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Generalized Hyers - Ulam Stability of Additive - Quadratic - Cubic - Quartic Functional Equation in Fuzzy Normed Spaces: A Direct Method

Research Article*

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Abstract: In this paper, the authors investigate the generalized Hyers-Ulam-stability of AQCQ functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)-4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$$

in fuzzy normed spaces using direct method.

MSC: 39B52, 39B82, 26E50, 46S50.

Keywords: Fuzzy normed space, Additive, quadratic, cubic and quartic functional equation, Generalized Hyers-Ulam-Rassias stability.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of S.M. Ulam [40] concerning the stability of group homomorphisms. D.H. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces.
Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [35] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [35] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 1982, J.M. Rassias [32] followed the innovative approach of the Th.M. Rassias theorem [35] in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in R$ with $p+q=1$.
In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [37] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 12, 16, 20]).

A.K. Katsaras [22] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space.

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Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 24, 41]. In particular, T. Bag and S.K. Samanta [8], following S.C. Cheng and J.N. Mordeson [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [9]. We use the definition of fuzzy normed spaces given in [8] and [27–30].

Definition 1.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

$(F1)$ $N(x,c) = 0$ *for* $c \leq 0$;

$(F2)$ $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;

$(F3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

$(F4)$ $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

$(F5)$ $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;

$(F6)$ *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 1.2. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.3. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 1.4. *A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 1.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

The stability of various functional equations in fuzzy normed spaces was investigated in [3, 4, 6, 17, 26–30, 38]. In this paper, the authors investigate the generalized Hyers-Ulam-Aoki-Rassias stability AQCQ functional equation

$$f(x+2y) + f(x-2y) = 4f(x+y) - 4f(x-y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \tag{1}$$

in the fuzzy normed vector space by direct method.

2. Fuzzy Stability Results: Direct Method

Throughout this section, assume that $X, (Z, N')$ and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. Now use the following notation for a given mapping $f : X \to Y$

$$D\ f(x,y) = f(x+2y) + f(x-2y) - 4f(x+y) + 4f(x-y) + 6f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y)$$

for all $x, y \in X$. Now, we investigate the generalized Ulam-Hyers stability of AQCQ functional equation (1).

Theorem 2.1. *Let $\beta \in \{-1, 1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2}\right)^\beta < 1$*

$$N'\left(\alpha\left(2^\beta y, 2^\beta y\right), r\right) \geq N'\left(d^\beta \alpha\left(y, y\right), r\right) \tag{2}$$

for all $y \in X$ and all $r > 0$, $d > 0$, and

$$\lim_{k\to\infty} N'\left(\alpha\left(2^{\beta k}x, 2^{\beta k}y\right), 2^{\beta k}r\right) = 1 \tag{3}$$

for all $x, y \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(D\ f(x,y), r\right) \geq N'\left(\alpha(x,y), r\right) \tag{4}$$

for all $r > 0$ and all $x, y \in X$. Then the limit

$$A(y) = N - \lim_{k\to\infty} \frac{f(2^{\beta k}y)}{2^{\beta k}} \tag{5}$$

exists for all $y \in X$ and the mapping $A : X \to Y$ is a unique additive mapping such that

$$N\left(f(2y) - 8f(y) - A(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2-d)r}{4}\right)\right\} \tag{6}$$

for all $y \in X$ and all $r > 0$.

Proof. First assume $\beta = 1$. Replacing (x, y) by (y, y) in (4), we get

$$N\left(f(3y) - 4f(2y) + 5f(y), r\right) \geq N'\left(\alpha(y,y), r\right) \tag{7}$$

for all $y \in X$ and all $r > 0$. Replacing x by $2y$ in (4), we obtain

$$N\left(f(4y) - 4f(3y) + 6f(2y) - 4f(y), r\right) \geq N'\left(\alpha(2y,y), r\right) \tag{8}$$

for all $y \in X$ and all $r > 0$. Now, from (7) and (8), we have

$$\begin{aligned} N\left(f(4y) - 10f(2y) + 16f(y), r\right) &\geq min\left\{N\left(4\left(f(3y) - 4f(2y) + 5f(y)\right), \frac{r}{2}\right), N\left(f(4y) - 4f(3y) + 6f(2y) - 4f(y), \frac{r}{2}\right)\right\} \\ &\geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \end{aligned} \tag{9}$$

for all $y \in X$ and all $r > 0$. Let $a : X \to Y$ be a mapping defined by $a(y) = f(2y) - 8f(y)$. Then we conclude that

$$N\left(a(2y) - 2a(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \tag{10}$$

for all $y \in X$ and all $r > 0$. Thus, we have

$$N\left(\frac{a(2y)}{2} - a(y), \frac{r}{2}\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{4}\right)\right\} \tag{11}$$

