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On the Star Coloring of Circulant Graphs

Research Article*

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Abstract: Let $G = (V, E)$ be an undirected simple graph. The star chromatic number of a graph G is the least number of colors needed to color the path on four vertices with three distinct colors. The aim of this paper is to determine the star chromatic number of some Circulant graphs.

Keywords: Proper coloring, Chromatic number, Star coloring, Star chromatic number, Circulant graphs, Harary graphs, Andra sfai graph, Cocktail Party graph, Musical graph, Crown graph.

1. Introduction

In this paper, we have taken the graphs to be finite, undirected and loopless. The path P of a graph is a walk in which no vertices are repeated. In 1973 the concept of star coloring was introduced by Grünbaum [4] and also he introduced the notion of star chromatic number. His works were developed further by Bondy and Hell [2]. According to them the star coloring is the proper coloring on the paths with four vertices by giving 3- distinct colors on it. In graph theory, a Circulant graph [5] is an undirected graph that has a cyclic group of symmetries that takes any vertex to any other vertex. In this paper we have established the star chromatic number of some Circulant graphs.

2. Definitions

Definition 2.1 (Vertex Coloring). *Let G be a graph and let V(G) be the set of all vertices of G and let{1,2,3..k}denotes the set of all colors which are assigned to each vertex of G. A proper vertex coloring of a graph is a mapping c: V(G) →{1, 2, 3 . . . .k} such that c(u) ≠ c(v) for all arbitrary adjacent vertices u, v ∈ V(G).*

Definition 2.2 (Chromatic Number). *If G has a proper vertex coloring then the chromatic number of G is the minimum number of colors needed to color G . The chromatic number of G is denoted by $\chi(G)$.*

Definition 2.3 (Star coloring). *A proper vertex coloring of a graph G is called star coloring [10], if every path of G on four vertices is not 2- colored.*

Definition 2.4 (Star Chromatic Number). *The star chromatic number is the minimum number of colors needed to star color G [1]and is denoted by $\chi_s(G)$.*

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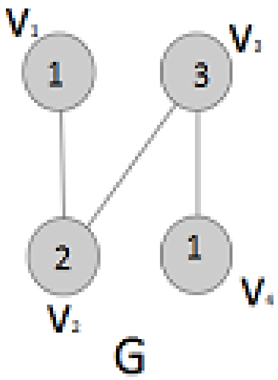


Let us consider the following example:

Let G be a path graph and V(G) $=\{v_1,v_2,v_3,v_4\}$. Here $c(v_1) = c(v_4) = 1$, $c(v_2) = 2$, $c(v_3) = 3$. Then the path $P_4$of G is

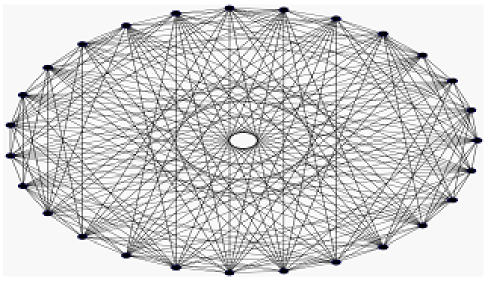
$v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4$

$c(P_4) = c(v_1) \rightarrow c(v_2) \rightarrow c(v_3) \rightarrow c(v_4)=$ 1-2-3-1.

$\chi_s(P_4) = 3 \Longrightarrow \chi_s(G) = 3$.

Definition 2.5 (Circulant graphs)**.** *A circulant graph [5] is a graph of n graph vertices in which the* i^{th} *vertex is adjacent to the* $(i+j)^{th}$ *and* $(i-j)^{th}$ *vertices for each j by the cyclic group of symmetry.*

Example 2.6 (Schläfli graph)**.**



Definition 2.7 (Harary Graphs)**.** *The structure of harary graphs [3] denoted by* $H_{(k,m)}$ *depends on the parities of k and m where k – vertex connectivity and m-number of vertices.*

<u>Case 1:</u> when k and m are even:

Let k = 2n, then $H_{(2n,m)}$ *is constructed as follows. It has vertices 0,1,2,. . . ,m-1 and two vertices i and j are joined if i-n = j ≤ i+n (when addition is taken modulo m) .* $H_{(4,8)}$ *is shown in fig(a).*

