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K-hub Number of a Graph

Ahmed Mohammad Nour[1,*] and M. Manjunatha[1]

1 PG Department of Mathematics, P.E.S College of Science, Arts and Commerce, Mandya, Karnataka, India.

Abstract: In this paper, we introduce the concept of $k-$hub set and $k-$hub number of a graph. We compute the $k-$hub number for some standard graphs, also we determined the $k-$hub number for corona of two graphs. Some bounds of $k-$hub number are established. Finally we characterize the structure of all graphs for which $h_k(G) = 1$.

MSC: 05C50, 05C99.

Keywords: Hub number, K-hub number, Hub set.

1. Introduction

Let $G = (V, E)$ be a graph such that G is a finite and undirected graph without loops and multiple edges. A graph G is called (p, q) graph if G is with p vertices and q edges. The degree of a vertex v in a graph G denoted by $deg(v)$ is the number of edges of G incident with v. Where $\delta(G)(\Delta(G))$ denotes the minimum (maximum) degree among the vertices of G, respectively [2]. An end vertex is a vertex of degree one, let E_n be the set of all end vertices of G. The difference between two sets A and B is denoted by $A \setminus B$. For $v \in V(G)$, the open neighbourhood of v is denoted by $N(v) = \{u \in V(G) : uv \in E(G)\}$, for $S \subseteq V(G)$, $N(S) = \bigcup_{v\in S} N(v)$, similarly the closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$, and $N[S] = N(S) \cup S$. See [2] for terminology and notations not defined here.

Walsh [11] introduced the theory of hub number in the year 2006, a hub set in a graph G is a set H of vertices in G such that any two vertices in $V(G) \setminus H$ are connected by a path whose all internal vertices lie in H. The hub number of G, denoted by $h(G)$, is the minimum size of a hub set of G. A hub set H_r of G is called a restrained hub set if for any two vertices $u, v \in V(G) \setminus H_r$, u and v are connected by a path whose all internal vertices not in H_r [6]. The contraction of a vertex x in G (denoted by G/x) as being the graph obtained by deleting x and putting a clique on the (open) neighbourhood of x, (note that this operation does not create multiple edges, if two neighbours of x are already adjacent, then they remain simply adjacent). For more details on the hub studies we refer to [3, 4, 7–10]. The corona $G \circ F$ of two graphs G and F is the graph obtained by taking one copy of G of order p and p copies of F, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of F. For every $v \in V(G)$, denoted by F_v the copy of F whose vertices are attached one by one to the vertex v [1]. The following results will be useful in the proof of our results.

Theorem 1.1 ([6])**.** *Let G be any graph. Then the set H_r is restrained hub set if and only if G/H_r is complete, and $G[V(G) \setminus H_r]$ is connected.*

* *E-mail: ahmadnoor01298@gmail.com*

Theorem 1.2 ([6]). *Let G be a graph with at least one end vertex, $h_r(G) = p - 2$ if and only if there exists minimum restrained hub set not containing an end vertex.*

Theorem 1.3 ([11]). *Let T be a tree with n vertices and l leaves. Then $h(G) = h_c(G) = p - l$.*

Theorem 1.4 ([3]). *For $k \geq 1$, if G is a connected graph with radius r, then $\gamma_k(G) \geq \frac{2r}{2k+1}$.*

Theorem 1.5 ([3]). *If G is a connected graph, then $\gamma_k^c(G) \leq (2k+1)\gamma_k - 2k$.*

2. Main Results

Definition 2.1. *Suppose that we have a graph G. Let $k \geq 1$ be an integer number, $S \subseteq V(G)$, and $x, y \in V(G)$. An $S - k-$path between x and y is a path whose all vertices are from S, except for k vertices from each end of the path which may not from the set S.*

Definition 2.2. *A set H is a $k-$hub set of G if for each $x, y \in (V(G) \setminus H)$, there is an $H - k-$path in G between x and y. The $k-$hub number of G is the minimum cardinality of a $k-$hub set of G, and denoted by $h_k(G)$. For $k = 1$, the $1-$hub number of G is precisely the hub number of G, and $h_1(G) = h(G)$.*

Definition 2.3. *Let H_k^c be a $k-$hub set of a graph G. Then H_k^c is called a connected $k-$hub set if and only if $G[H_k^c]$ is connected. The connected $k-$hub number of G is the minimum cardinality of a connected $k-$hub set of G, and denoted by $h_k^c(G)$. For $k = 1$, the connected $1-$hub number of G is precisely the connected hub number of G, and $h_1^c(G) = h_c(G)$.*

From the previous definitions, if H_k is a (connected) $k-$hub set of G, then it is also a (connected)$(k+1)-$hub set of G.