for all $y \in X$ and all $r > 0$. Replace y by $2^k y$ in (11), we get

$$N\left(\frac{a(2^{k+1}y)}{2^{k+1}} - \frac{f(2^k y)}{2^k}, \frac{r}{2^k 2}\right) \geq min\left\{N'\left(\alpha(2^k y, 2^k y), \frac{r}{8}\right), N'\left(\alpha(2^{k+1}y, 2^k y), \frac{r}{4}\right)\right\} \tag{12}$$

for all $y \in X$ and all $r > 0$. Using (2), $(F3)$ in (12), we arrive

$$N\left(\frac{a(2^{k+1}y)}{2^{k+1}} - \frac{a(2^k y)}{2^k}, \frac{r}{2^k 2}\right) \geq min\left\{N'\left(\alpha(y, y), \frac{r}{8d^k}\right), N'\left(\alpha(2y, y), \frac{r}{4d^k}\right)\right\} \tag{13}$$

for all $y \in X$ and all $r > 0$. Replacing r by $d^k r$ in (13), we get

$$N\left(\frac{a(2^{k+1}y)}{2^{k+1}} - \frac{a(2^k y)}{2^k}, \frac{d^k r}{2^k 2}\right) \geq min\left\{N'\left(\alpha(y, y), \frac{r}{8}\right), N'\left(\alpha(2y, y), \frac{r}{4}\right)\right\} \tag{14}$$

for all $y \in X$ and all $r > 0$. It is easy to see that

$$\frac{a(2^k y)}{2^k} - a(y) = \sum_{i=0}^{k-1}\left[\frac{a(2^{i+1}y)}{2^{i+1}} - \frac{a(2^i y)}{2^i}\right] \tag{15}$$

for all $y \in X$. From equations (14) and (15), we have

$$\begin{aligned} N\left(\frac{a(2^k y)}{2^k} - a(y), \sum_{i=0}^{k-1}\frac{d^i r}{2^i 2}\right) &\geq min\bigcup_{i=0}^{k-1} N\left\{\frac{a(2^{i+1}y)}{2^{i+1}} - \frac{a(2^i y)}{2^i}, \sum_{i=0}^{k-1}\frac{d^i r}{2^i 2}\right\} \\ &\geq min\bigcup_{i=0}^{k-1}\left\{N'\left(\alpha(y, y), \frac{r}{8}\right), N'\left(\alpha(2y, y), \frac{r}{4}\right)\right\} \\ &\geq min\left\{N'\left(\alpha(y, y), \frac{r}{8}\right), N'\left(\alpha(2y, y), \frac{r}{4}\right)\right\} \end{aligned} \tag{16}$$

for all $y \in X$ and all $r > 0$. Replacing x by $2^m x$ in (16) and using (2), $(F3)$, we obtain

$$N\left(\frac{a(2^{k+m}x)}{2^{(k+m)}} - \frac{a(2^m x)}{2^m}, \sum_{i=0}^{k-1}\frac{d^i\, r}{2^{i+m}2}\right) \geq min\left\{N'\left(\alpha(y, y), \frac{r}{8d^m}\right), N'\left(\alpha(2y, y), \frac{r}{4d^m}\right)\right\} \tag{17}$$

for all $y \in X$ and all $r > 0$ and all $m, k \geq 0$. Replacing r by $d^m r$ in (17), we get

$$N\left(\frac{a(2^{k+m}y)}{2^{(k+m)}} - \frac{a(2^m y)}{2^m}, \sum_{i=0}^{m+k-1}\frac{d^{i+m}\, r}{2^{i+m}2}\right) \geq min\left\{N'\left(\alpha(y, y), \frac{r}{8}\right), N'\left(\alpha(2y, y), \frac{r}{4}\right)\right\} \tag{18}$$

for all $y \in X$ and all $r > 0$ and all $m, k \geq 0$. Using $(F3)$ in (18), we obtain

$$N\left(\frac{a(2^{k+m}y)}{2^{(k+m)}} - \frac{a(2^m y)}{2^m}, r\right) \geq min\left\{N'\left(\alpha(y, y), \frac{r}{8\sum_{i=m}^{m+k-1}\frac{d^i}{2^i 2}}\right), N'\left(\alpha(2y, y), \frac{r}{4\sum_{i=m}^{m+k-1}\frac{d^i}{2^i 2}}\right)\right\} \tag{19}$$

for all $y \in X$ and all $r > 0$ and all $m, k \geq 0$. Since $0 < d < 2$ and $\sum_{i=0}^{k}\left(\frac{d}{2}\right)^i < \infty$, the cauchy criterion for convergence and $(F5)$ implies that $\{\frac{a(2^k y)}{2^k}\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $A(y) \in Y$. So one can define the mapping $A : X \to Y$ by $A(y) = N - \lim_{k\to\infty}\frac{a(2^k y)}{2^k}$ for all $y \in X$. Letting $m = 0$ in (19), we get

