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On Some generalization of Unitary Quasi-Equivalence of Operators

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Abstract: In this paper, we generalize the class of Unitary Quasi-Equivalence by extending this study to n-Unitary Quasi-Equivalence and investigate the properties of this class. We investigate the relation of this equivalence class to other relations.

Keywords: Unitary Quasi-Equivalence, n-metric equivalence, n-unitary Quasi-Equivalence, n-normal operators.

1. Introduction

Throughout this paper, H is a separable complex Hilbert space, $B(H)$ is the Banach algebra of all bounded linear operators. $T \in B(H)$ is normal if $T^*T = TT^*$, n-normal if $T^*T^n = T^nT^*$, projection if $T^2 = T$, hyponormal if $T^*T \geq TT^*$, quasinormal if $T(T^*T) = (T^*T)T$, n-hyponormal if $T^*T^n \geq T^nT^*$. Two operators $S, T \in B(H)$ are said to be Metrically equivalent if $S^*S = T^*T$ for more on this refer to [3], n-metrically equivalent if $S^*S^n = T^*T^n$ we refer the reader to [7] for more, Unitarily Quasi-Equivalent if there exists a unitary operator $U \in B(H)$ such that $S^*S = UT^*TU^*$ and $SS^* = UTT^*U^*$ [1] and nearly equivalent if $S^*S = UT^*TU^*$ [6]. Two operators $S, T \in B(H)$ are said to be n-Unitarily Quasi-Equivalent if there exists a unitary operator $U \in B(H)$ such that $S^*S^n = UT^*T^nU^*$ and $S^nS^* = UT^nT^*U^*$ for any positive integer n. We note that n-Unitarily Quasi-Equivalent operators are Unitarily Quasi-Equivalent operators when $n = 1$.

2. Main Results

Theorem 2.1. *n-Unitary Quasi-Equivalence is an equivalence relation.*

Proof. Let $S, T, P \in B(H)$. S is n-Unitarily Quasi-Equivalent to S since $S^*S^n = IS^*S^nI^*$ and $S^nS^* = IS^nS^*I^*$ for $I = U$. If S is n-Unitarily Quasi-Equivalent to T, then $S^*S^n = UT^*T^nU^*$ and $S^nS^* = UT^nT^*U^*$. Pre-multiplying and post-multiplying the two equations by U^* and U on both sides we end up with $T^*T^n = US^*S^nU^*$ and $T^nT^* = US^nS^*U^*$. Hence T is n-Unitarily Quasi-Equivalent to S. We now have to show that if S is n-Unitarily Quasi-Equivalent to T and T is n-Unitarily Quasi-Equivalent to P, then S is n-Unitarily Quasi-Equivalent to P. Now $S^*S^n = UT^*T^nU^*$ and $S^nS^* = UT^nT^*U^*$ and $T^*T^n = VP^*P^nV^*$ and $T^nT^* = VP^nP^*V^*$, where U and V are unitary operators. Then $S^*S^n = UT^*T^nU^* = UVP^*P^nV^*U^* = QP^*P^nQ^*$, for $Q = UV$, which is unitary. Equally $S^nS^* = UT^nT^*U^* = UVP^nP^*V^*U^* = QP^nP^*Q^*$,

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for $Q = UV$ which is unitary. This shows that S is n-Unitarily Quasi-Equivalent to P and hence n-Unitary Quasi-Equivalence is an equivalence relation. $\square$

Theorem 2.2. *Let $S, T \in B(H)$ be n-unitarily Quasi-equivalent. Then S is n-normal if and only if T is n-normal.*

Proof. Suppose that S and T are n-unitarily quasi-equivalent and also suppose that S is n-normal, then $T^*T^n = US^*S^nU^*$ and $T^nT^* = US^nS^*U^*$. Hence $T^*T^n = US^*S^nU^* = US^nS^*U^* = T^nT^*$. The converse is proved in the similar way. $\square$

Lemma 2.3. *Two operators $S, T \in B(H)$ are n-unitarily Quasi-equivalent if and only if $S^*S^n - S^nS^* = U(T^*T^n - T^nT^*)U^*$.*

