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Solving Delay Differential Equations of Fractional Order

Research Article*

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Abstract: In this paper, we implement Adomian decomposition method and Modified Laguerre wavelets method for solving numerically linear and non-linear delay differential equations of fractional order. The fractional derivative will be in the caputo sense. Computational work is fully supportive of compatibility of proposed algorithm and hence the same may be extended to other physical problems also. Some numerical examples are presented to illustrate the accuracy and ability of the proposed method.

Keywords: Adomian decomposition method, delay differential equations, Method of steps, Modified Laguerre wavelets method, fractional delay differential equations.

1. Introduction

Fractional delay differential equations (FDDEs) are widely used in ecology, physiology, physical sciences and many other areas of applied science. Fractional delay differential equations usually do not have analytic solutions and can only be solved by some numerical methods. In this paper, we shall use the Adomian Decomposition Method (ADM) and Modified Laguerre Wavelets Method (MLWM) to find the approximate solution of the Fractional Delay Differential Equation (FDDEs) with variable delays. The proposed method besides being simple, is so exact and sensible in the solved problems.

Definition 1.1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as*

$$I_y^{\alpha} f(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} f(t)\, dt \quad , \ \alpha > 0 \ , \ t > 0$$

$$I_y^0 f(y) = f(y)$$

Definition 1.2. *The Riemann-Liouville fractional derivative operator of order $\alpha > 0$ is defined as*

$$D_y^{\alpha} f(y) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dy^n} \int_0^y (y-t)^{n-\alpha-1} \ f(t)\, dt$$

where n is an integer and $n-1 < \alpha \leq n$.

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Definition 1.3. *The Caputo fractional derivative operator of order $\alpha > 0$ is defined as*

$$cD_y^\alpha\ f(y) = \frac{1}{\Gamma(n-\alpha)} \int_0^y (y-t)^{n-\alpha-1} \frac{d^n}{dy^n}\ f(y)\, dy$$

where n is an integer and $n-1 < \alpha \leq n$.

Result 1.4. *Caputo fractional derivative has a useful property*

$$I_y^\alpha cD_y^\alpha\ f(y) = f(y) - \sum_{k=0}^{n-1} f^{(k)}\left(0^+\right) \frac{y^k}{k!}$$

where n is an integer and $n-1 < \alpha \leq n$.

Result 1.5. *Caputo fractional derivative operator is a linear operation*

$$cD_y^\alpha\left(\lambda f(y) + \mu g(y)\right) = \lambda\ cD_y^\alpha f(y) + \mu\ cD_t^\alpha g(y)$$

where λ and μ are constants.

Result 1.6. *For the Caputo's derivative, also we have:*

$$cD_y^\alpha\ C = 0 \quad , \quad C \text{ is constant}$$

$$cD_y^\alpha y^n = \begin{cases} 0 & , \text{ for } n \in N_0 \text{ and } n \geq [\alpha] \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} y^{n-\alpha} & , \text{ for } n \in N \text{ and } n \geq [\alpha] \end{cases}$$

we use the ceiling function $[\alpha]$ to denote the smallest integer greater than or equal to α and $N = \{0,\ 1,\ 2,\ \ldots.\}$.

2. The Adomian Decomposition Method (ADM)

Let us consider the operator equation $Fu = G$, where

F – General nonlinear ordinary differential operator

G – Given function

Then F can be expressed as:

$$Lu + Ru + Nu = G \tag{1}$$

where

N – Nonlinear operator

L – The highest order derivative which is assumed to be invertible

R – Linear differential operator of order less than L

G – Non homogeneous term

Equation (1) can be written as

$$Lu = G - Ru - Nu \tag{2}$$

By applying the operator L^{-1} on both the sides and by using the given conditions, we obtain

$$u = h + L^{-1}G - L^{-1}Ru - L^{-1}Nu \tag{3}$$

where h is the solution of the homogeneous equation $Lu = 0$ with the initial-boundary conditions. Define the decomposition parameter σ as:

$$u = \sum_{n=0}^{\infty} \sigma^n u_n$$

then N(u) will be a function of $\sigma,\ u_0,\ u_1, \ldots$. By expanding N(u) in Maclurian series with respect to σ. We get

$$N(u) = \sum_{n=0}^{\infty} \sigma^n A_n$$

where

$$A_n = \frac{1}{n!} \frac{d^n}{d\sigma^n} \left[N\left(\sum_{k=0}^{n} \sigma^k u_k \right) \right]_{\sigma=0} \tag{4}$$