$$N\left(\frac{a(2^k y)}{2^k} - a(y), r\right) \geq min\left\{N'\left(\alpha(y, y), \frac{r}{8\sum_{i=0}^{k-1}\frac{d^i}{2^i 2}}\right), N'\left(\alpha(2y, y), \frac{r}{4\sum_{i=0}^{k-1}\frac{d^i}{2^i 2}}\right)\right\} \tag{20}$$

for all $y \in X$ and all $r > 0$. Letting $k \to \infty$ in (20) and using $(F6)$, we arrive

$$N\left(a(y) - A(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2-d)r}{4}\right)\right\}$$

for all $y \in X$ and all $r > 0$. To prove A satisfies the (1), replacing (x, y) by $(2^k x, 2^k y)$ in (4), respectively , we obtain

$$N\left(\frac{1}{2^k} Df(2^k x, 2^k y), r\right) \geq N'\left(\alpha(2^k x, 2^k y), 2^k r\right) \tag{21}$$

for all $r > 0$ and all $x, y \in X$. Now,

$$\begin{aligned}
&N\left(A(x+2y) + A(x-2y) - 4A(x+y) + 4A(x-y) + 6A(x) - A(2y) - A(-2y) + 4A(y) + 4A(-y), r\right) \\
&\geq min\left\{N\left(A(x+2y) - \frac{1}{2^k} f(x+2y), \frac{r}{10}\right), N\left(A(x-2y) - \frac{1}{2^k} f(x-2y), \frac{r}{10}\right),\right. \\
&\quad N\left(-4A(x+y) + 4\frac{1}{2^k} f(x+y), \frac{r}{10}\right), N\left(4A(x-y) - 4\frac{1}{2^k} f(x-y), \frac{r}{10}\right), \\
&\quad N\left(6A(x) - 6\frac{1}{2^k} f(x), \frac{r}{10}\right), N\left(-A(2y) + \frac{1}{2^k} f(2y), \frac{r}{10}\right), \\
&\quad N\left(-A(-2y) + \frac{1}{2^k} f(-2y), \frac{r}{10}\right), N\left(4A(y) - 4\frac{1}{2^k} f(y), \frac{r}{10}\right), \\
&\quad N\left(4A(-y) - 4\frac{1}{2^k} f(-y), \frac{r}{10}\right), N\left(\frac{1}{2^k} f(x+2y) + \frac{1}{2^k} f(x-2y) - \frac{1}{2^k} 4f(x+y)\right. \\
&\quad \left.\left. + \frac{1}{2^k} 4f(x-y) + \frac{1}{2^k} 6f(x) - \frac{1}{2^k} f(2y) - \frac{1}{2^k} f(-2y) + \frac{1}{2^k} 4f(y) + \frac{1}{2^k} 4f(-y), \frac{r}{10}\right)\right\}
\end{aligned} \tag{22}$$

for all $x, y \in X$ and all $r > 0$. Using (21) and $(F5)$ in (22), we arrive

$$\begin{aligned}
&N\left(A(x+2y) + A(x-2y) - 4A(x+y) + 4A(x-y) + 6A(x) - A(2y) - A(-2y) + 4A(y) + 4A(-y), r\right) \\
&\geq min\left\{1, 1, 1, 1, 1, 1, 1, 1, 1, \left\{N'\left(\alpha(2^k y, 2^k y), \frac{(2-d)2^k r}{8}\right), N'\left(\alpha(2.2^k y, 2^k y), \frac{(2-d)2^k r}{4}\right)\right\}\right\} \\
&\geq min\left\{N'\left(\alpha(2^k y, 2^k y), \frac{(2-d)2^k r}{8}\right), N'\left(\alpha(2.2^k y, 2^k y), \frac{(2-d)r}{42^k}\right)\right\}
\end{aligned} \tag{23}$$

for all $x, y \in X$ and all $r > 0$. Letting $k \to \infty$ in (23) and using (3), we see that

$$N\left(A(x+2y) + A(x-2y) - 4A(x+y) + 4A(x-y) + 6A(x) - A(2y) - A(-2y) + 4A(y) + 4A(-y), r\right) = 1 \tag{24}$$

for all $x, y \in X$ and all $r > 0$. Using $(F2)$ in the above inequality gives

$$A(x+2y) + A(x-2y) = 4A(x+y) - 4A(x-y) - 6A(x) + A(2y) + A(-2y) - 4A(y) - 4A(-y)$$

for all $x, y \in X$. Hence A satisfies the cubic functional equation (1). In order to prove $A(y)$ is unique, let $A'(y)$ be another additive functional equation satisfying (1) and (6). Hence,

