A Decagonal Fuzzy Number and its Vertex Method

Research Article[*]

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Abstract: The decision data of human judgements with preferences are often vague so that the usual ways of using crisp values are in adequate in many real life situation. Also using fuzzy numbers such as triangular, trapezoidal are not suitable in few cases where the uncertainties arises in ten different points. Therefore, Decagonal Fuzzy Number (DFN) and its arithmetic operations based on α-cut are introduced. Also, the linguistic terms can be expressed in Decagonal fuzzy numbers and a vertex method is defined to calculate the distance between two Decagonal fuzzy numbers.

Keywords: Fuzzy Arithmetic, Linguistic values, Decagonal Fuzzy Number (DFN), α cut.
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1. Introduction

In 1965, fuzzy sets were introduced by Zadeh [1], L.A to provide natural way of dealing with problems in which the source of imprecision and vagueness occurs. It can be applied in many fields such as artificial intelligence, control system, decision making, expert system etc. Fuzzy numbers and their fuzzy operations are seeds of fuzzy number theory. Fuzzy number and their arithmetic operation for modeling expert system, Cognitive computational models, measurement, knowledge, intelligence .The concept of fuzzy number has been defined as a fuzzy subset of real line by Dubois, D and Prade, H [2]. A fuzzy number is a quantity whose values are precise, rather than exact as in the case with single valued numbers. To deal imprecise in real life situation, many researchers used triangular and trapezoidal fuzzy number .Also hexagonal, nonagonal, decagonal fuzzy numbers have been introduced to clear the vagueness .Most of the researcher has focused on uncertain linguistic term in group decision making processes. In decision making problem experts may provide uncertain linguistic term to express their opinion when they have no clear idea and lack of information. The uncertain linguistic term is frequently used as input in decision analysis activities. So far linguistic values are usually represented as fuzzy numbers such as triangular, trapezoidal. But it is complex to restrict the membership functions to take triangular, trapezoidal when vagueness arises in ten different points. Therefore, in this paper a new form of decagonal Fuzzy Number (DFN) is proposed under uncertain linguistic environment. This paper is organized as follows: section one presents introduction. Basic definition of fuzzy number and linguistic terms are given in section two, section three provides decagonal fuzzy number and it linguistic values. New operation for addition, multiplication, division of decagonal fuzzy number α cut is proposed in section four. Section five proposes vertex method to calculate the distance between two decagonal fuzzy numbers. Finally, conclusion and future directions are given.

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2. Basic Definitions and Notations

In this section, some basic definitions of fuzzy set theory and fuzzy numbers are reviewed.

Definition 2.1. *A fuzzy set $\tilde{A}$ in X is characterized by a membership function $\mu_{\tilde{A}}(x)$ which associates each point in X; to a real number in the interval [0, 1]: The value of $\mu_{\tilde{A}}(x)$ represents "grade of membership" of $x \in \mu_{\tilde{A}}(x)$: More general representation for a fuzzy set is $\tilde{A} = (x; \mu_{\tilde{A}}(x))$.*

Definition 2.2. *The α-cut of the fuzzy set A of the universe of discourse X is defined as $\tilde{A}_\alpha = \{x \in X \ / \ \mu_{\tilde{A}}(x) \geq \alpha\}$, where $\alpha \in [0, 1]$.*

Definition 2.3. *A fuzzy set $\tilde{A}$ defined on the set of real numbers R is said to be a fuzzy number if its membership function $\tilde{A} : R \to [0, 1]$ has the following characteristics.*

(1). $\tilde{A}$ is convex $\mu_{\tilde{A}}(x_1 + (1 - \lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$, $\forall \ x \in [0, 1]$, $\lambda \in [0, 1]$.

(2). $\tilde{A}$ is normal i.e; there exists an $x \in R$ such that if $\max \mu_{\tilde{A}}(x) = 1$.

(3). $\tilde{A}$ is piecewise continuous.