Remark 2.4. *Let G be any graph, then $h_j(G) \leq h_i(G)$, for all $i \leq j$.*

Lemma 2.5. *Let G be a connected graph. Then $h_k(G) = h_k^c(G) = 0$, if and only if $k \geq \lceil \frac{d(G)+1}{2} \rceil$.*

Proof. Let G be a connected graph, by contradiction, let $h_k(G) = 0$, and $k \leq \lceil \frac{d(G)+1}{2} \rceil - 1$, take $x, y \in V(G)$ such that $d(x, y) = d(G)$. Now, there is $xy-$ path whose all vertices lie in H_k, except for k vertices in the tails of the path, where H_k is a minimum k^{th} hub set of G, since $H_k = \phi$, all the vertices of the path are outside H_k. Therefore:

$$
\begin{aligned}
d(x, y) &\leq 2k - 1 \\
&\leq 2(\lceil \frac{d(G)+1}{2} \rceil - 1) - 1 \\
&\leq d(G) - 1,
\end{aligned}
$$

and that is a contradiction. Conversely, let $k \geq \lceil \frac{d(G)+1}{2} \rceil$, so $d(G) \leq 2k - 1$. Now, let $H_k = \phi$, and $x, y \in V(G) \setminus H_k$. Then $d(x, y) \leq d(G) \leq 2k - 1$, so the minimum path between x and y is $H_k - k-$path between them, thus H_k is a $k-$hub set of G, hence $h_k(G) = 0$. □

Theorem 2.6. *Let G be a graph. Then $h_k(G) = 1$ if and only if G has the following conditions:*

(1). $d(G) \geq 2k$.

(2). $V(G) = A \dot\cup B \dot\cup \{v\}$, where $\{v\}$ is the $k-$hub set of G.

(3). For every $x \in B, d(x, v) \leq k$.

(4). For every pair $(x, y) \in A \times (A \cup B), d(x, y) \leq 2k - 1$.

Proof. Let G be a graph, and $h_k(G) = 1$ with a $k-$hub set $\{v\}$, if $d(G) < 2k$, then by Lemma 2.5, $h_k(G) = 0$, and that a contradiction, this proves the first condition. To show conditions 2 and 3, take $B = N_k(v)$, and $A = V(G) \setminus (A \cup \{v\})$, now for the 4^{th} condition, let $(x, y) \in A \times (A \cup B)$, if $d(x, y) > 2k - 1$, then by definition of A, $d(x, v) \geq k$, so v is not in any $\{v\} - k-$ path between x and any other vertex. Therefore, there is a path between x and y consists from at most $2k$ vertices. Thus $d(x, y) \leq 2k - 1$. The converse is trivial. $\square$

Theorem 2.7. *Let G be a tree. Then $h_2(G) = h(F)$, where $F \cong G[V(G) \setminus E_n(G)]$.*

Proof. Let G be a tree, and $F \cong G[V(G) \setminus E_n(G)]$, its clear that the set A of all non leaf vertices of F forms a $2-$hub set for the graph G, and no proper sub set of A is a $2-$hub set of G, since every vertex in A is a cut vertex. To complete the proof, we need to show that we can't find a minimum $2-$hub set of G contained in A. So, let S be a minimum $2-$hub set of G which contains a vertex out side A(say x). Since the vertices of A forms a $2-$hub set of G, S must exclude one vertex w from A. Choose a vertex y such that y is the nearest vertex to x in the $xw-$path, where $y \in A \setminus S$. Then $S' = (S \setminus \{x\}) \cup \{y\}$ is also a $2-$ hub set, since any $S - 2$-path between y and any other vertex z can be extended to be a $S' - 2$-path through x and z. Hence we remove a vertex from $V(G) \setminus A$, without adding another, we can repeat this process to find a minimum $2-$ hub set containing no vertices of $V(G) \setminus A$. However the only such set is A, so A must be minimum. Thus