$$\begin{aligned}
N(A(y) - A'(y), r) &\geq min\left\{N\left(\frac{A(2^k y)}{2^k} - \frac{f(2^k y)}{2^k}, \frac{r}{2}\right), N\left(\frac{f(2^k y)}{2^k} - \frac{A(2^k y)}{2^k}, \frac{r}{2}\right)\right\} \\
&\geq min\left\{N'\left(\alpha(2^k y, 2^k y), \frac{2^k(2-d)r}{8}\right), N'\left(\alpha(2^k 2y, 2^k y), \frac{2^k(2-d)r}{4}\right)\right\} \\
&\geq min\left\{N'\left(\alpha(y, y), \frac{2^k(2-d)r}{8d^k}\right), N'\left(\alpha(2y, y), \frac{2^k(2-d)r}{4d^k}\right)\right\}
\end{aligned}$$

for all $y \in X$ and all $r > 0$. Since

$$\lim_{k\to\infty} \frac{2^k(2-d)r}{8d^k} = \infty \text{ and } \lim_{k\to\infty} \frac{2^k(2-d)r}{4d^k} = \infty,$$

we obtain

$$\lim_{k\to\infty} N'\left(\alpha(y,y), \frac{2^k(2-d)r}{8d^k}\right) = 1 \text{ and } \lim_{k\to\infty} N'\left(\alpha(2y,y), \frac{2^k(2-d)r}{4d^k}\right) = 1$$

for all $y \in X$ and all $r > 0$. Thus

$$N(A(y) - A'(y), r) = 1$$

for all $y \in X$ and all $r > 0$, hence $A(y) = A'(y)$. Therefore $A(y)$ is unique. For $\beta = -1$, we can prove the result by a similar method. □

From Theorem 2.1, we obtain the following corollaries concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (1).

Corollary 2.2. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N(D\,f(x,y),r) \geq \begin{cases} N'(\epsilon, r), & \\ N'(\epsilon\{||x||^s + ||y||^s\}, r), & s \neq 1; \\ N'(\epsilon\{||x||^s||y||^s\}, r), & s \neq \frac{1}{2}; \\ N'\left(\epsilon\left(||x||^s||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{1}{2}; \end{cases} \tag{25}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N(f(2y) - 8f(y) - A(y), r) \geq \begin{cases} \min\left\{N'\left(\epsilon, \frac{|2|r}{8}\right), N'\left(\epsilon, \frac{|2|r}{4}\right)\right\} \\ \min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{4|2^s-2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{2|2^s-2|}\right)\right\} \\ \min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right)\right\} \\ \min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right)\right\} \end{cases} \tag{26}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.3. *Let $\beta \in \{-1, 1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2^3}\right)^\beta < 1$*

$$N'\left(\alpha\left(2^\beta y, 2^\beta y\right), r\right) \geq N'\left(d^\beta \alpha(y,y), r\right) \tag{27}$$

for all $y \in X$ and all $r > 0$, $d > 0$, and

$$\lim_{k\to\infty} N'\left(\alpha\left(2^{\beta k}x, 2^{\beta k}y\right), 2^{\beta k}r\right) = 1 \tag{28}$$

for all $x, y \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N(Df(x,y), r) \geq N'(\alpha(x,y), r) \tag{29}$$

for all $r > 0$ and all $x, y \in X$. Then the limit

$$C(y) = N - \lim_{k\to\infty} \frac{a(2^{\beta k}y)}{2^{3k\beta}} \tag{30}$$

exists for all $y \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping such that

$$N\left(f(2y) - 2f(y) - C(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2^3-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^3-d)r}{4}\right)\right\} \tag{31}$$

for all $y \in X$ and all $r > 0$.

Proof. It is easy to see from (9) that

$$\begin{aligned} N\left(f(4y) - 2f(2y) - 8f(2y), r\right) &\geq min\left\{N\left(4\left(f(3y) - 4f(2y) + 5f(y)\right), \frac{r}{2}\right), N\left(f(4y) - 4f(3y) + 6f(2y) - 4f(y), \frac{r}{2}\right)\right\} \\ &\geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \end{aligned} \tag{32}$$

for all $y \in X$ and all $r > 0$. Let $h : X \to Y$ be a mapping defined by $h(y) = f(2y) - 2f(y)$. Then we conclude that

$$N\left(h(2y) - 8h(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \tag{33}$$

for all $y \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.1. □

The following corollary is an immediate consequence of Theorem 2.3 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.4. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 3; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq \frac{3}{2}; \\ N'\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{3}{2}; \end{cases} \tag{34}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique Cubic mapping $C : X \to Y$ such that