Definition 2.4. *A triangular fuzzy number $\tilde{A}$ denoted by (a_1, a_2, a_3), and the membership function is defined as*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & for \quad a_1 \leq x \leq a_2 \\ \frac{a_3 - x}{a_3 - a_2}, & for \quad a_2 \leq x \leq a_3 \\ 0, & elsewhere \end{cases}$$
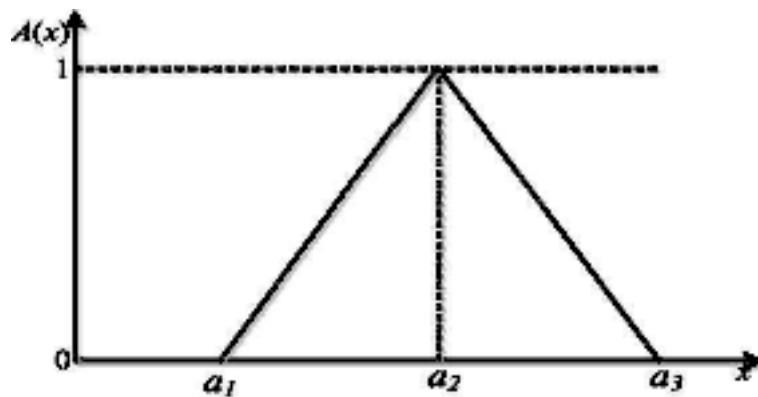

Figure 1. **Triangular fuzzy number**

Definition 2.5. *A Linguistic variable/term is a variable whose value is not crisp number but word or sentence in a natural language.*

Definition 2.6. *A trapezoidal fuzzy number $\tilde{A}$ can be defined as (a_1, a_2, a_3, a_4), and the membership function is defined as*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & for \ a_1 \leq x \leq a_2 \\ 1, & for \ a_2 \leq x \leq a_3 \\ \frac{a_4 - x}{a_4 - a_3}, & for \ a_3 \leq x \leq a_4 \\ 0, & elsewhere \end{cases}$$
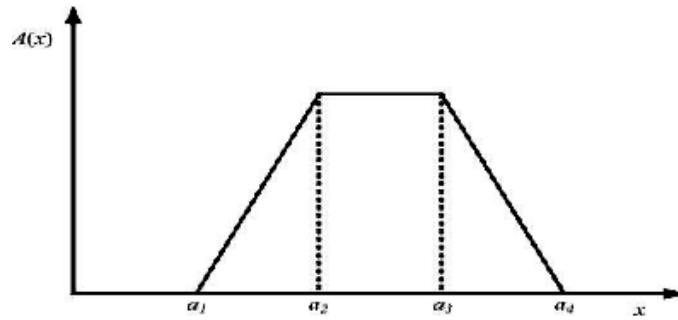
Figure 2.   **Trapezoidal fuzzy number**

3.   Decagonal Fuzzy Number (DFN)

Definition 3.1. *A Decagonal fuzzy number $\tilde{D}$ denoted as $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$, and the membership function is defined as*

$$
\mu_{\tilde{D}}(x) = \begin{cases}
\frac{1}{4}\frac{(x-a_1)}{(a_2-a_1)}, & a_1 \leq x \leq a_2 \\
\frac{1}{4} + \frac{1}{4}\frac{(x-a_2)}{(a_3-a_2)}, & a_2 \leq x \leq a_3 \\
\frac{1}{2} + \frac{1}{4}\frac{(x-a_3)}{(a_4-a_3)}, & a_3 \leq x \leq a_4 \\
\frac{3}{4} + \frac{1}{4}\frac{(x-a_4)}{(a_5-a_4)}, & a_4 \leq x \leq a_5 \\
1, & a_5 \leq x \leq a_6 \\
1 - \frac{1}{4}\frac{(x-a_6)}{(a_7-a_6)}, & a_6 \leq x \leq a_7 \\
\frac{3}{4} - \frac{1}{4}\frac{(x-a_7)}{(a_8-a_7)}, & a_7 \leq x \leq a_8 \\
\frac{1}{2} - \frac{1}{4}\frac{(x-a_8)}{(a_9-a_8)}, & a_8 \leq x \leq a_9 \\
\frac{1}{4}\frac{(x-a_9)}{(a_{10}-a_9)}, & a_9 \leq x \leq a_{10} \\
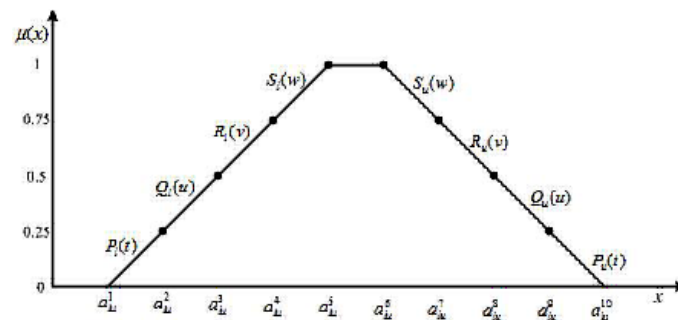0, & otherwise.
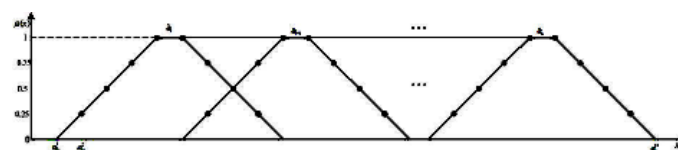\end{cases}
$$