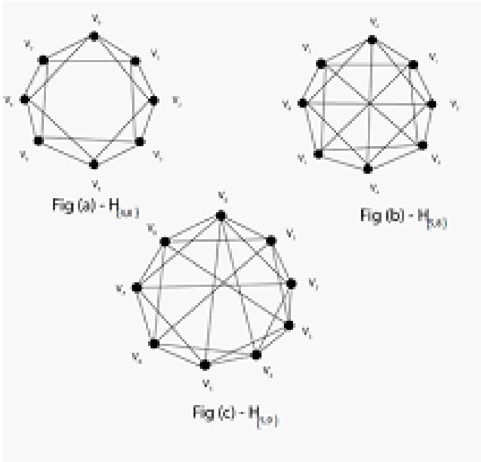
<u>Case 2:</u> when k is odd and m is even:

Let k = 2n+1, then $H_{(2n+1,m)}$ *is constructed by first drawing* $H_{(2n,m)}$ *and then adding edges joining vertex i to vertex i+(m/2) for 1= i =* m/2. $H_{(5,8)}$ *is shown in fig(b).*

<u>Case 3:</u> when k and m are odd:

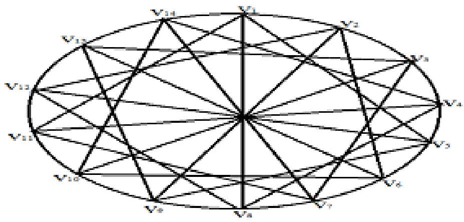
Let k = 2n+1, then $H_{(2n+1,m)}$ *is constructed by first drawing* $H_{(2n,m)}$ *and then adding edges joining vertex 0 to the vertices (m-1)/2 and (m+1)/2 and vertex i to vertex i+(m+1)/2 for 1= i= (m-1)/2 .* $H_{(5,9)}$ *is shown in fig(c).*

Example 2.8.



Definition 2.9 (Andrásfai Graphs)**.** *The n-Andrásfai graph [6] is a circulant graph on 3n-1 vertices and whose indices are given by the integers 1,..., 3n-1 that are congruent to 1 (mod 3). The Andrásfai graphs have graph diameter 2 for* $n = 1$*and is denoted by* A_n.

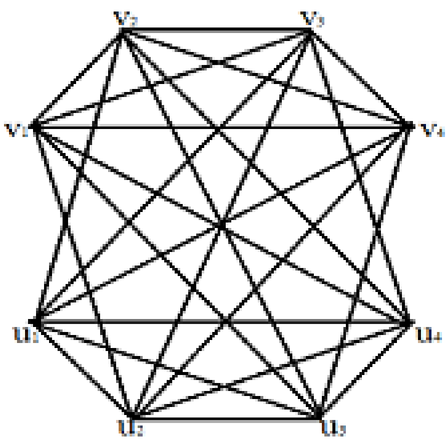
Example 2.10.



A_5

Definition 2.11 (Cocktail party Graph)**.** *The cocktail party graph [7] is the graph consisting of two rows of paired vertices in which all vertices except the paired ones are connected with an edge and is denoted by* Cp_n.

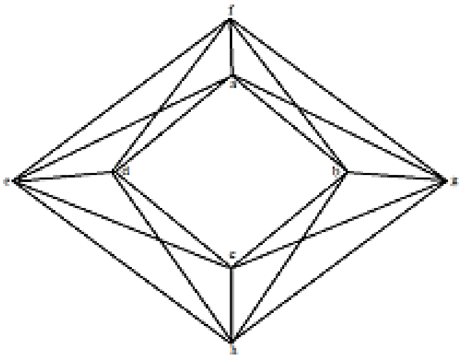
Example 2.12.



Cp_4

Definition 2.13 (musical Graph)**.** *The musical graph [8] of order n consists of two parallel copies of cycle graphs* C_n *in which all the paired vertices and its neighborhood vertices are connected with an edge and is denoted by* $M_{2n}\forall$ $n = 3$.

Example 2.14.



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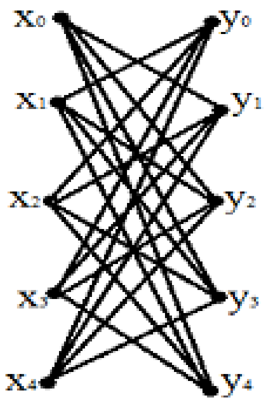
Definition 2.15 (crown Graph)**.** *The crown graph* S_n^0 *[9] for an integer* $n \geq 3$ *is the graph with vertex set* $\{x_0, x_1, x_2, \ldots, x_{n-1}, y_0, y_1, y_2, \ldots, y_{n-1}\}$ *and edge set* $\{(x_i, y_j) : 0 = i, j = n-1, i \neq j\}$. S_n^0 *is therefore equivalent to the complete bipartite graph* $K_{n,n}$ *with horizontal edges removed*

Example 2.16.



