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A Mathematical Model for Effect of Couple-stress in Fluid Through the Blood Vessel with Mild Stenosis

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Abstract: A mathematical model is developed to study the effect of couple stress on the blood flow through two-dimensional symmetric arteries with stenosis. The present model is consists of a core region to be a couple stress fluid and the peripheral layer of Newtonian fluid. Analytical expressions have been obtained for resistance to flow and wall shear stress with the height of stenosis. The numerical values are derived from these analytic expressions. The effect of various parameters on these flows has been examined and depicted through graphs for different values. The information of blood flow could be very useful in the development of new diagnosis tool for the arterial disease.

MSC: 76Z05, 92C35.

Keywords: Physiological flow, stenosis, couple stress fluid, peripheral layer.

1. Introduction

Arteries play an important role in the transport and regular of blood in the cardiovascular system. A pressure gradient is produced as the heart pumps, which blood will flow in the full human body. Among the cardiovascular diseases, stroke and atherosclerosis are closely related to abnormality and disorder of blood flow characteristics in a human body, on account of this, blood-related problems have shown more interest in medical research. Atherosclerosis is the abnormal growth in the arterial wall thickness. This may be caused by unhealthy living conditions such as lack of physical activity, improper dietary habits and exposure to tobacco smoke. It is always resulted by the serious change in blood flow, pressure distribution, flow resistance and wall shear stress.

In view of this, many researchers have developed various mathematical models for flow through stenosed artery like Young [19], Shukla et al. [12], Zendehbudi and Moayery [20] and Radhakrishnamacharya et al. [10]. In all these studies, blood has been considered as a Newtonian fluid. The non-Newtonian behaviour of blood is mostly due to the suspension of red blood cells in plasma. The noticeable feature of accurate flow behaviour cannot be predicted by the classical Newtonian theory. The couple stress fluid theory enlarged by Stokes [18] represents the simplest generalization of the classical viscous fluid that bears couple stresses and the body couples. The Two-layered poiseuille flow model for blood flow through arteries

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of small diameter and arterioles have been developed by Chaturani [3].The study of the effects of couple stresses on the blood flow through the thin artery with mild stenosis has been carried out by Sinha and Singh [15] and Shrivastava [17] considered the flow of couple stress fluid through stenotic blood channels. Sankad and Radhakrishnamacharya [11], and Naeem et al. [9] researched inverse solutions for unsteady in-compressible couple fluid flow. It is observed that when blood flows through the narrow tubes, there exists a cell free plasma layer near the wall by Bugliarello and Hyden [2]. In view of their studies, it is noticeable to represent the flow of blood through tubes by a two layers model. Kumar et al. [7] studied on a mathematical model for different shapes of stenosis and slip velocity at the wall through mild stenosis artery. Shukla et al. [12] have considered a model on the effect of peripheral layer viscosity on blood flow through the artery with mild stenosis. Mishra et al. [8] are also studied on a mathematical model for the study of blood flow through a channel with permeable walls. The study on the effect of couple stresses on the pulsatile flow through a constricted annulus by Srinivasacharya et al. [16]. Keeping these in mind, in this paper we have developed the effects of the peripheral layer on couple stress fluid flow through the artery with mild stenosis. The governing equations of stenosed artery have been solved and analytical formulations for the resistance to flow and the wall shear stress at the maximum height of stenosis have been developed and shown graphically.

2. Mathematical Analysis

In the present analysis, we considered the steady and incompressible fluid flow in a channel, with Newtonian fluid in the peripheral layer and couple stress fluid in the core region. For mathematical convenience, we take the artery to be a long cylindrical tube with cartesian coordinate x-axis coinciding with the centerline of the channel and y-axis normal to it. The geometry of the stenosis in the wall can be expressed, Shukla et al. [14]:

$$R(x) = \begin{cases} R_0 - \frac{\delta_s}{2}\left(1 + \cos\frac{2\pi}{L_0}[x - d_s - \frac{L_0}{2}]\right), & d \le x \le d_s + L_0 \\ R_0, & \text{otherwise} \end{cases} \tag{1}$$

where R_0 is mean half width of the non-stenotic region of the channel, $R(x)$ is the radius of stenotic region, L is the length of the channel, L_0 is the length of the stenosis, d_s is the location of stenosis and δ_s is maximum height of stenosis (Figure 1).