Figure 3.   **The Decagonal fuzzy number**

Figure 4.   **The decagonal fuzzy number from uncertain linguistic term $[s_l; s_u]$**

4.   Generalized Decagonal Fuzzy Number (DFN)

Definition 4.1. *A Decagonal fuzzy number denoted as $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}; w)$, and the membership function is defined as*

$$\mu_{\tilde{D}}(x) = \begin{cases} \frac{1}{4}\frac{(x-a_1)}{(a_2-a_1)}, & a_1 \leq x \leq a_2 \\[2mm] \frac{1}{4} + \frac{1}{4}w\frac{(x-a_2)}{(a_3-a_2)}, & a_2 \leq x \leq a_3 \\[2mm] \frac{1}{2} + \frac{1}{4}w\frac{(x-a_3)}{(a_4-a_3)}, & a_3 \leq x \leq a_4 \\[2mm] \frac{3}{4} + \frac{1}{4}w\frac{(x-a_4)}{(a_5-a_4)}, & a_4 \leq x \leq a_5 \\[2mm] 1, & a_5 \leq x \leq a_6 \\[2mm] 1 - \frac{1}{4}w\frac{(x-a_6)}{(a_7-a_6)}, & a_6 \leq x \leq a_7 \\[2mm] \frac{3}{4} - \frac{1}{4}w\frac{(x-a_7)}{(a_8-a_7)}, & a_7 \leq x \leq a_8 \\[2mm] \frac{1}{2} - \frac{1}{4}w\frac{(x-a_8)}{(a_9-a_8)}, & a_8 \leq x \leq a_9 \\[2mm] \frac{1}{4}\frac{(x-a_9)}{(a_{10}-a_9)}, & a_9 \leq x \leq a_{10} \\[2mm] 0, & otherwise. \end{cases}$$

Linguistic terms	Linguistic values
No influence	(0, 0, 0, 0, 0, 0.3, 0.6, 0.9, 1.2, 1.5)
Very Low influence	(0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0)
Low influence	(1.8, 2.1, 2.4, 2.7, 3.0, 3.3, 3.6, 3.9, 4.2, 4.5)
Medium	(3.3, 3.6, 3.9, 4.2, 4.5, 4.8, 5.1, 5.4, 5.7, 6.0)
High influence	(4.8, 5.1, 5.4, 5.7, 6.0, 6.3, 6.6, 6.9, 7.2, 7.5)
Very High influence	(6.3, 6.6, 6.9, 7.2, 7.5, 7.8, 7.8, 7.8, 7.8, 7.8)

Table 1.   **Decagonal Fuzzy linguistic scale**

Furthermore, in light of the research work by Kaufman, A and Zadeh, LA [10], the arithmetic operations of decagonal fuzzy number and vertex method are provided in the following section.

5.   Arithmetic Operations Of Decagonal Fuzzy Numbers (DFN)

In this section, arithmetic operations of (DFN) based on α-cut method is reviewed.

5.1.   Arithmetic Operation on α-cut

Definition 5.1. *A Decagonal fuzzy number $\tilde{D}$ can also be defined as $\tilde{D} = P_l(t)$, $Q_l(u)$, $R_l(v)$, $S_l(w)$, $P_u(t)$, $Q_u(u)$, $R_u(v)$, $S_u(w)$, $t \in [0, 0.25]$, $u \in [0.25, 0.5]$, $v \in [0.5, 0.75]$ and $w \in [0, 0.1]$, where*