S_5^0

3. Preliminaries

Theorem 3.1 (Star Coloring of Cycle [4])**.** *Let* C_n *be the cycle with* $n \geq 3$ *vertices. Then* $\chi_s(C_n) = \begin{cases} 4, & \text{when } n = 5 \\ 3, & \text{otherwise} \end{cases}$

Theorem 3.2 (Star Coloring of complete graph [4])**.** *If* K_n *is the complete graph with n vertices. Then* $\chi_s(k_n) = n$, $\forall\, n \geq 3$.

Theorem 3.3 (Star Coloring of path graph [4])**.** *Let* P_n *be the path graph with n vertices. Then* $\chi_s(P_n) = 3$, $\forall\, n \geq 4$.

4. Star Coloring of Some Graphs Formed From the Cartesian Product of Simple Graphs

In this section we have obtained the star chromatic number of various graphs formed from the Cartesian product of simple graphs

4.1. Star Chromatic Number of Harary Graphs

Theorem 4.1. *The star chromatic number of Harary graphs when both k and m are even. Let $H(_{(k,m)})$ be a Harary graph with k=2n and m=4n, then star chromatic number* $\chi_s(H_{(k,m)})= \begin{cases} 3, & when\ n=1 \\ (k+2), & otherwise \end{cases}$ *, $\forall\ n=1$.*

Proof. Let $H_{(k,m)}$ be a harary graph [3] with m vertices, where k = 2n and m = 4n. Therefore the vertex set of $H_{(k,m)}$ is given by $V(H_{(k,m)})=\{v_0, v_1, v_2, \ldots, v_{m-1}\}$ we know that the graph $H_{(k,m)}$ contains m vertices and the degree of each vertex is k. By the definition of harary graph we know that any two vertices v_i and v_j are joined if i-n= j = i+n (when addition is taken modulo m). Let us now star color the graph $H_{(k,m)}$

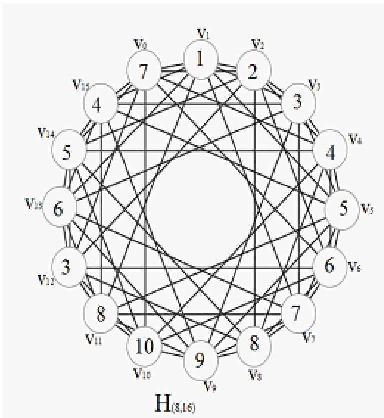
<u>Case 1:</u> for n = 1, we have k= 2 and m=4 then $H_{(2,4)}$ has four vertices and each vertex is of degree 2. We know that a graph with four vertices and each vertex is of degree 2 is cycle C_4. Therefore the graph $H_{(2,4)}$is the graph $C_4$and we know that $\chi_s(C_4) = 3$ [4] therefore $\chi_s(H_{(2,4)}) = 3$. Hence $\chi_s(H_{(k,m)})=3$ for n=1.

<u>Case 2:</u> for $n = 2$ and k, m are even and $H_{(k,m)}$ has 4n vertices. Let us now see the procedure to star color the graph $H_{(k,m)}$ as follows.

- Assign colors from 1, 2, 3,... ,k+2 to vertices v_1, v_2,... to v_{k+2} vertices respectively.
- Assign even colors from the color set {1, 2, ..., k+2} to the vertices with odd indices and assign odd colors from the color set {1, 2, ..., k+2} to the vertices with even indices.

Here one can observe that there is no possibility for any path on four vertices to be bicolored. Thus there is a valid star coloring and Therefore χ_s $(H_{(k,m)})$ = k+2 $\forall\ n = 2$. When k=2n and m=4n. Therefore $\chi_s(H_{(k,m)})= \begin{cases} 3, & when\ n=1 \\ (k+2), & otherwise \end{cases}$ □

Example 4.2. *when n=4, we have k=8 and m=16*



here $c(v_0)=c(v_7)=7$, $c(v_1)=1$ $c(v_2)=2$ $c(v_3)=3, \ldots,$ $c(v_{10})=10, \ldots$ $c(v_{10})=10, \ldots,$ $c(v_{15})=4$. Clearly this graph accepts valid star coloring. Therefore $\chi_s(H_{(8,16)})=10$. (8+2).