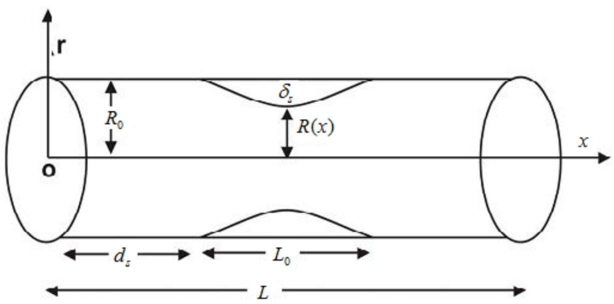

Figure 1: Geometry of arterial disease

The appropriate equations describing the flow in the central region and peripheral layer are given as Shrinavasacharya and Shrikanth [16]:

$$\rho_1\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right] = -\nabla p + \mu_1\nabla^2 v - \eta\nabla^4 v;\ 0 \le y \le R_1(x) \tag{2}$$

$$\rho_2\left[\frac{\partial v_1}{\partial t}+(v_1\cdot\nabla)v_1\right]=-\nabla p+\mu_2\nabla^2 v_1;\ R_1(x)\le y\le R(x) \tag{3}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, ρ_1 and ρ_2 are the densities and $v(r,s)$ is the velocity vectors of the fluids in the central region and $v_1(r_1,s_1)$ is the velocity vector of the fluids in peripheral layer, p is the pressure of fluid, μ_2 is the viscosity plasma in the peripheral layer, μ_1 is the viscosity coefficient of the fluid in the core region, t is time and λ is the couple stress fluid viscosity. In this paper, we are neglecting body forces and taking the restrictions for mild stenosis Young [19]. Now the equation (2) and (3) becomes:

$$\frac{dp}{dx}=\mu_1\frac{\partial^2 r}{\partial y^2}-\eta\frac{\partial^4 r}{\partial y^4};\ 0\le y\le R_1(x) \tag{4}$$

$$\frac{dp}{dx}=\mu_2\frac{\partial^2 r_1}{\partial y^2};\ R_1(x)\le y\le R(x) \tag{5}$$

Boundary Conditions: The boundary conditions are

$$\frac{\partial r}{\partial x}=0\quad at\quad y=0 \tag{6}$$

$$r_1=\frac{-R_o\sqrt{D}}{\alpha_1}\frac{\partial r_1}{\partial y}\quad at\ y=\pm R(x) \tag{7}$$

$$\frac{\partial^2 r}{\partial y^2}=0\quad at\ y=\pm R_1(x) \tag{8}$$

$$r=s\ \ and\ \ \tau_1=\tau_2\quad at\ \ y=\pm R_1(x) \tag{9}$$

where τ_1 and τ_2 are shear stress of the central and peripheral layers respectively, D is the permeability parameter and $\alpha_1$is the slip parameter. Here equation (7) is the Saffman's slip boundary condition by Bhatt and Sacheti [1] and equation (8) indicates the vanishing of couple stress. On solving equations (4) and (5) with boundary conditions (6) to (9) we get the following system of equation:

$$r(y)=\frac{R_0^2}{2\mu_2}\frac{dp}{dx}\left[\left(\frac{R(x)}{R_0}\right)^2-\overline{\mu}_2\left(\frac{y}{R_0}\right)^2-(1-\overline{\mu}_2)\left(\frac{R_1(x)}{R_0}\right)^2-\frac{2\mu_2}{c^2}+\frac{2\sqrt{D}}{\alpha_1}\left(\frac{R_1(x)}{R_0}\right)-\frac{2D}{\alpha_1^2}+\frac{2\mu_2\cos R(x)\left(c\frac{y}{R_0}\right)}{c^2\cos R(x)\left(c\frac{R_1(x)}{R_0}\right)}\right] \tag{10}$$

for $0\le y\le R_1(x)$

$$r_1(y)=-\frac{R_0^2}{2\mu_2}\frac{dp}{dx}\left[\left(\frac{R(x)}{R_0}\right)^2-\left(\frac{y}{R_0}\right)^2+\frac{2\sqrt{D}}{\alpha_1}\left(\frac{y}{R_0}\right)-\frac{2D}{\alpha_1^2}\right]\ \text{for}\ R_1(x)\le y\le R(x) \tag{11}$$

where $\overline{\mu}_2=\frac{\mu_2}{\mu_1}$ and $c=R_0\left(\frac{\mu_1}{\eta}\right)^{\frac{1}{2}}$ is the couple stress parameter.