$$
\begin{aligned}
h_2(G) &= |V(G)| - (|E_n(G) \cup E_n(F)|) \\
&= |V(G)| - (|E_n(G)| + |E_n(F)|) \ (since E_n(G) \cap E_n(F) = \phi) \\
&= (|V(G)| - |E_n(G)|) - |E_n(F)| \\
&= |V(F)| - |E_n(F)| \\
&= h(F) \quad \text{(by Theorem 1.3)}.
\end{aligned}
$$

$\square$

Note that, if T is tree, then by using the same idea in the previous proof, and since any graph constructed by deleting the end vertices of tree, is a tree, we get the following corollary.

Corollary 2.8. *Let $T(p, q)$ be a tree. Then $h_k(T) = h_{k-1}(T_1)$, where $T_1 \cong T[V(T) \setminus E_n(T)]$.*

Corollary 2.9. *Let T be a tree, then $h_k(T) = p - \sum_{i=0}^{k-1} |E_n(T_k)|$, where $T_i \cong T[V(T_{i-1}) \setminus E_n(T_{i-1})]$, and $T_0 \cong T$.*

Proof. Let T be a tree, and $T_i \cong T[V(T_{i-1}) \setminus E_n(T_{i-1})]$, where $T_0 \cong T$, and since $E_n(T_i) \subseteq V(T_i)$, so $|V(T_i) \setminus E_n(T_i)| = |V(T_i)| - |E_n(T_i)|$, and we get that:

$$
\begin{aligned}
|V(T_k)| &= |V(T_{k-1})| - |E_n(T_{k-1})| \\
&= |V(T_{k-2})| - |E_n(T_{k-2})| - |E_n(T_{k-1})| \\
&= \ldots \\
&= |V(T)| - \sum_{k=0}^{k-1} |E_n(T_k)|. \qquad (*)
\end{aligned}
$$

Now by Corollary 2.8, we get that:

$$
\begin{aligned}
h_k(T) &= h_{k-1}(T_1) \\
&= h_{k-2}(T_2)
\end{aligned}
$$

$$
\begin{aligned}
&= ...\\
&= h_1(T_{k-1})\\
&= |V(T_{k-1})| - |E_n(T_{k-1})|\\
&= |V(T)| - \sum_{k=0}^{k-2} |E_n(T_k)| - |E_n(T_{k-1})| \quad by\ (*)\\
&= p - \sum_{k=0}^{k-1} |E_n(T_k)|.
\end{aligned}
$$

□

Theorem 2.10. *Let C_n be a cycle. Then*

$$
h_k(C_n) = \begin{cases} 0 & , \ if\ k \geq \lceil \frac{\lfloor \frac{n}{2} \rfloor + 1}{2} \rceil; \\ n - 3k & , \ otherwise. \end{cases}
$$

Proof. Let C_n be any cycle of order n, now we have to discuss the following cases:

Case 1: $k \geq \lceil \frac{\lfloor \frac{n}{2} \rfloor + 1}{2} \rceil$. Then by Lemma 2.5, $h_k(C_n) = 0$ since $d(C_n) = \lfloor \frac{n}{2} \rfloor$.

Case 2: $k < \lceil \frac{\lfloor \frac{n}{2} \rfloor + 1}{2} \rceil$. Then by Lemma 2.5, $h_k(C_n) \neq 0$. Now let $v_1, v_2, v_3, ..., v_n$ be a path in the cycle C_n, H_k be a $k-$hub set of C_n, and let A be any component of $C_n[V(G) \setminus H_k]$, and m be the number of components. Now we need to prove that $h_k(C_n) \geq n - 3k$, by showing that $|V(C_n) \setminus H_k| \leq 3k$. So we have to discuss the following subcases:

Subcase 2.1: $|A| \leq k-1$. If $|V(C_n) \setminus (H_k \cup A)| \leq 2k+1$, then the result holds. While if not, then without loss of generality let $A = \{v_1, v_2, v_3, ..., v_t\}, t \leq k-1$, and enumerate the vertices in $V(C_n) \setminus (H_k \cup A)$ by $w_1, w_2, w_3, ..., w_q$, where $q \geq 2k+2$ such that for any two vertices $w_i = v_s$ and $w_j = v_r$, then $i < j$ if $s < r$ for all $i, j = 1, 2, 3, ..., q$. So, there is no $H_k - k-$path between w_1 and w_{2k+1}, a contradiction.