$$P_l(t) = \frac{1}{4}\frac{(x-a_1)}{(a_2-a_1)},$$

$$Q_l(u) = \frac{1}{4} + \frac{1}{4}\frac{(x-a_2)}{(a_3-a_2)},$$

$$R_l(v) = \frac{1}{2} + \frac{1}{4}\frac{(x-a_3)}{(a_4-a_3)},$$

$$S_l(w) = \frac{3}{4} + \frac{1}{4}\frac{(x-a_4)}{(a_5-a_4)},$$

$$S_u(w) = 1 - \frac{1}{4}\frac{(x-a_6)}{(a_7-a_6)},$$

$$R_u \, (\text{v}) = \frac{3}{4} - \frac{1}{4} \frac{(x - a_7)}{(a_8 - a_7)},$$

$$Q_u \, (\text{u}) = \frac{1}{2} - \frac{1}{4} \frac{(x - a_8)}{(a_9 - a_8)},$$

$$P_u \, (\text{t}) = \frac{1}{4} \frac{(a_{10} - x)}{(a_{10} - a_9)}$$

Here, $P_l(t)$, $Q_l(u)$, $R_l(v)$, $S_l(w)$, is bounded and continuous increasing function over [0, 0.25], [0.25, 0.5], [0.5, 0.75] and [0, 0.1] respectively. $P_u(t)$, $Q_u(u)$, $R_u(v)$, $S_u(w)$, is bounded and continuous increasing function over [0, 0.25], [0.25, 0.5], [0.5, 0.75] and [0, 0.1] respectively.

Definition 5.2. *The α-cut of the fuzzy set of the universe of discourse X is defined as $\tilde{D}_\alpha = \{x \in X \, / \, \mu_{\tilde{A}}(x) \geq \alpha\}$, where $\alpha \in [0,1]$.*

$$\tilde{D}_\alpha = \begin{cases} [P_l(\alpha), P_u(\alpha)], & for \ \alpha \in [0, 0.25) \\ [Q_l(\alpha), Q_u(\alpha)], & for \ \alpha \in [0.25, 0.5) \\ [R_l(\alpha), R_u(\alpha)], & for \ \alpha \in [0.5, 0.75) \\ [S_l(\alpha), S_u(\alpha)], & for \ \alpha \in [0.75, 1] \end{cases}$$

Definition 5.3. *If $P_l(x) = \alpha$ and $P_u(x) = \alpha$ then α-cut operations interval $\tilde{D}_\alpha$ is obtained as*

(1). $[P_l(\alpha), P_u(\alpha)] = [4\alpha(a_2 - a_1) + a_1, -4\alpha(a_{10} - a_9) + a_{10}]$. Similarly, we can obtain α-cut operations interval D_α for $[Q_l(\alpha), Q_u(\alpha)]$, $[R_l(\alpha), R_u(\alpha)]$ and $[S_l(\alpha), S_u(\alpha)]$ as follows

(2). $[Q_l(\alpha), Q_u(\alpha)] = [4\alpha(a_3 - a_2) + 2a_2 - a_3, -4\alpha(a_9 - a_8) + 2a_9 - a_8]$

(3). $[R_l(\alpha), R_u(\alpha)] = [4\alpha(a_4 - a_3) + 3a_3 - 2a_4, -4\alpha(a_8 - a_7) + 3a_8 - 2a_7]$

(4). $[S_l(\alpha), S_u(\alpha)] = [4\alpha(a_5 - a_4) + 4a_4 - 3a_5, -4\alpha(a_7 - a_6) + 4a_7 - 3a_6]$

Hence, α- cut of Decagonal Fuzzy Number

$$\tilde{D}_\alpha = \begin{cases} [4a(a_2 - a_1) + a_1, -4a(a_{10} - a_9) + a_{10}], & for \ \alpha \in [0, 0.25) \\ [4a(a_3 - a_2) + 2a_2 - a_3, -4a(a_9 - a_8) + 2a_9 - a_8], & for \ \alpha \in [0.25, 0.5) \\ [4a(a_4 - a_3) + 3a_3 - 2a_4, -4a(a_8 - a_7) + 3a_8 - 2a_7], & for \ \alpha \in [0.5, 0.75) \\ [4a(a_5 - a_4) + 4a_4 - 3a_5, -4a(a_7 - a_6) + 4a_7 - 3a_6], & for \ \alpha \in [0.75, 1] \end{cases}$$

Theorem 5.4. *If $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$ are the Decagonal fuzzy numbers, then $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is also Decagonal fuzzy number*

$$\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6, a_7 + b_7, a_8 + b_8, a_9 + b_9, a_{10} + b_{10}).$$