Flow Rate: The flow rate Q, which is defined as

$$Q=2\pi\left[\int_0^{R_1} r dy+2\int_{R_1}^{R} r_1 dy\right] \tag{12}$$

with the help of equation (10) and (11), we get

$$\begin{aligned}Q=-\frac{2}{3}\frac{R_0^3}{2\mu_2}\bigg[\frac{dp}{dx}\left(\frac{R(x)}{R_0}\right)^3-(1-\overline{\mu}_2)\left(\frac{R_1}{R_0}\right)^3-\frac{3\overline{\mu}_2}{c^2}\left(\frac{R_1(x)}{R_0}\right)-\frac{3\sqrt{D}}{2\alpha_1}\left(\left(\frac{R_1}{R_0}\right)^2+\left(\frac{R(x)}{R_0}\right)^2\right)\\-\frac{3D}{\alpha_1}\left(\frac{R(x)}{R_0}\right)+\frac{3\overline{\mu}_2}{c^3}\tan R(x)c\left(\frac{R_1(x)}{R_0}\right)\bigg]\end{aligned} \tag{13}$$

Introducing the following dimensionless quantities $x' = \frac{x}{L}$, $L_0' = \frac{L_0}{L}$ and $d' = \frac{d}{L}$. The geometry of the interface between the peripheral and the core region is taken by Shukla et al. [14]

$$R_1(x) = \begin{cases} \gamma R_0 - \frac{\delta_i}{2}\left(1 + \cos\frac{2\pi}{L_0}[x - d_s - \frac{L_0}{2}]\right), & d_s \le x \le d_s + L_0 \\ \gamma R_0, & \text{otherwise} \end{cases} \tag{14}$$

where γ is the ratio of the central mean half width to the artery and δ_i is the maximum bulging of the interface. In equation (1), (13) and from (14) the pressure gradient can be obtained as:

$$\frac{dp}{dx} = -\frac{3}{2}\frac{\mu_2}{R_0^3}\frac{Q}{A} \tag{15}$$

where

$$A = \left(\frac{R(x)}{R_0}\right)^3 - (1-\overline{\mu}_2)\left(\frac{R_1(x)}{R_0}\right)^3 - \frac{3\overline{\mu}_2}{c^2}\left(\frac{R_1(x)}{R_0}\right) + \frac{3\sqrt{D}}{2\alpha_1}\left(\left(\frac{R_1(x)}{R_0}\right)^2 + \left(\frac{R(x)}{R_0}\right)^2\right) - \frac{3D}{\alpha_1}\left(\frac{R(x)}{R_0}\right) + \frac{3\overline{\mu}_2}{c^3}\tan R(x) c\left(\frac{R_1(x)}{R_0}\right)$$

Integrating equation (15), we get pressure gradient:

$$\Delta p = \frac{3}{2}\frac{\mu_2 Q}{R_0^2}\int_0^1 \frac{dx}{A} \tag{16}$$

Resistance Parameter: The resistance to flow is denoted by λ and defined as follows:

$$\lambda = \frac{\Delta p}{Q} = \frac{3}{2}\frac{\mu_2 Q}{R_0^2}\int_0^1 \frac{dx}{A} \tag{17}$$

Shear stress: The wall shear stress at $y = R(x)$ is given by:

$$\tau_w = -\mu_2 \frac{\partial r_1}{\partial y} \tag{18}$$

where τ_w is the shearing stress at the wall. The wall shear stress at the maximum height of stenosis (at $x = d + \frac{L_0}{2}$):

$$\tau_m = -\frac{3}{2}\frac{\mu_2 Q}{R_0^2}\frac{K_1}{K_2} \tag{19}$$

where τ_m is the shearing stress at the maximum height.