Here the components of A_n are the so called Adomian polynomials they can be generated for each nonlinearity, for example, If $N(u) = f(u)$ then the Adomian polynomials are given as:

$$\begin{aligned}
A_0 &= f(u_0) \\
A_1 &= u_1 f'(u_0) \\
A_2 &= u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0) \\
A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0) \\
&\vdots
\end{aligned}$$

We parameterize equation (3) in the form:

$$u = h + L^{-1}G - \sigma L^{-1}Ru - \sigma L^{-1}Nu \tag{5}$$

where σ is just an identifier for collecting terms in a suitable way such that u_n depends on u_0 , u_1 , $u_2, \ldots, u_n$ and we will later set σ=1.

$$\sum_{n=0}^{\infty} \sigma^n u_n = h + L^{-1}G - \sigma L^{-1}R \sum_{n=0}^{\infty} \sigma^n u_n - \sigma L^{-1} \sum_{n=0}^{\infty} \sigma^n A_n \tag{6}$$

Equating the coefficients of equal powers ofσ, we get

$$\begin{aligned}
u_0 &= h + L^{-1}G \\
u_1 &= -L^{-1}(Ru_0) - L^{-1}(A_0) \\
u_2 &= -L^{-1}(Ru_1) - L^{-1}(A_1)
\end{aligned}$$

In general,

$$u_n = -L^{-1}(Ru_{n-1}) - L^{-1}(A_{n-1}) \quad , \ n \geq 1 \tag{7}$$

Finally, an N-terms that approximate solution is given by:

$$\varphi_N(T) = \sum_{n=0}^{N-1} u_n(T) \quad , \quad N \geq 1$$

and the exact solution is

$$u(t) = \lim_{N \to \infty} \varphi_N(t).$$

3. The Approach

We shall approximate the solution of the following FDDEs:

$$cD_t^{\alpha} u(t) = N(t, u(t), u(\varphi(t))) \tag{8}$$

$$u(t) = \psi(t) \quad , \ -\tau \leq t \leq 0 \tag{9}$$

$$u^i(0) = u_0^i \quad , \quad i = 0, 1, 2, \ldots, n-1$$

where $n - 1 < \alpha \leq n$. Operating I_t^{α} to the both sides of equation (8), we have

$$u(t) = I_t^{\alpha} N\left(t, u(t), u(\varphi(t)) + \sum_{k=0}^{n-1} y(0^+) \frac{t^k}{k!}\right) \tag{10}$$

Adomian's method defined the solution y(t) by the series

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \tag{11}$$

So that, the components u_n will be determined recursively. Moreover, the method defines the nonlinear term $N(t, u(t), u(\varphi(t))$ by the Adomian polynomials

$$N(t, u(t), u(\varphi(t)) = \sum_{n=0}^{\infty} A_n \tag{12}$$

where A_n are Adomian polynomials that can be generated for all forms of nonlinearity as

$$A_n = \frac{1}{n!} \frac{d^n}{d\sigma^n} [N\left(t, \sum_{j=0}^{\infty} \sigma^j u_j(t), \sum_{j=0}^{\infty} \sigma^j u_j(\varphi(t))\right)]_{\sigma=0} \tag{13}$$

Substituting equations (11) and (12) into equation (10), we get

$$\sum_{n=0}^{\infty} u_n(t) = \sum_{k=0}^{n-1} u^{(k)}(0^+) \frac{t^k}{k!} + I_t^{\alpha}\left(\sum_{n=0}^{\infty} A_n\right) \tag{14}$$

To determine the components $u_n(x)$, $n \geq 0$. First we identify the zero component $u_0(x)$ by the terms $\sum_{k=0}^{n-1} u^{(k)}(0^+)$ and (t) and $I_t^{\alpha} f(t)$ where f(t) represents the non-homogeneous term of $N(t, u(t), u(\varphi(t)))$. In other words, the method introduces the recursive relation:

$$u_0(t) = \psi(t) + \sum_{k=0}^{n-1} u(0^+) \frac{t^k}{k!} + I_t^{\alpha} f(t) \tag{15}$$

$$u_{n+1}(t) = I_t^{\alpha} A_n \quad , \ n \geq 0 \tag{16}$$

Example 3.1. *Solve the Fractional Delay Differential Equation of the form*

$$D_t^{\alpha} \ u(t) = \ \frac{2}{3} u(t) + u\left(\frac{t}{2}\right) - t^2 + 2 \ \ , \ \ 0 \leq t \leq 1 \ \ , \ \ 1 < \alpha \leq 2 \tag{17}$$

Subject to the initial condition $u(0) = 0$.