Proof. The membership function of DFN $\tilde{C} = \tilde{A} \oplus \tilde{B}$ can be found by α-cut method with the transform $z = x + y$, α-cut membership function of $\tilde{A}(x)$ is,

$$x \in \begin{cases} [4a(a_2 - a_1) + a_1, -4a(a_{10} - a_9) + a_{10}], & for \ \alpha \in [0, 0.25) \\ [4a(a_3 - a_2) + 2a_2 - a_3, -4a(a_9 - a_8) + 2a_9 - a_8], & for \ \alpha \in [0.25, 0.5) \\ [4a(a_4 - a_3) + 3a_3 - 2a_4, -4a(a_8 - a_7) + 3a_8 - 2a_7], & for \ \alpha \in [0.5, 0.75) \\ [4a(a_5 - a_4) + 4a_4 - 3a_5, -4a(a_7 - a_6) + 4a_7 - 3a_6], & for \ \alpha \in [0.75, 1] \end{cases}$$

α- cut membership function of $\tilde{B}$ (y) is,

$$y \in \begin{cases} [4a\,(b_2 - b_1) + b_1, -4a\,(b_{10} - b_9) + b_{10}], & for\ \alpha \in [0, 0.25) \\ [4a(b_3 - b_2) + 2b_2 - b_3, -4a(b_9 - b_8) + 2b_9 - b_8], & for\ \alpha \in [0.25, 0.5) \\ [4a(b_4 - b_3) + 3b_3 - 2b_4, -4a(b_8 - b_7) + 3b_8 - 2b_7], & for\ \alpha \in [0.5, 0.75) \\ [4a(b_5 - b_4) + 4b_4 - 3b_5, -4a(b_7 - b_6) + 4b_7 - 3b_6], & for\ \alpha \in [0.75, 1] \end{cases}$$

$$z = x + y \in \begin{cases} [4a\,(a_2 - a_1) + a_1, -4a\,(a_{10} - a_9) + a_{10}] + \\ [4a\,(b_2 - b_1) + b_1, -4a\,(b_{10} - b_9) + b_{10}], & for\ \alpha \in [0, 0.25) \\ [4a\,(a_3 - a_2) + 2a_2 - a_3, -4a\,(a_9 - a_8) + 2a_9 - a_8] + \\ [4a(b_3 - b_2) + 2b_2 - b_3, -4a(b_9 - b_8) + 2b_9 - b_8], & for\ \alpha \in [0.25, 0.5) \\ [4a\,(a_4 - a_3) + 3a_3 - 2a_4, -4a\,(a_8 - a_7) + 3a_8 - 2a_7] + \\ [4a(b_4 - b_3) + 3b_3 - 2b_4, -4a(b_8 - b_7) + 3b_8 - 2b_7], & for\ \alpha \in [0.5, 0.75) \\ [4a\,(a_5 - a_4) + 4a_4 - 3a_5, -4a\,(a_7 - a_6) + 4a_7 - 3a_6] + \\ [4a(b_5 - b_4) + 4b_4 - 3b_5, -4a(b_7 - b_6) + 4b_7 - 3b_6], & for\ \alpha \in [0.75, 1] \end{cases}$$

The membership function of $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is,

$$\mu_{\tilde{C}}(x) = \begin{cases} \frac{z - (a_1 + b_1)}{4[(a_2 - a_1) + (b_2 - b_1)]}, & a_1 + b_1 \leq z \leq a_2 + b_2 \\ \frac{z - 2(a_2 + b_2) + (a_3 + b_3)}{4[(a_3 - a_2) + (b_3 - b_2)]}, & a_2 + b_2 \leq z \leq a_3 + b_3 \\ \frac{z - 3(a_3 + b_3) + 2(a_4 + b_4)}{4[(a_4 - a_3) + (b_4 - b_3)]}, & a_3 + b_3 \leq z \leq a_4 + b_4 \\ \frac{z - 4(a_4 + b_4) + 3(a_5 + b_5)}{4[(a_5 - a_4) + (b_5 - b_4)]}, & a_4 + b_4 \leq z \leq a_5 + b_5 \\ 1, & a_5 + b_5 \leq z \leq a_6 + b_6 \\ \frac{z - 4(a_7 + b_7) + 3(a_6 + b_6)}{4[(a_6 - a_7) + (b_6 - b_7)]}, & a_6 + b_6 \leq z \leq a_7 + b_7 \\ \frac{z - 3(a_8 + b_8) + 2(a_7 + b_7)}{4[(a_7 - a_8) + (b_7 - b_8)]}, & a_7 + b_7 \leq z \leq a_8 + b_8 \\ \frac{z - 2(a_9 + b_9) + 1(a_8 + b_8)}{4[(a_8 - a_9) + (b_8 - b_9)]}, & a_8 + b_8 \leq z \leq a_9 + b_9 \\ \frac{z - (a_{10} + b_{10})}{4[(a_9 - a_{10}) + (b_9 - b_{10})]}, & a_9 + b_9 \leq z \leq a_{10} + b_{10} \\ 0, & otherwise \end{cases}$$