Subcase 2.2: $|A| \geq k$ and $m \geq 4$. Let $A_i, i = 1, 2, ..., t$ are the components of $C_n[V(C_n) \setminus H_k]$, $t \geq 4$, then there is two vertices $x \in V(A_i)$, $y \in V(A_j)$ for some choices of i and j, such that there is no $H_k - k-$path between them, a contradiction.

Subcase 2.3: $|A| \geq k$ and $m = 3$, and any component of them say $|A_1| \geq k+1$. Then let $A_1 = \{v_1, v_2, v_3, ..., v_t\}, t \geq k+1$, thus there is no H_k − k-path between v_1 (or v_t), and some vertices in A_2, and that is a contradiction. Therefore, $|A_1| = k$, so $|V(G) \setminus H_k| = A_1 + A_2 + A_3 = 3k$.

Subcase 2.4: $|A| \geq k$ and $m = 2$. If $|A_1| \geq k+1$ and $|A_2| \geq k+1$. Then let $A_1 = \{v_1, v_2, v_3, ..., v_s\}, s \geq k+1$, and let $A_2 = \{w_1, w_2, w_3, ..., w_t\}, t \geq k+1$, as the way of enumeration on subcase 2.1, so there is no $H_k - k-$path between the vertices v_1 and w_1, thus one of them say A_2 has just k vertices. Now, if $|A_1| \geq 2k+1$, then let $A_1 = \{v_1, v_2, v_3, ..., v_{2k+1}\}$, thus there is no $H_k - k$ path between v_1 and v_{2k+1}, so $|V(G) \setminus H_k| \leq A_1 + A_2 \leq 2k + k = 3k$.

Subcase 2.5: $|A| \geq k$ and $m = 1$. Assume $|A| \geq 3k+1$, let $A_1 = \{v_1, v_2, v_3, ..., v_{3k+1}\}$, thus there is no $H_k - k-$path between the vertices v_1 and v_{2k+1} a contradiction. Therefore, $|V(G) \setminus H_k| \leq |A_1| \leq 3k$.

From the previous cases we get that for any $k-$hub set H_k of C_n, $|V(C_n) \setminus H_k| \leq 3k$, so $h_k(C_n) \geq n - 3k$, now take $H_k = \{v_{3k+1}, v_{3k+2}, v_{3k+3}, ..., v_n\}$, this set is a $k-$hub set of C_n and its minimum since $|H_k| = n - 3k$. Hence the assertion follows. □

Note that by previous proof, if H_k is a minimum $k-$hub set of a cycle C_n, then it has one of the following shapes, included in Figure 1, where black(white) vertex means that the vertex belongs(dose not belong) to H_k, since $G[H_k]$ is connected with same order, $h_k(C_n) = h_k^c(C_n)$.

Lemma 2.11. *If H_k is a $k-$hub set of a graph G, then $d(G/H_k) \leq 2k-1$, moreover the converse is true if and only if $k = 1$.*

Proof. Let H_k be a $k-$hub set of a graph G, if $d(G/H_k) \geq 2k$, then take $x, y \notin H_k$ such that $d(x, y) \geq 2k$, thus every $xy-$path has at least one vertex not in H_k other than k vertices in every tail of the path, hence H_k is not a $k-$hub set of G, and that is a contradiction, so $d(G/H_k) \leq 2k - 1$.
Now if $k = 1$ the converse is true, if $k \geq 2$ then we have the following counter example: $G \cong P_{2k+1} = v_1, v_2, ..., v_{2k+1}$, and $H_k = \{v_2\}$. □