$$K_1 = 1 - \left(\frac{\delta_s}{R_0}\right) + \frac{\sqrt{D}}{\alpha_1}$$

$$K_2 = \left(1 - \frac{\delta_s}{R_0}\right)^3 - (1-\overline{\mu}_2)\left(1 - \frac{\delta_i}{R_0}\right)^3 - 3\frac{\mu_2}{c^2}\left(1 - \frac{\delta_i}{R_0}\right) - \frac{3\sqrt{D}}{\alpha_1^2}\left(1 - \frac{\delta_s}{R_0}\right) + \frac{3\sqrt{D}}{2\alpha_1}\left\{\left(1 - \frac{\delta_s}{R_0}\right)^2 + \left(1 - \frac{\delta_i}{R_0}\right)^2\right\} + \frac{3\overline{\mu}_2}{c^3}\tan R(x)\left\{c\left(1 - \frac{\delta_i}{R_0}\right)\right\}\Bigg]$$

Using by Shukla et al. [14], $R_1(x) = \gamma R(x)$ and $\delta_i = \gamma\delta_s$ for the dimensionless resistance to flow $\overline{\lambda}$and shear stress $\overline{\tau}_2$ can be written as:

$$\overline{\lambda} = \overline{\mu}_2 K_3 \int_0^1 \frac{dx}{k_4} \tag{20}$$

where $\overline{\lambda} = \frac{\lambda}{\lambda_0}$,

$$K_3 = 1 - \frac{3}{c^2} + \frac{3\sqrt{D}}{\alpha_1} - \frac{3\sqrt{D}}{\alpha_1^2} - \frac{3}{c^2}\tan R(x)(c)$$

$$K_4 = \left(1-(1-\overline{\mu})\gamma^3\right)\left(\frac{R(x)}{R_0}\right)^3 + \frac{3}{2}\frac{\sqrt{D}}{2\alpha_1}(\gamma^2+1)\left(\frac{R(x)}{R_0}\right)^3 - 3\left(\frac{\overline{\mu}_2\gamma}{c^2}+\frac{D}{\alpha_1^2}\right)\left(\frac{R(x)}{R_0}\right) + \frac{3\overline{\mu}_2}{c^3}\tan R(x)\left\{c\gamma\left(\frac{R(x)}{R_0}\right)\right\}$$

and

$$\overline{\tau}_m = \overline{\mu}_2 K_3 \int_0^1 \frac{dx}{K_5} \tag{21}$$

where $\tau_m = \frac{\tau_m}{\tau_0}$.

$$K_5 = \left(1+\frac{\sqrt{D}}{\alpha_1}\right)\left(1-(1-\overline{\mu})\gamma^3\right)\left(\frac{R(x)}{R_0}\right)^3\left(1-\frac{\delta_s}{R_0}\right)^3 - 3\left(\frac{\overline{\mu}_2\gamma}{c^2}+\frac{D}{\alpha_1^2}\right)\left(1-\frac{\delta_s}{R_0}\right)$$
$$+\frac{3\sqrt{D}}{2\alpha_1}(\gamma^2+1)\left(1-\frac{\delta_s}{R_0}\right)^2 + \frac{3\overline{\mu}_2}{c^3}\tan R(x)\left\{c\gamma\left(1-\frac{\delta_s}{R_0}\right)\right\}$$

The term λ_0 is the resistance to flow and$\tau_0$is the wall stress in the absence of peripheral layer.

3. Results and Discussion

The present model consists of a core region to be a couple stress fluids and the peripheral layer of Newtonian fluid. Analytical expressions have been obtained for resistance to flow and wall shear stress with the height of stenosis. The expression for resistance to the flow is given by equation (20) and wall shear stress is given by equation (21). These results are based on the numerical solution using MATLAB software for different values which are shown by graphically. The effects of various parameters on the resistance to flow have been presented from Figures 2-7, and it can also be observed that the resistance to flow increase with the height of the stenosis. We can notice the effect of the viscosity $(\overline{\mu}_2)$ on resistance to flow $(\overline{\lambda})$ in Figure 2. Similarly we can also notice in Figure 3 that is showing the effect of the permeability parameter (D) on resistance to flow $(\overline{\lambda})$, the effect of (γ) on resistance to flow $(\overline{\lambda})$ in Figure 4, the effect of length of stenosis (L_0) on resistance to flow $(\overline{\lambda})$ in Figure 5, the effect of the effect of couple stress parameter (c) on resistance to flow $(\overline{\lambda})$ in Figure 6, and the effect of slip parameter (α_1) on resistance to flow $(\overline{\lambda})$ in Figure 7. Now the wall shear stress increases with the maximum height of the stenosis from Figures 8-12. We can see the effect of the viscosity $(\overline{\mu}_2)$ on wall shear stress $(\overline{\tau}_m)$ in Figure 8. Similarly and we can observe the effect the permeability parameter (D) on wall shear stress (τ_m) is shown in Figure 9, the effect of ratio of central mean half width to the artery (γ) on wall shear stress (τ_m) is represented in Figure 10, the effect of length of stenosis (L_o) on resistance to flow $(\overline{\lambda})$ in Figure 11, and the effect of couple stress parameter (c) on wall shear stress at maximum height (τ_m) is represented in Figure 12.