$$N\left(f(2y) - 2f(y) - C(y), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{|7|}\right), N'\left(\epsilon, \frac{2r}{|7|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{|2^s - 2^3|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{2r}{|2^s - 2^3|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s} - 2^3|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s} - 2^3|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s} - 2^3|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{2r}{|2^{2s} - 2^3|}\right)\right\} \end{cases} \tag{35}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.5. *Let $\beta \in \{-1, 1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some d with the condition given (2) and (27) and $0 < \left(\frac{d}{2}\right)^\beta < 1$, $0 < \left(\frac{d}{2^3}\right)^\beta < 1$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq N'\left(\alpha(x,y), r\right) \tag{36}$$

for all $r > 0$ and all $x, y \in X$. Then there exists a additive mapping $A : X \to Y$ and unique cubic mapping $C : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(y) - A(y) - C(y), r\right) \geq \min\left\{N'\left(\alpha(y,y), \frac{(2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2-d)r}{4}\right), N'\left(\alpha(y,y), \frac{(2^3-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^3-d)r}{4}\right)\right\} \tag{37}$$

for all $y \in X$ and all $r > 0$.

Proof. By Theorems 2.1 and 2.3, there exists a unique additive function $A_1 : X \to Y$ and a unique cubic function $C_1 : X \to Y$ such that

$$N\left(f(2y) - 8f(y) - A_1(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2-d)r}{4}\right)\right\} \tag{38}$$

for all $y \in X$ and all $r > 0$ and

$$N\left(f(2y) - 2f(y) - C_1(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2^3-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^3-d)r}{4}\right)\right\} \tag{39}$$

for all $y \in X$ and all $r > 0$. Now from (38) and (39), one can see that

$$\begin{aligned} N\left(f(y) + \frac{1}{6}A_1(y) - \frac{1}{6}C_1(y), 2r\right) &\geq \min\left\{N\left(\frac{f(2y)}{6} - \frac{8}{6}f(y) - \frac{1}{6}A_1(y), \frac{r}{6}\right), N\left(\frac{f(2y)}{6} - \frac{2}{6}f(y) - \frac{1}{6}C_1(y), \frac{r}{6}\right)\right\} \\ &\geq \min\left\{N\left(f(2y) - 8f(y) - A_1(y), r\right), N\left(f(2y) - 2f(y) - C_1(y), r\right)\right\} \\ &\geq \min\left\{N'\left(\alpha(y,y), \frac{(2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2-d)r}{4}\right), N'\left(\alpha(y,y), \frac{(2^3-d)r}{8}\right), \right. \\ &\qquad \left. N'\left(\alpha(2y,y), \frac{(2^3-d)r}{4}\right)\right\} \end{aligned}$$

for all $y \in X$ and all $r > 0$. Thus we obtain (37) by defining $A(y) = \frac{-1}{6}A_1(y)$ and $C(y) = \frac{1}{6}C_1(y)$ for all $y \in X$ and all $r > 0$. □

The following corollary is an immediate consequence of Theorem 2.5 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.6. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 1, 3; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq \frac{1}{2}, \frac{3}{2}; \\ N'\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{1}{2}, \frac{3}{2}; \end{cases} \tag{40}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique Cubic mapping $C : X \to Y$ such that

$$N\left(f(x) - A(x) - C(x), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|2|r}{8}\right), N'\left(\epsilon, \frac{|2|r}{4}\right), N'\left(\epsilon, \frac{r}{|7|}\right), N'\left(\epsilon, \frac{2r}{|7|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{4|2^s-2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{2|2^s-2|}\right), \right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{|2^s-2^3|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{2r}{|2^s-2^3|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), \right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2^3|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), \right. \\ \qquad \left. N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{2r}{|2^{2s}-2^3|}\right)\right\} \end{cases} \tag{41}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.7. *Let $\beta \in \{-1,1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2^2}\right)^{\beta} < 1$*

$$N'\left(\alpha\left(2^{\beta}y, 2^{\beta}y\right), r\right) \geq N'\left(d^{\beta}\alpha\left(y,y\right), r\right) \tag{42}$$

for all $y \in X$ and all $r > 0$, $d > 0$, and

$$\lim_{k\to\infty} N'\left(\alpha\left(2^{\beta k}x, 2^{\beta k}y\right), 2^{\beta k}r\right) = 1 \tag{43}$$

for all $x, y \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(Df(x,y), r\right) \geq N'\left(\alpha(x,y), r\right) \tag{44}$$

for all $r > 0$ and all $x, y \in X$. Then the limit

$$Q_2(y) = N - \lim_{k\to\infty} \frac{q(2^{\beta k}y)}{2^{2k\beta}} \tag{45}$$

exists for all $y \in X$ and the mapping $Q_2 : X \to Y$ is a unique quadratic mapping such that

$$N\left(f(2y) - 16f(y) - Q_2(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2^2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^2-d)r}{4}\right)\right\} \tag{46}$$

for all $y \in X$ and all $r > 0$.