Hence, addition rule is proved. Therefore we have $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6, a_7 + b_7, a_8 + b_8, a_9 + b_9, a_{10} + b_{10})$ is a Decagonal Fuzzy Number. $\qquad \square$

Example 5.5. *Let us consider two DFN $\tilde{A} = (1.52, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6)$ and $\tilde{B} = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$. Then, the addition of these two DFN is defined as $\tilde{A} \oplus \tilde{B} = (3.5, 5, 6.5, 8, 9.5, 11, 12.5, 14, 15.5, 17)$ with membership function as follows*

$$\mu_{\tilde{C}}(x) = \begin{cases} \frac{1}{4}\frac{(x - 3.5)}{(1.5)}, & 3.5 \leq x \leq 5 \\ \frac{1}{4}\frac{(x - 3.5)}{(1.5)}, & 5 \leq x \leq 6.5 \\ \frac{1}{4}\frac{(x - 3.5)}{(1.5)}, & 6.5 \leq x \leq 8 \\ \frac{1}{4}\frac{(x - 3.5)}{(1.5)}, & 8 \leq x \leq 9.5 \\ 1, & 9.5 \leq x \leq 11 \\ -\frac{1}{4}\frac{(x - 17)}{(1.5)}, & 11 \leq x \leq 12.5 \\ -\frac{1}{4}\frac{(x - 17)}{(1.5)}, & 12.5 \leq x \leq 14 \\ -\frac{1}{4}\frac{(x - 17)}{(1.5)}, & 14 \leq x \leq 15.5 \\ -\frac{1}{4}\frac{(x - 17)}{(1.5)}, & 15.5 \leq x \leq 17 \\ 0, & otherwise. \end{cases}$$

Theorem 5.6. *If $\tilde{A}= (a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_9,\ a_{10})$ and $\tilde{B}= (b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6,\ b_7,\ b_8,\ b_9,\ b_{10})$ are the decagonal fuzzy numbers, then $\tilde{P} = \tilde{A} \otimes \tilde{B}$ is also decagonal fuzzy number*

$$\tilde{A} \otimes \tilde{B} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$$

Proof. The membership function of DFN $\tilde{P} = \tilde{A} \otimes \tilde{B}$ can be found by α-cut method with the transform $z = x \times y$

$$
z = x \times y \in
\begin{cases}
[4a\,(a_2 - a_1) + a_1, -4a\,(a_{10} - a_9) + a_{10}] + \\
[4a\,(b_2 - b_1) + b_1, -4a\,(b_{10} - b_9) + b_{10}], & for\ a \in [0, 0.25) \\
[4a\,(a_3 - a_2) + 2a_2 - a_3, -4a\,(a_9 - a_8) + 2a_9 - a_8] + \\
[4a(b_3 - b_2) + 2b_2 - b_3, -4a(b_9 - b_8) + 2b_9 - b_8], & for\ a \in [0.25, 0.5) \\
[4a\,(a_4 - a_3) + 3a_3 - 2a_4, -4a\,(a_8 - a_7) + 3a_8 - 2a_7] + \\
[4a(b_4 - b_3) + 3b_3 - 2b_4, -4a(b_8 - b_7) + 3b_8 - 2b_7], & for\ a \in [0.5, 0.75) \\
[4a\,(a_5 - a_4) + 4a_4 - 3a_5, -4a\,(a_7 - a_6) + 4a_7 - 3a_6] + \\
[4a(b_5 - b_4) + 4b_4 - 3b_5, -4a(b_7 - b_6) + 4b_7 - 3b_6], & for\ a \in [0.75, 1]
\end{cases}
$$