Theorem 4.3. *The star chromatic number of Harary graphs when k is odd and m is even. Let $H_{(k,m)}$ be the harary graph with k=2n+1 and m=2n+2, then the star chromatic number $\chi_s(H_{(k,m)})=$ m where k=2n+1 and m=2n+2, $\forall\ n = 1$.*

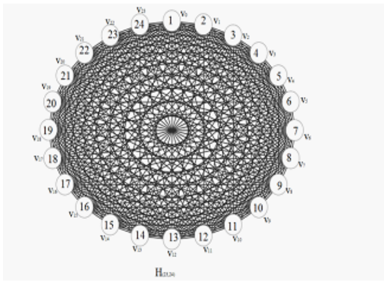
Proof. Let $H_{(k,m)}$ be the harary graph with the parities k and m and take that k is odd and m is even. Let us take k=2n+1 and =2n+2 $\forall\ n = 1$. By the definition of harary graphs [3] for the case k- odd and m- even is constructed as follows :

- First $H_{(2n,2n+2)}$ is constructed.

- Then edges are joined from vertex i to vertex i+$\left(\frac{2n+2}{2}\right)$ $for\ 1 = i = \left(\frac{2n+2}{2}\right)$.

Let us now assign colors to star color the graph. We can observe that the harary graph $H_{(2n+1,2n+2)}$to be a complete graph with 2n+2 vertices. Since every pair of vertices is adjacent to each other. In a complete graph, since all the vertices are adjacent, each vertex receives different colors. Thus for any vertex v_i it's neighborhood vertices are assigned with distinct colors. Therefore for any path on four vertices is not bicolored thus the star chromatic number is equal to the chromatic number. And we know that the star chromatic number of the complete graph is equal to the number of vertices in it [4]. (i.e.) $\chi_s(K_n)$=n. Since the harary graph $H_{(2n+1,2n+2)}$ is a complete graph, the above discussion also holds for the harary graph $H_{(2n+1,2n+2)}$. Thus the star chromatic number of the harary graph $H_{(2n+1,2n+2)}$ is equal to the number of vertices. Here m denotes the number of vertices. Therefore $\chi_s(H_{(k,m)})$= m (i.e.) $\chi_s(H_{(2n+1,2n+2)})$= 2n+2 $\forall\ n = 1$. □

Example 4.4. *When $n = 11$ then $k = 2 * 11 + 1 = 23$ and $m = 2 * 11 + 2 = 24$.*



Here $c(v_0)$=1, $c(v_1)$=2, $c(v_2)$=3,. . . , $c(v_{23})$=24. Clearly the graph accepts valid star coloring. Therefore $\chi_s(H_{(23,24)})$=24.

Theorem 4.5 (Star chromatic number of Andrásfai graphs)**.** *Let A_n be the Andrásfai graph with k vertices where k=3n-1. Then the star chromatic number of A_n is given by* $\chi_s(A_n) = \begin{cases} 4, & when\ n = 2 \\ k-2, & otherwise \end{cases}$

Proof. We know that Andrásfai graph is a circulant graph on 3n-1 vertices, the degree of each vertex is n and any two vertices v_i and v_j are joined by taking 1mod (**??**) [7]. Let us now star color the graph A_n.

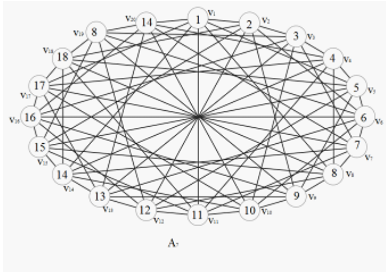
Case1: for n = 2, we have k=3(**??**)-1 = 5 (i.e.) k=5. Clearly it is the cycle with five vertices and we know that $\chi_s(C_5)$= 4. Thus $\chi_s(A_2)$= $\chi_s(C_5)$= 4. Therefore $\chi_s(A_n)$ =4 for n=2.