Corollary 2.12. *Let G be a graph, then $h_k(G) \geq d(G) - 2k + 1$.*

Proof. Let G be a graph and H_k be a $k-$hub set of G, by Lemma 2.11, $d(G/H_k) \leq 2k - 1$, and by walsh every single vertex contraction decrease the diameter by at most one, so we need at least $d(G) - (2k - 1)$ contractions, to reach the diameter of G/H_k. Therefore $h_k(G) \geq d(G) - 2k + 1$. □

Theorem 2.13. *Let G be a graph, and $H_k^c \subseteq V(G)$ such that $G[H_k^c]$ is connected. Then H_k^c is a connected $k-$hub of G set if and only if $d(G/H_k^c) \leq 2k - 1$ and for every vertex $x \notin N_k[H_k^c]$, $d_{G-G[H_k^c]}(x, u) \leq 2k - 1$, where $u \notin H_k^c$.*

Proof. Let G be a graph, and H_k^c be a connected $k-$hub set of G, and there is a vertex $x \notin N_k[H_k^c]$, with $d_{G-G[H_k^c]}(x, u) \geq 2k$, for some vertex $u \notin H_k^c$. Let P be a $H_k^c - k-$path between x and u, if the path contains any vertex from H_k^c, then the $x-$tail from the path has more than k vertices are not from the set H_k^c, a contradiction, while if the path does not contain any vertex from H_k^c, then the path has at most $2k$ vertices, thus $d_{G-G[H_k^c]}(x, u) \leq 2k - 1$, which contradicts our hypothesis. Therefor, $d_{G-G[H_k^c]}(x, u) \leq 2k - 1$, and by Lemma 2.11, $d(G/H_k^c) \leq 2k - 1$.
Conversely, suppose that there is $H_k^c \subseteq V(G)$ such that $G[H_k^c]$ is connected, $d(G/H_k^c) \leq 2k - 1$ and for every vertex $x \notin N_k[H_k^c]$, $d_{G-G[H_k^c]}(x, u) \leq 2k - 1$, where $u \notin H_k^c$. Now, take $w, z \in V(G) \setminus H_k^c$, we have to discuss the following cases:
Case 1: $w, z \in N_k[H_k^c]$. So there is a path $w, w_1, w_2, ..., w_n$, where $w_n \in H_k^c$, and $n \leq k$, also a path $z, z_1, z_2, ..., z_m$, where $z_m \in H_k^c$, and $m \leq k$, and a path $w_n, c_1, c_2, ..., c_t, z_m$, whose all vertices lies in H_k^c since $G[H_k^c]$ is connected. Therefore, the path $w, w_1, w_2, ..., w_n, c_1, c_2, ..., c_t, z_m, z_{m-1}, ..., z$, is a $H_k^c - k-$path between w and z.
Case 2: $w \notin N_k[H_k^c]$, or $z \notin N_k[H_k^c]$. By assumption $d_G(w, z) \leq d_{G-G[H_k^c]}(w, z) \leq 2k - 1$, so the minimum path between z and w in G is a $H_k^c - k-$path.
Therefore, in both cases we found a $H_k^c - k-$path between any two vertices $w, z \in V(G) \setminus H_k^c$, hence H_k^c is a connected $k-$hub set of G. □

Theorem 2.14. *Let G be a graph, and $H_c \subseteq V(G)$ such that $G[H_c]$ is connected. Then the following are equivalent:*

(1). H_c is a connected hub set of G.

(2). for every vertex $x \notin N[H_c]$, x is adjacent to u, where $u \notin H_c$.

(3). G/H_c is complete graph.

Proof. (1) ⇒ (2). Let H_c be a connected hub set of G, and let $x \notin N[H_c]$. Then by Theorem 2.13, $d_{G-G[H_c]}(x, u) \leq 1$, where $u \notin H_c$, thus x is adjacent to u, where $u \notin H_c$.
(2) ⇒ (3). Assume that, for every vertex $x \notin N[H_c]$, x is adjacent to u, where $u \notin H_c$. Then take $u, v \in V(G/H_c)$, if $u, v \in N[H_c]$, then by definition of G/H_c, $uv \in E(G/H_c)$, while if $u \notin N[H_c]$ or $v \notin N[H_c]$, then by assumption u is adjacent to v, hence G/H_c is complete graph.
(3) ⇒ (1). Let G/H_c is complete graph. Then by Theorem 1.1, H_c is a connected hub set of G. □