So, the membership function of $\tilde{P} = \tilde{A} \otimes \tilde{B}$ is,

$$
\mu_{\tilde{P}}(x) \in
\begin{cases}
\frac{-B_1 + \sqrt{B_1^2 - 4A_1(a_1 b_1 - z)}}{2A_1}, & a_1 b_1 \leq z \leq a_2 b_2 \\
\frac{-B_2 + \sqrt{B_2^2 - 4A_2(4a_2 b_2 - 2(a_2 b_3 + a_3 b_2) + a_3 b_3 - z)}}{2A_2}, & a_2 b_2 \leq z \leq a_3 b_3 \\
\frac{-B_3 + \sqrt{B_3^2 - 4A_3(9a_3 b_3 - 6(a_3 b_4 + a_4 b_3) + 4a_4 b_4 - z)}}{2A_3}, & a_3 b_3 \leq z \leq a_4 b_4 \\
\frac{-B_4 + \sqrt{B_4^2 - 4A_4(16a_4 b_4 - 12(a_4 b_5 + a_5 b_4) + 9a_5 b_5 - z)}}{2A_4}, & a_4 b_4 \leq z \leq a_5 b_5 \\
1, & a_5 b_5 \leq z \leq a_6 b_6 \\
\frac{B_6 - \sqrt{B_6^2 - 4A_6(9a_6 b_6 - 12(a_6 b_7 + a_7 b_6) + 16a_7 b_7 - z)}}{2A_6}, & a_6 b_6 \leq z \leq a_7,\ b_7 \\
\frac{B_7 - \sqrt{B_7^2 - 4A_7(4a_7 b_7 - 6(a_7 b_8 + a_8 b_7) + 9a_8 b_8 - z)}}{2A_7}, & a_7 b_7 \leq z \leq a_8 b_8 \\
\frac{B_8 - \sqrt{B_8^2 - 4A_8(a_8 b_8 - 2(a_8 b_9 + a_9 b_8) + 4a_9 b_9 - z)}}{2A_8}, & a_8 b_8 \leq z \leq a_9 b_9 \\
\frac{B_9 - \sqrt{B_9^2 - 4A_9(a_{10} b_{10} - z)}}{2A_9}, & a_9 b_9 \leq z \leq a_{10} b_{10} \\
0, & otherwise.
\end{cases}
$$

Hence, Multiplication rule is proved. Therefore we have $\tilde{A} \otimes \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4, a_5 b_5, a_6 b_6, a_7 b_7, a_8 b_8, a_9 b_9, a_{10} b_{10})$ is a decagonal Fuzzy Number. $\square$

Example 5.7. *Let us consider two DFN $\tilde{A} = (1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6)$ and $\tilde{B} = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$. Then, the multiplication of these two DFN is defined as $\tilde{A} \otimes \tilde{B} = (3, 6, 10, 15, 21, 28, 36, 45, 55, 66)$ with membership function as follows*

$$
\mu_{\tilde{P}}(x) =
\begin{cases}
\frac{1}{4}\frac{(x-3)}{(3)}, & 3 \leq x \leq 6 \\
\frac{1}{4}\frac{(x-2)}{(4)}, & 6 \leq x \leq 10 \\
\frac{1}{4}\frac{(x)}{(5)}, & 10 \leq x \leq 15 \\
\frac{1}{4}\frac{(x+3)}{(6)}, & 15 \leq x \leq 21 \\
1, & 21 \leq x \leq 28 \\
-\frac{1}{4}\frac{(x-60)}{(8)}, & 28 \leq x \leq 36 \\
-\frac{1}{4}\frac{(x-63)}{(9)}, & 36 \leq x \leq 45 \\
-\frac{1}{4}\frac{(x-65)}{(10)}, & 45 \leq x \leq 55 \\
-\frac{1}{4}\frac{(x-66)}{(11)}, & 55 \leq x \leq 66 \\
0, & otherwise.
\end{cases}
$$

Theorem 5.8. *If $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$ are the Decagonal Fuzzy Numbers, then $\tilde{D} = \tilde{A} \oslash \tilde{B}$ is also decagonal fuzzy number*

Proof. The membership function of DFN can be found by α-cut method with the transform $z = x \div y$,

$$
z = x \div y \in
\begin{cases}
[4a\,(a_2 - a_1) + a_1, -4a\,(a_{10} - a_9) + a_{10}] \div \\
\quad [4a\,(b_2 - b_1) + b_1, -4a\,(b_{10} - b_9) + b_{10}], & for\ \alpha \in [0, 0.25) \\[4pt]
[4a\,(a_3 - a_2) + 2a_2 - a_3, -4a\,(a_9 - a_8) + 2a_9 - a_8] \div \\
\quad [4a(b_3 - b_2) + 2b_2 - b_3, -4a(b_9 - b_8) + 2b_9 - b_8], & for\ \alpha \in [0.25, 0.5) \\[4pt]
[4a\,(a_4 - a_3) + 3a_3 - 2a_4, -4a\,(a_8 - a_7) + 3a_8 - 2a_7] \div \\
\quad [4a(b_4 - b_3) + 3b_3 - 2b_4, -4a(b_8 - b_7) + 3b_8 - 2b_7], & for\ \alpha \in [0.5, 0.75) \\[4pt]
[4a\,(a_5 - a_4) + 4a_4 - 3a_5, -4a\,(a_7 - a_6) + 4a_7 - 3a_6] \div \\
\quad [4a(b_5 - b_4) + 4b_4 - 3b_5, -4a(b_7 - b_6) + 4b_7 - 3b_6], & for\ \alpha \in [0.75, 1]
\end{cases}
$$