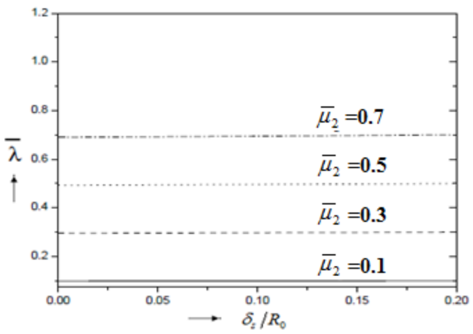

Figure 2: Effect of $\overline{\mu}_2$ on the variation of resistance to flow $\overline{\lambda}$ with stenosis length $\frac{\delta_s}{R_0}$

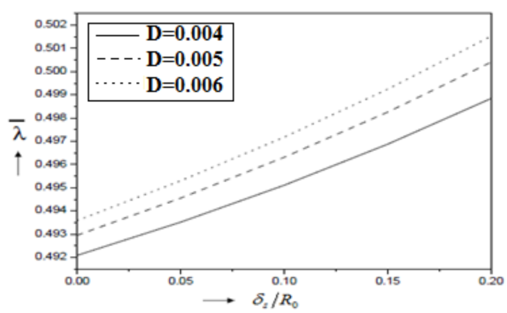

Figure 3: Effect of D on resistance to flow $\overline{\lambda}$ with $\frac{\delta_s}{R_0}$

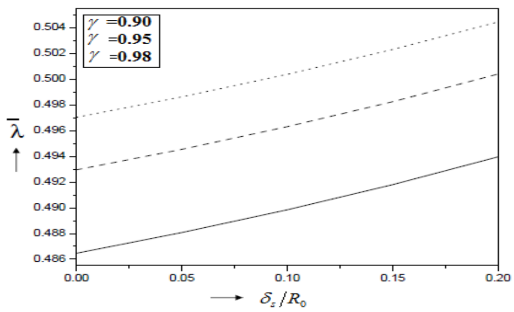

Figure 4: Effect of γ on resistance to flow $\overline{\lambda}$ with $\frac{\delta_s}{R_0}$

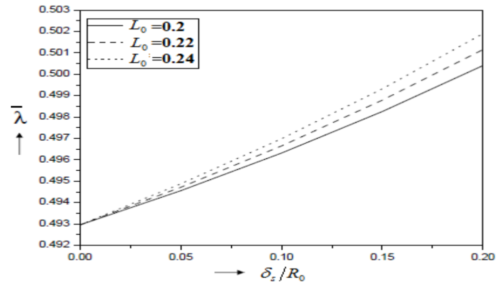

Figure 5: Effect of L_0 on resistance to flow $\overline{\lambda}$ with $\frac{\delta_s}{R_0}$

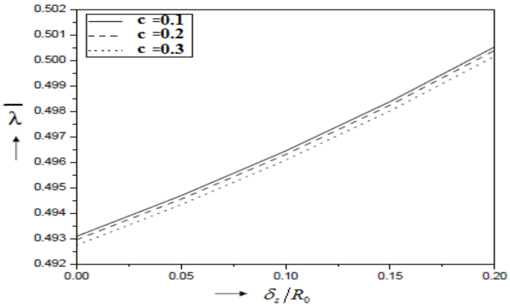

Figure 6: Effect of on 'c' on resistance to flow $\overline{\lambda}$ with $\frac{\delta_s}{R_0}$

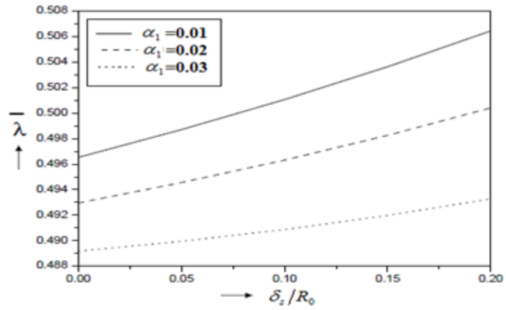

Figure 7: Effect α_1 on resistance to flow $\overline{\lambda}$ with $\frac{\delta_s}{R_0}$