Theorem 2.4. *Let $S, T \in B(H)$ be n-unitarily Quasi-equivalent. Then S is n-hyponormal if and only if T is n-hyponormal.*

Proof. The proof follows from Lemma , $S^*S^n - S^nS^*$ is unitarily equivalent to $T^*T^n - T^nT^*$. If $S^*S^n - S^nS^* \geq 0$, then $T^*T^n - T^nT^* = U(S^*S^n - S^nS^*)U^* \geq 0$. This shows that n-Unitary quasi-equivalence preserves n-hyponormality. $\square$

We note that n-unitary quasi-equivalence preserves n-quasinormality and n-binormality of operators, this follows from Theorem 2.4, and the fact that these classes are contained in the class of n-hyponormal operators.

Lemma 2.5. *$T \in B(H)$ is n-unitarily equivalent to a unitary operator if and only if it is a unitary operator.*

Proof. Suppose that $T^n = PU^nP^*$, where $U, P \in B(H)$ are unitary operators. Then, we have $T^*T^n = PU^*P^*PU^nP^* = I$ and $T^nT^* = PU^nP^*PU^*P^* = I$. $\square$

Lemma 2.5 can be extended to the class of n-unitarily Quasi-equivalent of operators.

Theorem 2.6. *$T \in B(H)$ is n-unitarily Quasi-equivalent to a unitary operator if and only if it is a unitary operator.*

Proof. Let $T \in B(H)$ is n-unitarily Quasi-equivalent to a unitary operator $P \in B(H)$, then there exists a unitary operator $U \in B(H)$ such that $T^*T^n = U(P^*P^n)U^* = I$ and $T^nT^* = U(P^nP^*)U^* = I$. This implies that $T^*T^n = T^nT^*$. $\square$

The converse follows from Lemma 2.5.

Theorem 2.7. *If $S, T \in B(H)$ are both Self and 2-Unitarily quasi-equivalent, then S^3 and T^3 are unitarily equivalent.*

Proof. The proof follows directly from the definitions; $S^*S^2 = UT^*T^2U^*$ and $S^2S^* = UT^2T^*U^*$, then using the self adjoint property of S and T we have; $S^3 = UT^3U^*$. Hence the proof. $\square$

Theorem 2.8. *Let $S, T \in B(H)$ be n-unitarily Quasi-equivalent. Then $\|S^n\| = \|T^n\|$.*

Proof. $\|S^n\|^2 = \|S^*S^n\| = \|UT^*T^nU^*\| = \|T^*T^n\| = \|T^n\|^2$. Taking square root on both sides of the equation we get the intended result. $\square$

Proposition 2.9. *Let $T \in B(H)$, then we have*

(i). $Ker(T^*T^n) = Ker(T^n)$.

(ii). $\overline{Ran(T^nT^*)} = \overline{Ran(T^n)}$.

Proof.

(i).
$$\begin{aligned} Ker(T^*T^n) &= \{\xi \in H : T^*T^n\xi = 0\} \\ &= \{\xi \in H : T^n\xi = 0\} \\ &= Ker(T^n) \end{aligned}$$

(ii). $\overline{Ran(T^nT^*)} = \overline{\{\xi \in H : \xi = T^nT^*x, x \in H\}}$

$$= \overline{\{\xi \in H : \xi = T^n(T^*x)\}}$$

$$= \overline{Ran(T^n)}.$$

□

Theorem 2.10. *If $S, T \in B(H)$ are n-unitarily Quasi-equivalent, then $Ker(S^n) = Ker(T^n)$ and $\overline{Ran(\|S^n\|)} = \overline{Ran(\|T^n\|)}$.*

Proof. The proof follows from Proposition 2 and the definition of n-unitary quasi-equivalence of operators. □

Corollary 2.11. *If $S, T \in B(H)$ are n-unitarily Quasi-equivalent and S^n is injective, then T^n is injective.*

We note that n-unitarily quasi-equivalence unlike n-metric equivalence preserves injectivity of operators.

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