Proof. It is easy to see from (9) that

$$N\left(f(3y) - 6f(2y) + 15f(y), r\right) \geq N'\left(\alpha\left(y,y\right), r\right) \tag{47}$$

for all $y \in X$ and all $r > 0$. Replacing x by $2y$ in (9), we obtain

$$N\left(f(4y) - 4f(3y) + 4f(2y) + 4f(y), r\right) \geq N'\left(\alpha\left(2y,y\right), r\right) \tag{48}$$

for all $y \in X$ and all $r > 0$. It follows from (47) and (48) that

$$\begin{aligned} N\left(f(4y) - 20f(2y) + 64f(2y), r\right) &\geq min\left\{N\left(4\left(f(3y) - 24f(2y) + 60f(y)\right), \frac{r}{2}\right), N\left(f(4y) - 4f(3y) + 4f(2y) + 4f(y), \frac{r}{2}\right)\right\} \\ &\geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \end{aligned} \tag{49}$$

for all $y \in X$ and all $r > 0$. Let $q_2 : X \to Y$ be a mapping defined by $q_2(y) = f(2y) - 16f(y)$. Then we conclude that

$$N\left(q_2(2y) - 4q_2(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \tag{50}$$

for all $y \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.1.

□

The following corollary is an immediate consequence of Theorem 2.7 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.8. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \quad \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 2; \\ N'\left(\epsilon\left\{||x||^s||y||^s\right\}, r\right), & s \neq 1; \\ N'\left(\epsilon\left(||x||^s||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq 1; \end{cases} \tag{51}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a such that

$$N\left(f(2y) - 16f(y) - Q_2(y), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{2|-3|}\right), N'\left(\epsilon, \frac{r}{|-3|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{2|2^s-2^2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{|2^s-2^2|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s}+\frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right)\right\} \end{cases} \tag{52}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.9. *Let $\beta \in \{-1, 1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2^4}\right)^\beta < 1$*

$$N'\left(\alpha\left(2^\beta y, 2^\beta y\right), r\right) \geq N'\left(d^\beta \alpha\left(y, y\right), r\right) \tag{53}$$

for all $y \in X$ and all $r > 0$, $d > 0$, and

$$\lim_{k\to\infty} N'\left(\alpha\left(2^{\beta k}x, 2^{\beta k}y\right), 2^{\beta k}r\right) = 1 \tag{54}$$

for all $x, y \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(Df(x,y), r\right) \geq N'\left(\alpha(x,y), r\right) \tag{55}$$

for all $r > 0$ and all $x, y \in X$. Then the limit

$$Q_4(y) = N - \lim_{k\to\infty} \frac{q_4(2^{\beta k}y)}{2^{4k\beta}} \tag{56}$$

exists for all $y \in X$ and the mapping $Q_4 : X \to Y$ is a unique quartic mapping such that

$$N\left(f(2y) - 4f(y) - Q_4(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2^4-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^4-d)r}{4}\right)\right\} \tag{57}$$

for all $y \in X$ and all $r > 0$.

Proof. It is easy to see from (49)

$$N\left(f(4y) - 4f(2y) - 16f(2y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \tag{58}$$

for all $y \in X$ and all $r > 0$. Let $q_4 : X \to Y$ be a mapping defined by $q_4(y) = f(2y) - 4f(y)$. Then we conclude that

$$N\left(q_4(2y) - 16q_4(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \tag{59}$$

for all $y \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.1. □

The following corollary is an immediate consequence of Theorem 2.9 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.10. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y),r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 4; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq 2; \\ N'\left(\epsilon\left(||x||^s||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq 2; \end{cases} \tag{60}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quartic mapping $Q_4 : X \to Y$ such that

$$N\left(f(2y) - 4f(y) - Q_4(y), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{2r}{|15|}\right), N'\left(\epsilon, \frac{4r}{|15|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{2r}{|2^s - 2^4|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{4r}{|2^s - 2^4|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s} - 2^4|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{4r}{|2^{2s} - 2^4|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s} - 2^4|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{4r}{|2^{2s} - 2^4|}\right)\right\} \end{cases} \tag{61}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.11. *Let $\beta \in \{-1, 1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some d with the condition given (42) and (53) and $0 < \left(\frac{d}{2^2}\right)^\beta < 1$, $0 < \left(\frac{d}{2^4}\right)^\beta < 1$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq N'\left(\alpha(x,y), r\right) \tag{62}$$

for all $r > 0$ and all $x, y \in X$. Then there exists a quadratic mapping $Q_2 : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} N\left(f(y) - Q_2(y) - Q_4(y), r\right) \geq \min&\left\{N'\left(\alpha(y,y), \frac{(2^2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^2-d)r}{4}\right),\right. \\ &\left. N'\left(\alpha(y,y), \frac{(2^4-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^4-d)r}{4}\right)\right\} \end{aligned} \tag{63}$$

for all $y \in X$ and all $r > 0$.