Theorem 2.15. *Let G and F be two connected graphs, then*

$$h^c_{(k+1)}(G\circ F)=\begin{cases}\gamma^c_k(G), & if\ \gamma^c_k(G)\leq h^c_k(G)(1+|V(F)|);\\ h^c_k(G)(1+|V(F)|), & if\ \gamma^c_k(G)>h^c_k(G)(1+|V(F)|).\end{cases}$$

Proof. Let G and F be two graphs, and let H_{k+1} be a connected $(k+1)-$hub set of $G\circ F$, by definition of corona and Theorem 2.13, $H_k = H_{k+1}\setminus V(F)$, is a connected $k-$hub set of G. Therefore, to construct any connected $(k+1)-$hub set of $G\circ F$, the construction must start with $k-$hub set of G. Now, let H_k be any hub set of G, then we have to discuss the following cases:

Case 1: $V(G)\setminus N_k(H_k)\neq\phi$. In this case, one of the following two ways must be followed to construct a connected $(k+1)-$hub set of $G\circ F$.

First way: Since there exist $x\in(V(G)\setminus N_k(H_k))$, so there is no $H_{k+1}-(k+1)-$path between x and any vertex y in $V(F_v)$, where $v\in H_k$. Therefore, $\bigcup\limits_{v\in H_k}V(F_v)\subset H_{k+1}$, thus $H_k\bigcup\bigcup\limits_{v\in H_k}V(F_v)\subseteq H_{k+1}$, hence $h^c_{k+1}(G\circ F)\geq h^c_k(G)+h^c_k(G)|V(F)| = h^c_k(G)(1+|V(F)|)$.

Second way: Add vertices from $V(G)$ to H_k, in order to get a connected set H'_k, such that $V(G)\setminus N_k(H'_k)=\phi$, this constructs a connected $k-$ domination set of G, in the same time it is a connected $(k+1)-$hub set of $G\circ F$. Therefore, $h^c_{k+1}(G\circ F)\geq|H'_k|\geq\gamma^c_k(G)$.

Case 2: $V(G)\setminus N_k(H_k)=\phi$. Then H_k is a connected $k-$ domination set of G, hence it follows the second way on case 1.

The both lower bounds are hold by taking $H_{k+1}=H_k\bigcup\limits_{v\in H_k}V(F_v)$, where H_K is a minimum $k-$hub set of G for the first way, and by taking $H_{k+1}=D_k$, where D_k is a connected $k-$dominating set of G for second way. Therefore, $h^c_{k+1} = min\{\gamma^c_k(G), h^c_k(G)(1+|V(F)|)\}$. Thus

$$h^c_{(k+1)}(G\circ F)=\begin{cases}\gamma^c_k(G), & \text{if } \gamma^c_k(G)\leq h^c_k(G)(1+|V(F)|);\\ h^c_k(G)(1+|V(F)|), & \text{if } \gamma^c_k(G)>h^c_k(G)(1+|V(F)|).\end{cases}$$

□

3. Bounds of $k-$hub Number

Proposition 3.1. *Let G be a graph, then $h_k(G)\leq p-|M_k(G)|$, where $M_k(G)=max\{|N_k(v)|, v\in V(G)\}$.*

Proof. Let G be a graph, with $M_k(G)=|N_k(v)|$, for some vertex $v\in V(G)$. Then the set $H_k=(V(G)\setminus N_k(v))$, is a $k-$hub set of G, thus $h_k(G)\leq|H_k(G)|=p-M_k(G)$. □

Proposition 3.2. *If F is a spanning sub graph of G, then $h_k(F)\geq h_k(G)$.*

Proposition 3.3. *Let G be a connected graph, then $\gamma^c_k(G)-k\leq h^c_k(G)\leq\gamma^c_k(G)$.*