Case 2: for n >2. Since $A_n$is a circulant graph [5] all the vertices are adjacent to each other in a symmetrical manner. So it is possible to color the vertices with k-2 colors. Suppose if we color the vertices with less than k-2 colors, we can observe that either certain paths on four vertices are bicolored or certain vertices which are adjacent to each other are assigned with same color. Therefore assigning colors less than k-2 colors is neither a proper coloring nor a proper star coloring. So assign colors 1,2,3. . . , k-2 to the vertices consecutively to all the k-2 vertices. There are 2 vertices that are left uncolored. Assign any two colors from the color set 1,2,3,. . . ,k-2 randomly by checking the possible paths that are not to be bicolored from those two vertices. This is a valid star coloring. Hence $\chi_s(A_n)$ =k-2. From case 1 and case 2, $\chi_s(A_n) = \begin{cases} 4, & when\ n = 2 \\ k-2, & otherwise \end{cases}$ □

Example 4.6. *for n=7 we have k=3(**??**)-1 =20.*

Here $c(v_1)$=1, $c(v_2)$=2, $c(v_3)$=3,. . . , $c(v_{18})$=18, $c(v_{19})$=8, $c(v_{20})$=14. Clearly the graph accepts valid star coloring. Therefore $\chi_s(A_n)$=18 (20-2).

Theorem 4.7. *Star chromatic number of cocktail party graphs] Let Cp_n be the cocktail party graph with m vertices where m=2n. Then the star chromatic number of Cp_n is given by $\chi_s(Cp_n)$= 2n-1 $\forall\ n \geq 2$.*

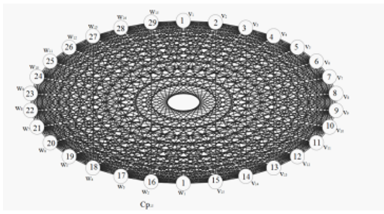


Proof. The cocktail party graph is formed from two rows of paired vertices in which all the vertices are connected except the paired ones [8]. Here v(Cp$_n$)= 2n. Let A= {v$_1$, v$_2$, v$_3$,. . . , v$_n$} be one set of vertices an let B= {w$_1$, w$_2$, w$_3$,. . . , w$_n$}be another set of vertices. Here for any i, v$_i$ is connected to all the vertices of the vertex set A and to all the vertices of the vertex set B except wi. In this graph v$_1$is non adjacent to w$_1$, v$_2$is non adjacent to w$_2$, etc. Let us now see the procedure to star color the graph Cp$_n$:

- Assign colors from 1 to n successively to all the vertices of the vertex set A. (i.e.) assign colors 1,2,..n to the vertices v$_1$, v$_2$, v$_3$,. . . , v$_n$.
- Assign color 1 to vertex w$_1$.
- Then assign colors from n+1 to 2n-1 to the remaining vertices of the vertex set of B (i.e.) n+1, n+2, . . . ,2n-1 to the vertices w$_2$, w$_3$,. . . , w$_n$.

Clearly no path on four vertices is bicolored, which is a valid star coloring. Hence χ_s(Cp$_n$)= 2n-1 $\forall\ n = 2$. □

Example 4.8. *when n= 15 we have m=30.*



*Here c(v$_1$)=1, c(v$_2$)=2, c(v$_3$)=3,. . . ., c(v$_{15}$)=15, c(w$_1$)=1, c(w$_2$)=16, c(w$_3$)=17,. . . ., c(w$_{15}$)=29. Clearly the graph accepts valid star coloring. Therefore χ_s(Cp$_{15}$)= 29 (2*15 -1).*

Theorem 4.9 (Star chromatic number of musical graph)**.** *Let M_n be the musical graph with k vertices where k=2n. Then the star chromatic number of M_n is given by $\chi_s(M_n)$ = 6 $\forall\ n = 3$.*

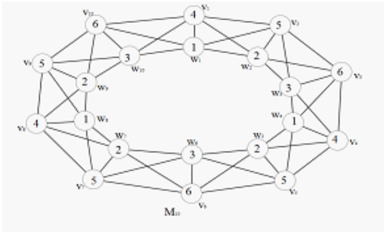
Proof. From the definition of musical graph [10], it consists of two parallel cycles C$_n$. Let V= {v$_1$, v$_2$, v$_3$,. . . , v$_n$} and be the vertex set of the exterior cycle and let W= {w$_1$, w$_2$, w$_3$,. . . , w$_n$} be the vertex set of the interior cycle. For any i, v$_i$is adjacent to w$_i$ and N(w$_i$). Let us star color the graph M$_n$by the following procedure: Color the vertices w$_1$, w$_2$, w$_3$,. . . , w$_n$ with color 1, 2, 3 accordingly to satisfy the star coloring condition except for the cycle C$_5$ (since χ_s(C$_5$) = 4).