Proof. By Theorems (??) and (??), there exists a unique quadratic function $Q_{2_1} : X \to Y$ and a unique quartic function $Q_{4_1} : X \to Y$ such that

$$N\left(f(2y) - 16f(y) - Q_{2_1}(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2^2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^2-d)r}{4}\right)\right\} \tag{64}$$

for all $y \in X$ and all $r > 0$ and

$$N\left(f(2y) - 4f(y) - Q_{4_1}(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{(2^4-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^4-d)r}{4}\right)\right\} \tag{65}$$

for all $y \in X$ and all $r > 0$. Now from (64) and (65), one can see that

$$\begin{aligned} &N\left(f(y) + \frac{1}{12}Q_{2_1}(y) - \frac{1}{12}Q_{4_1}(y), 2r\right) \\ &\geq \min\left\{N\left(\frac{f(2y)}{12} - \frac{16}{12}f(y) - \frac{1}{12}Q_{2_1}(y), \frac{r}{12}\right), N\left(\frac{f(2y)}{12} - \frac{4}{12}f(y) - \frac{1}{12}Q_{4_1}(y), \frac{r}{12}\right)\right\} \\ &\geq \min\left\{N\left(f(2y) - 16f(y) - Q_{2_1}(y), r\right), N\left(f(2y) - 4f(y) - Q_{4_1}(y), r\right)\right\} \\ &\geq \min\left\{N'\left(\alpha(y,y), \frac{(2^2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^2-d)r}{4}\right), N'\left(\alpha(y,y), \frac{(2^4-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^4-d)r}{4}\right)\right\} \end{aligned}$$

for all $y \in X$ and all $r > 0$. Thus we obtain (37) by defining $Q_2(y) = \frac{-1}{12} Q_{2_1}(y)$ and $Q_4(y) = \frac{1}{12} Q_{4_1}(y)$ for all $y \in X$ and all $r > 0$. □

The following corollary is an immediate consequence of Theorem 2.11 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.12. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 2,4; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq 1,2; \\ N'\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq 1,2; \end{cases} \tag{66}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$N\left(f(y) - Q_2(y) - Q_4(y), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{2|3|}\right), N'\left(\epsilon, \frac{r}{|3|}\right), N'\left(\epsilon, \frac{2r}{|15|}\right), N'\left(\epsilon, \frac{4r}{|15|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{2|2^s-2^2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{|2^s-2^2|}\right),\right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{2r}{|2^s-2^4|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{4r}{|2^s-2^4|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right),\right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right),\right. \\ \qquad \left. N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right)\right\} \end{cases} \tag{67}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.13. *Let $\beta \in \{-1, 1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some d with the condition given (2), (27), (42), (53) and $0 < \left(\frac{d}{2}\right)^\beta < 1$, $0 < \left(\frac{d}{2^2}\right)^\beta < 1$, $0 < \left(\frac{d}{2^3}\right)^\beta < 1$ and $0 < \left(\frac{d}{2^4}\right)^\beta < 1$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq N'\left(\alpha(x,y), r\right) \tag{68}$$

for all $r > 0$ and all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$, a unique cubic cubic mapping $C : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} &N\left(f(y) - A(y) - Q_2(y) - C(y) - Q_4(y), r\right) \\ &\geq \min\left\{N'\left(\alpha(y,y), \frac{(2-d)r}{16}\right), N'\left(\alpha(-y,-y), \frac{(2-d)r}{16}\right), N'\left(\alpha(2y,y), \frac{(2-d)r}{8}\right),\right. \\ &\quad N'\left(\alpha(-2y,-y), \frac{(2-d)r}{8}\right), N'\left(\alpha(y,y), \frac{(2^3-d)r}{16}\right), N'\left(\alpha(-y,-y), \frac{(2^3-d)r}{16}\right), \\ &\quad N'\left(\alpha(2y,y), \frac{(2^3-d)r}{8}\right), N'\left(\alpha(-2y,-y), \frac{(2^3-d)r}{8}\right), N'\left(\alpha(y,y), \frac{(2^2-d)r}{16}\right), \\ &\quad N'\left(\alpha(-y,-y), \frac{(2^2-d)r}{16}\right) N'\left(\alpha(2y,y), \frac{(2^2-d)r}{8}\right), N'\left(\alpha(-2y,-y), \frac{(2^2-d)r}{8}\right), \\ &\quad \left. N'\left(\alpha(y,y), \frac{(2^4-d)r}{16}\right), N'\left(\alpha(-y,-y), \frac{(2^4-d)r}{16}\right), N'\left(\alpha(2y,y), \frac{(2^4-d)r}{8}\right), N'\left(\alpha(-2y,-y), \frac{(2^4-d)r}{8}\right)\right\} \end{aligned}$$

for all $y \in X$ and all $r > 0$.