Solution:

The exact solution of the above system is $u(t) = t^2$. According to equations (15) and (16), we have

$$u_0(t) = \frac{2}{\Gamma(\alpha+1)} t^{\alpha} - \frac{2}{\Gamma(\alpha+3)} t^{\alpha+2}$$
$$u_{n+1}(t) = I_t^{\alpha}\left(\frac{2}{3} u_n(t) + u_n\left(\frac{t}{2}\right)\right)$$

For n = 0, 1, 2. we have,

$$u_1(t) = I_t^{\alpha}\left(\frac{2}{3} u_0(t) + u_0\left(\frac{t}{2}\right)\right)$$
$$u_2(t) = I_t^{\alpha}\left(\frac{2}{3} u_1(t) + u_1\left(\frac{t}{2}\right)\right)$$
$$u_3(t) = I_t^{\alpha}\left(\frac{2}{3} u_2(t) + u_2\left(\frac{t}{2}\right)\right)$$

The following table represent the approximate solution of (17) using Adomian Decomposition Method (ADM) up to three terms for different values of α with a comparison with the exact solution when $\alpha = 2$.

t	Exact	ADM$\alpha = 1.5$	ADM$\alpha = 1.75$	ADM$\alpha = 2$
0	0	0	0	0
0.1	0.01	0.048	0.022	0.01
0.2	0.04	0.137	0.075	0.04
0.3	0.09	0.255	0.153	0.09
0.4	0.16	0.398	0.254	0.16
0.5	0.25	0.564	0.377	0.25
0.6	0.36	0.753	0.521	0.36
0.7	0.49	0.963	0.687	0.49
0.8	0.64	1.195	0.873	0.64
0.9	0.81	1.1448	1.080	0.809
1.0	1	1.723	1.307	0.999

Table 1. The Approximate solution of the above example using different values of α with a comparison with the exact solution when α=2.

4. Laguerre Wavelets

Wavelets [2], [7] and [8] constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. Let a be a dilation parameter and b be a translation parameter when both vary continuously, we get the family of continuous wavelets as

$$\varphi_{a,b}(x) = |a|^{-\frac{1}{2}} \varphi\left(\frac{x-b}{a}\right), \quad a, b \in R, \ a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, we have the following family of discrete wavelets

$$\varphi_{k,n}(x) = |a|^{\frac{k}{2}} \varphi\left(a_0^k x - nb_0\right), \ k, n \in Z$$

form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\varphi_{k,n}(x)$ form an orthonormal basis. The Laguerrewavelets $\varphi_{n,m}(x) = \varphi(k, n, m, x)$ involve four arguments $n = 1, 2, \ldots, 2^{k-1}$, k is any positive integer, m is the degree of the Laguerre polynomials and it is the normalized time. They are defined on the interval as

$$\varphi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} \widetilde{L_m}\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

where

$$\widetilde{L_m}(x) = \frac{1}{m!} L_m(x), \quad m = 0, 1, 2, \ldots, M-1 \tag{19}$$

In equation (19) the coefficients are used for orthonormality. Here $L_m(x)$ are the Laguerre polynomials of degree m with respect to the weight function w(x)=1 on the interval $[0, \infty]$ and satisfy the following recursive formula

$$L_0(x) = 1$$

$$L_1(x) = 1 - x$$

And so

$$L_{m+2}(x) = \frac{((2m+3-x)L_{m+1}(x) - (m+1)L_m(x))}{m+2}, \; m = 0, 1, 2, \ldots .$$