1. Color the vertices v$_1$, v$_2$, v$_3$,. . . , v$_n$with colors 4, 5, 6 consecutively by satisfying the star coloring condition

Hence we observe that no path on four vertices is bicolored. This is a proper star coloring. Hence χ_s(M$_n$) = 6 $\forall\ n = 3$. □

Example 4.10.

Here c(v$_1$)=4, c(v$_2$)=5, c(v$_3$)=6,. . . ., c(v$_{10}$)=6, c(w$_1$)=1, c(w$_2$)=2, c(w$_3$)=3,. . . ., c(w$_{10}$)=3. Clearly the graph accepts valid star coloring. Therefore χ_s(M$_{10}$)= 6.



Theorem 4.11 (Star chromatic number of crown graph)**.** *Let S_n^0 be the crown graph with $n = 3$ vertices. Then the star chromatic number of S_n^0 is given by $\chi_s(S_n^0) = n \; \forall \; n = 3$.*

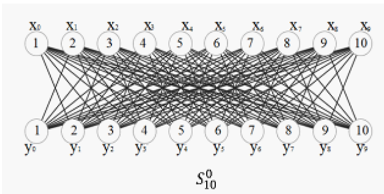
Proof. The crown graph S_n^0 [9] has 2n vertices. Let V(S_n^0) be the vertex set of S_n^0. Let V(S_n^0) be bipartitioned into two vertex subsets X and Y such that X=$\{x_0, x_1, \ldots, x_{n-1}\}$ and Y= $\{y_0, y_1, \ldots, y_{n-1}\}$. Here S_n^0 is equivalent to a complete bipartite graph in which the paired edges ($x_i y_i$) are removed. The procedure is to star color the graph S_n^0 is given as follows:

1. Assign colors 1, 2, 3,. . . , n to the vertices $x_0, x_1, \ldots, x_{n-1}$ [4].

2. Assign colors 1, 2, 3,. . . , n to the vertices $y_0, y_1, \ldots, y_{n-1}$ [4].

Since x_i and y_i are non adjacent there is no possibility for any path on four vertices to be bicolored. This is valid star coloring. Hence $\chi_s(S_n^0) =$ n $\forall \; n = 3$. □

Example 4.12.



Here $c(x_0)$=1, $c(x_1)$=2, $c(x_2)$=3,. . . . , $c(x_9)$=10, $c(y_0)$=1, $c(y_1)$=2, $c(y_2)$=3,. . . . , $c(y_9)$=10. Clearly the graph accepts valid star coloring. Therefore $\chi_s(S_{10}^0)$= 10

5. Conclusion

In this paper we determined the star chromatic number of some circulant graphs. This work can be further extended for various circulant graphs.

References

[1] Eckhard Steffen and Xuding Zhu, *Star chromatic number of graphs*, Combinatorica, 16(3)(1996), 439-448.

[2] N.Biggs, E.Lloyd and R.Wilson, *Graph Theory*, Oxford University Press, (1986), 17361936.

[3] J.A.Bondy and P.Hell, *A note on star chromatic number*, J. Graph Theory, 14(1990), 1340-1344.

[4] Guillaume Fertin and Andre Raspaud, *Bruce Reed Star coloring of graphs*, Graph Theoretic ideas in Computer Science, 27 th International Workshop, WG 2001, Springer Lecture Notes in Computer Science, (2204), 140-153.

[5] V.Vilfred, R.Balakrishnan, G.Sethuraman, Wilson and J.Robin, *On circulant graphs*, in Graph Theory and its Applications (Anna University, Chennai, March 1416, 2001), Alpha Science, (2004), 3436

[6] C.Godsil and G.Royle, *The Andrsfai Graphs, Colouring Andrsfai Graphs, and A Characterization*, Algebraic Graph Theory, New York, Springer-Verlag, (2001).

[7] N.L.Biggs, *Algebraic Graph Theory*, 2nd ed. Cambridge, England: Cambridge University Press, (1993).

[8] A.E.Brouwer, A.M.Cohen and A.Neumaier, *Distance-Regular Graphs*, New York: Springer-Verlag, (1989).