Proof. Let $f_{ac}(y) = \frac{f_o(y)-f_o(-y)}{2}$ for all $y \in X$. Then $f_{ac}(0) = 0$ and $f_o(-y) = -f_o(y)$ for all $y \in X$. Hence

$$N\left(Df_{ac}(x,y), r\right) \geq min\left\{N'\left(\alpha(x,y), \frac{r}{2}\right), N'\left(\alpha(-x,-y), \frac{r}{2}\right)\right\} \tag{69}$$

for all $y \in X$ and all $r > 0$. By Theorem (**??**), there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned} N\left(f_{ac}(y) - A(y) - C(y), r\right) \geq \min\Bigg\{&N'\left(\alpha(y,y), \frac{(2-d)r}{8}\right), N'\left(\alpha(-y,-y), \frac{(2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2-d)r}{4}\right), \\ &N'\left(\alpha(-2y,-y), \frac{(2-d)r}{4}\right), N'\left(\alpha(y,y), \frac{(2^3-d)r}{8}\right), N'\left(\alpha(-y,-y), \frac{(2^3-d)r}{8}\right), \\ &N'\left(\alpha(2y,y), \frac{(2^3-d)r}{4}\right), N'\left(\alpha(-2y,-y), \frac{(2^3-d)r}{4}\right)\Bigg\} \end{aligned} \tag{70}$$

for all $y \in X$ and all $r > 0$. Also, let $f_{qq}(y) = \frac{f_e(y)+f_e(-y)}{2}$ for all $y \in X$. Then $f_{qq}(0) = 0$ and $f_o(-y) = f_o(y)$ for all $y \in X$. Hence

$$N\left(Df_{qq}(x,y), r\right) \geq min\left\{N'\left(\alpha(x,y), \frac{r}{2}\right), N'\left(\alpha(-x,-y), \frac{r}{2}\right)\right\} \tag{71}$$

for all $y \in X$ and all $r > 0$. By Theorem (**??**), there exists a unique quadratic mapping $Q_2 : X \to Y$, and a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned} N\left(f_{qq}(y) - Q_2(y) - Q_4(y), r\right) \geq \min\Bigg\{&N'\left(\alpha(y,y), \frac{(2^2-d)r}{8}\right), N'\left(\alpha(-y,-y), \frac{(2^2-d)r}{8}\right), N'\left(\alpha(2y,y), \frac{(2^2-d)r}{4}\right), \\ &N'\left(\alpha(-2y,-y), \frac{(2^2-d)r}{4}\right), N'\left(\alpha(y,y), \frac{(2^4-d)r}{8}\right), N'\left(\alpha(-y,-y), \frac{(2^4-d)r}{8}\right), \\ &N'\left(\alpha(2y,y), \frac{(2^4-d)r}{4}\right), N'\left(\alpha(-2y,-y), \frac{(2^4-d)r}{4}\right)\Bigg\} \end{aligned} \tag{72}$$

for all $y \in X$ and all $r > 0$. Define a function $f(y)$ by

$$f(y) = f_{ac}(y) + f_{qq}(y) \tag{73}$$

for all $y \in X$. Combining (73), (70) and (72) we arrive our result. □

Corollary 2.14. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \quad \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 1,3,2,4; \\ N'\left(\epsilon\left\{||x||^s||y||^s\right\}, r\right), & s \neq \frac{1}{2}, \frac{3}{2}, 2, 4; \\ N'\left(\epsilon\left(||x||^s||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{1}{2}, \frac{3}{2}, 2, 4; \end{cases} \tag{74}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique Cubic mapping $C : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping

$Q_4 : X \to Y$ *such that*

$$N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right)$$
$$\geq \begin{cases} (i) N'\left(\epsilon, \frac{|2|r}{8}\right), N'\left(\epsilon, \frac{|2|r}{4}\right), N'\left(\epsilon, \frac{r}{|7|}\right), N'\left(\epsilon, \frac{2r}{|7|}\right), N'\left(\epsilon, \frac{r}{2|3|}\right), N'\left(\epsilon, \frac{r}{|3|}\right), \\ \quad N'\left(\epsilon, \frac{2r}{|15|}\right), N'\left(\epsilon, \frac{4r}{|15|}\right) \\ (ii) N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{4|2^s-2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{2|2^s-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{|2^s-2^3|}\right), \\ \quad N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{2r}{|2^s-2^3|}\right), N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{2|2^s-2^2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{|2^s-2^2|}\right), \\ \quad N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{2r}{|2^s-2^4|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{4r}{|2^s-2^4|}\right) \\ (iii) N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), \\ \quad N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2^3|}\right), N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right), \\ \quad N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right) \\ (iv) N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), \\ \quad N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{2r}{|2^{2s}-2^3|}\right), \\ \quad N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right), \\ \quad N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right) \end{cases} \quad (75)$$

for all $y \in X$ *and all* $r > 0$.

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