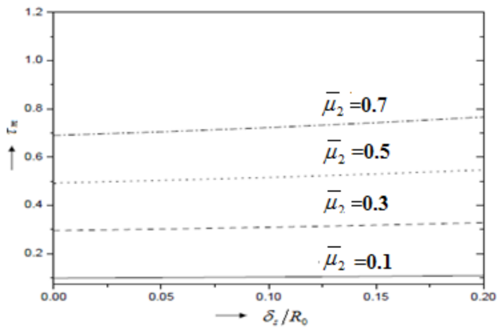

Figure 8: Effect $\overline{\mu}_2$ on wall shear stress at maximum height (τ_m) with $\frac{\delta_s}{R_0}$

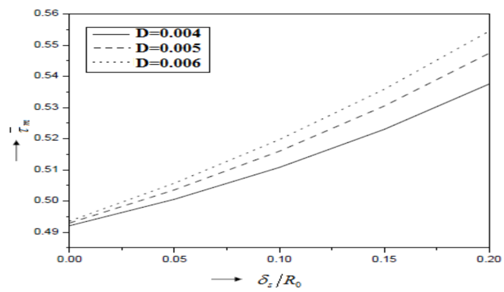

Figure 9: Effect of D on wall shear stress at maximum height (τ_m) with $\frac{\delta_s}{R_0}$

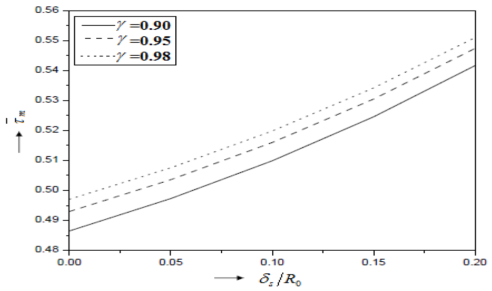

Figure 10: Effect of γ on wall shear stress at maximum height (τ_m) with $\frac{\delta_s}{R_0}$

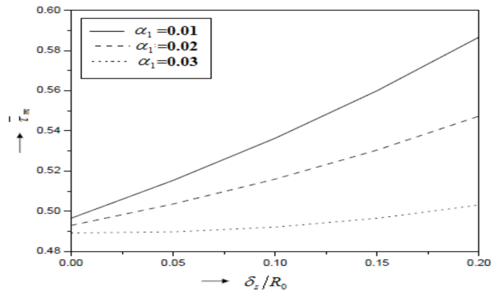

Figure 11: Effect α_1 on wall shear stress at maximum height τ_m with $\frac{\delta_s}{R_0}$

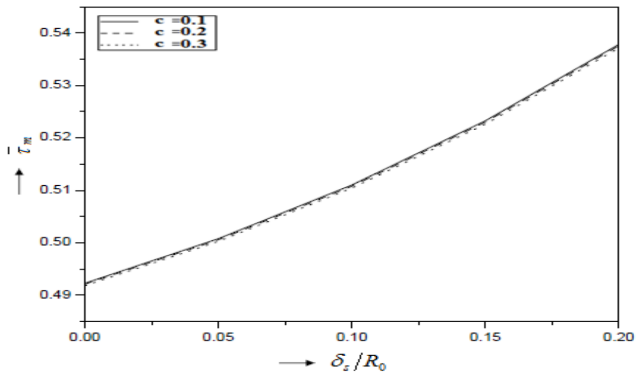

Figure 12: Effect of c on wall shear stress at maximum height τ_m with $\frac{\delta_s}{R_0}$

4. Conclusion

A mathematical model for the effect of couple stress fluid on the blood flow through an artery with mild stenosis has been investigated. We derived the resistance to flow $\overline{\lambda}$ and wall shear stress at different parameters and it is observed that the resistance to flow is increased as the maximum height of the stenosis increases, and wall shear stress increases as stenosis size increases. Here we studied the blood flow is represented by a couple stress fluid model on the wall shear stress. Increased shear stress may cause high blood pressure. Therefore, for medical purpose, this type of model is very important to reduce the wall shear stress to control cardiovascular system and various arterial diseases.

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