c_2 + s^3c_3)d(Fa,GFa) \\
&+ (s^3c_2 + s^4c_3)d(a^*,Fa^*) + (s^3c_2 + s^3c_3)d(Fa^*,GFa^*)
\end{aligned}$$

$$d(a,a^*) \leq (s^2+s)d(A,B) + s^2(c_1 + 2s^2c_3)d(a,a^*) + (s^3c_2 + s^4c_3 + s^3c_2 + s^3c_3 + s^3c_2 + s^4c_3 + s^3c_2 + s^3c_3)d(A,B)$$

$$d(a,a^*) \leq (s^2+s)d(A,B) + s^2(c_1 + 2s^2c_3)d(a,a^*) + (4s^3c_2 + 2s^3c_3 + 2s^4c_3)d(A,B)$$

$$d(a,a^*) \leq (s^2 + s + 2s^3(2c_2 + c_3 + sc_3))d(A,B) + s^2(c_1 + 2s^2c_3)d(a,a^*)$$

$$d(a,a^*) - s^2(c_1 + 2s^2c_3)d(a,a^*) \leq (s + s^2 + 2s^3(2c_2 + c_3 + sc_3))d(A,B)$$

$$d(a,a^*)(1 - s^2(c_1 + 2s^2c_3)) \leq (s + s^2 + 2s^3(2c_2 + c_3 + sc_3))d(A,B)$$

$$d(a,a^*) \leq \frac{s + s^2 + 2s^3(2c_2 + c_3 + sc_3)}{1 - s^2(c_1 + 2s^2c_3)} d(A,B)$$

This completes the proof of the theorem. □

Corollary 2.3. *Let X be a complete b- metric space with $S \geq 1$. Let A and B be nonempty closed subsets. Let $T : A \longrightarrow B$ ane $S : B \longrightarrow A$ satisfy the following conditions for nonnegative number $k < 1/2$.*

(1). S is non expansive

(2). $d(Tu, Tv) \leq k(d(u, STu) + d(v, STv))$ for all $u, v \in\in A$

(3). The pair (S, T) forms a weak k- cyclic contraction.

Then there exists elements $x \in A$ and $y \in B$

$$d(x, Tx) = d(A, B)$$

$$d(y, Sy) = d(A, B)$$

$$d(x, y) = d(A, B)$$

If x_0 is any fixed element in A, $x_{2n+1} = Tx_{2n}$, and $x_{2n} = Sx_{2n-1}$, then the sequence$x_{2n}$and x_{2n+1} converge to some best proximity points of T and S, respectively. Further, if x^* is another best proximity point of T, then

$$d(x, x^*) \leq 2(1 + 2K)d(A, B)). \tag{1}$$

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