5. Modified Laguerre Wavelet Method (MLWM)

Let us consider the delay differential equation of the form

$$u^{\alpha}(x) = f(u) + g(x)\, u\left(\frac{x}{a} - c\right), \; 0 < x < b, \; 1 < \alpha < 2, \tag{20}$$

$$u(x) = p(x), \; -b \leq x \leq 0$$

is a given continuous linear or nonlinear function. First we have to use the method of step to convert the delay differential equation (20) to non-homogeneous ordinary differential equation by using initial function, equation (20) implies,

$$u^{\alpha}(x) = f(u) + g(x)\, p\left(\frac{x}{a} - c\right), \; 0 < x < b, \; 1 < \alpha \leq 2 \tag{21}$$

which is a fractional differential equation and the solution of the equation (4) can be expanded as a Laguerre wavelets series as follows

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \varphi_{n,m}(x)$$

We approximate $\varphi_{n,m}(x)$ by the truncated series

$$u_{k,M}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm} \varphi_{n,m}(x) \tag{21a}$$

Then a total number of $2^{k-1}M$ conditions should exist for determination of $2^{k-1}M$ coefficients. Since two conditions are furnished by the initial conditions, namely

$$u_{k,M}(0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm} \varphi_{n,m}(0) = p(0) \tag{22}$$

$$\frac{d}{dx} u_{k,M}(0) = \frac{d}{dx} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm} \varphi_{n,m}(0) = p'(0)$$

$2^{k-1}M - 2$, we see that there should be extra conditions to recover the unknown coefficients. These conditions can be obtained by substituting equation (21) in equation (20),

$$\frac{d^n}{dx^n} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-3} C_{nm} \varphi_{n,m}(x) = f\left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-3} C_{nm} \varphi_{n,m}(x) \right) + g(x)\, p\left(\frac{x}{a} - c\right) \tag{23}$$

We, now assume equation (23) is exact at $2^{k-1}M-3$ points as follows

$$\frac{d^n}{dx^n}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3}C_{nm}\varphi_{n,m}\left(x_i\right)=f\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3}C_{nm}\varphi_{n,m}\left(x_i\right)\right)+g\left(x_i\right)p\left(\frac{x_i}{a}-c\right) \tag{24}$$

The best choice of the points are the zeros of the shifted Laguerre polynomials of degree $2^{k-1}M-2$ in the interval, that is,

$$x_i=\frac{s_i+1}{2}$$

where

$$s_i=\cos\left(\frac{(2i-1)\,\pi}{2^{k-1}M-1}\right),\quad i=1,2,\ldots,2^{k-1}M-2$$

Combine equations (22) and (24) to obtain $2^{k-1}M$ linear equations from which we can compute values for the unknown coefficients.

Example 5.1. *Solve the Fractional Delay Differential Equation of the form*

$$u^{\alpha}\left(t\right)=\frac{2}{3}\,u\left(t\right)+u\left(\frac{t}{2}\right)-t^2+2,\quad 0\leq t\leq 1,\quad 1<\alpha\leq 2$$

Subject to the initial conditions $(0)=0$ *and* $\dot{u}\left(0\right)=0$.

Solution: The exact solution of the above system is $u\left(t\right)=t^2$. The following table shows the comparison of the absolute error between exact solution and M=5, approximate solution by Modified Laguerre Wavelet Method (MLWM)

t	Exact solution	Solution by proposed method	Error in proposed method M=5
0.0	0.000000000000	-0.00000000141	1.41421 E-09
0.1	0.010000000000	0.01000004758	4.75800 E-08
0.2	0.040000000000	0.04000009693	9.69300 E-08
0.3	0.090000000000	0.09000014701	1.47010 E-07
0.4	0.160000000000	0.16000019820	1.98200 E-07
0.5	0.250000000000	0.25000025090	2.50900 E-07
0.6	0.360000000000	0.36000030550	3.05500 E-07
0.7	0.490000000000	0.49000036240	3.62400 E-07
0.8	0.640000000000	0.64000042200	4.22000 E-07
0.9	0.810000000000	0.81000048480	4.84800 E-07
1.0	1.000000000000	1.00000055100	5.51000 E-07

Table 2. The comparion of the absolute error between exact solution and M=5, approximate solution by Modified Lagurre Wavelet method

6. Conclusion

In this paper, we have been used the Adomian Decomposition Method (ADM) and Modified Laguerre Wavelets Method (MLWM) for solving variable order delay differential equations of fractional order. Two examples were solved in the view of the ADM and MLWM with good approximation and agreement with the exact solution.

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