Proof. Let G be a connected graph, the upper bound is trivial, since any connected distance $k-$domination set is a $k-$hub set. To show lower bound, let H^c_k be a minimum connected $k-$hub set of G, if $N_k[H^c_k]=V(G)$, then H^c_k is a connected distance $k-$domination set, and thus $h^c_k(G)\geq\gamma^c_k(G)\geq\gamma^c_k(G)-k$, while if not, then take $v\in[N_t(H^c_k)\setminus N_{t-1}(H^c_k)]$, where $N_{t+1}(H^c_k)=N_t(H^c_k)$, and take $v_1\in N(H^c_k)$, let the minimum path between v_1 and v be $v_1,v_2,...,v_k,v_{k+1},...,v_t$, take the set $D=\{v_1,v_2,...,v_k\}$. Therefore, by Lemma 2.11, and definition of D, we get that, for every vertex $y\in(V(G)\setminus N_k[H^c_k])$, there is $x\in D$, such that $d(x,y)\leq k$, and since $G[D\cup H^c_k]$ is connected, the set $D\cup H^c_k$ is connected distance $k-$domination set of G, thus:

$$\gamma^c_k(G)\leq|D\cup H^c_k|$$

$$= |D| + |H_k^c| \text{ since } D \cap H_k^c = \phi$$
$$= k + h_k^c(G).$$

Therefore $\gamma_k^c(G) - k \leq h_k^c(G)$. □

Corollary 3.4. *Let G and F be two connected graphs. Then we have the following properties:*

(1). If $k \geq \lceil \frac{d(G)+1}{2} \rceil$. Then $h_{(k+1)}^c(G \circ F) = 0$.

(2). If $h_k^c(G) = \gamma_k^c(G)$, then $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G)$.

(3). If $k \geq |V(F)| h_k^c(G)$, then $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G)$.

Proof. Let G and F be two connected graphs.

(1). Let $k \geq \lceil \frac{d(G)+1}{2} \rceil$, then by Theorem 2.5, $h_{(k+1)}^c(G \circ F) = h_k^c(G) = 0 < \gamma_k^c(G)$.

(2). Let $h_k^c(G) = \gamma_k^c(G)$. Then $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G) < \gamma_k^c(G)(1 + |V(F)|) = h_k^c(G)(1 + |V(F)|)$.

(3). Let $k \geq |V(F)| h_k^c(G)$. Then by proposition 3.3, $\gamma_k^c(G) \leq h_k^c(G) + k \leq h_k^c(G) + |V(F)| h_k^c(G) = h_k^c(G)(1 + |V(F)|)$. Thus $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G)$. □

Corollary 3.5. *Let G and F be two connected graphs. Then we have the following properties:*

(1). If $k \geq \lceil \frac{d(G)+1}{2} \rceil$. Then $h_{(k+1)}^c(G \circ F) = 0$.

(2). If $h_k^c(G) = \gamma_k^c(G)$, then $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G)$.

(3). If $k \geq |V(F)| h_k^c(G)$, then $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G)$.

Proof. Let G and F be two connected graphs.

(1). Let $k \geq \lceil \frac{d(G)+1}{2} \rceil$, then by Theorem 2.5, $h_{(k+1)}^c(G \circ F) = h_k^c(G) = 0 < \gamma_k^c(G)$.

(2). Let $h_k^c(G) = \gamma_k^c(G)$. Then $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G) < \gamma_k^c(G)(1 + |V(F)|) = h_k^c(G)(1 + |V(F)|)$.

(3). Let $k \geq |V(F)| h_k^c(G)$. Then by Proposition 3.3, $\gamma_k^c(G) \leq h_k^c(G) + k \leq h_k^c(G) + |V(F)| h_k^c(G) = h_k^c(G)(1 + |V(F)|)$. Thus $h_{(k+1)}^c(G \circ F) = \gamma_k^c(G)$. □

Theorem 3.6. *Let G be a connected graph, then $(2k+1)\gamma_k - 2k \geq h_k(G) \geq \dfrac{2r(G)}{2k+1}$.*

Proof. Let G be a connected graph, then by theorem 1.4, and by Proposition 3.3, we get that: $h_k(G) \geq \gamma_k(G) \geq \frac{2r(G)}{2k+1}$, and by Theorem 1.5, with Proposition 3.3, we get that $h_k(G) \leq h_k^c(G) \leq \gamma_k^c(G) \leq (2k+1)\gamma_k - 2k$. □

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