$$
\mu_{\tilde{D}}(x) =
\begin{cases}
\dfrac{zb_{10} - a_1}{4[(a_2 - a_1) + z(b_{10} - b_9)]}, & \dfrac{a_1}{b_{11}} \le z \le \dfrac{a_2}{b_{10}} \\[8pt]
\dfrac{z(2b_9 - b_8) - (2a_2 - a_3)}{4[(a_3 - a_2) + z(b_9 - b_8)]}, & \dfrac{a_2}{b_9} \le z \le \dfrac{a_3}{b_8} \\[8pt]
\dfrac{z(3b_8 - 2b_7) - (3a_3 - 2a_4)}{4[(a_4 - a_3) + z(b_8 - b_7)]}, & \dfrac{a_3}{b_8} \le z \le \dfrac{a_4}{b_7} \\[8pt]
\dfrac{z(4b_7 - 3b_6) - (4a_4 - 3a_5)}{4[(a_5 - a_4) + z(b_7 - b_6)]}, & \dfrac{a_4}{b_7} \le z \le \dfrac{a_5}{b_6} \\[8pt]
1, & \dfrac{a_5}{b_6} \le z \le \dfrac{a_6}{b_5} \\[8pt]
\dfrac{(4a_7 - 3a_6) - (4b_4 - 3b_5)z}{4[(a_7 - a_6) + (b_5 - b_4)z]}, & \dfrac{a_6}{b_5} \le z \le \dfrac{a_7}{b_4} \\[8pt]
\dfrac{(2b_9 - b_8) - (2a_2 - a_3)z}{4[(a_8 - a_7) + (b_4 - b_3)z]}, & \dfrac{a_7}{b_4} \le z \le \dfrac{a_8}{b_3} \\[8pt]
\dfrac{(2a_9 - a_8) - (2b_2 - b_3)z}{4[(a_9 - a_8) + (b_3 - b_2)z]}, & \dfrac{a_8}{b_3} \le z \le \dfrac{a_9}{b_2} \\[8pt]
\dfrac{(a_{10} - zb_1)}{4[(a_{10} - a_9) + (b_2 - b_1)z]}, & \dfrac{a_9}{b_2} \le z \le \dfrac{a_{10}}{b_1} \\[8pt]
0, & otherwise
\end{cases}
$$

Hence, division rule is proved. Therefore we have $\tilde{A} \oslash \tilde{B} = \left(\frac{a_1}{b_{10}}, \frac{a_2}{b_9}, \frac{a_3}{b_8}, \frac{a_4}{b_7}, \frac{a_5}{b_6}, \frac{a_6}{b_5}, \frac{a_7}{b_4}, \frac{a_8}{b_3}, \frac{a_9}{b_2}, \frac{a_{10}}{b_1} \right)$ is a decagonal Fuzzy Number. $\square$

Example 5.9. *Let us consider two DFN $\tilde{A} = (1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6)$ and $\tilde{B} = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$. Then, the division of these two DFN is defined as $\tilde{A} \oslash \tilde{B} = (0.13, 0.20, 0.27, 0.37, 0.50, 0.66, 0.90, 1.25, 1.83, 3.00)$ with membership function as follows*

$$
\mu_{\tilde{A}}(x) \le
\begin{cases}
\frac{1}{4} \frac{(x - 0.13)}{(0.07)}, & 0.13 \le x \le 0.20 \\[6pt]
\frac{1}{4} \frac{(x - 0.13)}{(0.07)}, & 0.20 \le x \le 0.27 \\[6pt]
\frac{1}{4} \frac{(x - 0.07)}{(0.1)}, & 0.27 \le x \le 0.37 \\[6pt]
\frac{1}{4} \frac{(x - 0.02)}{(0.13)}, & 0.37 \le x \le 0.50 \\[6pt]
1, & 0.50 \le x \le 0.66 \\[6pt]
1 - \frac{1}{5} \frac{(x - a_6)}{(a_7 - a_6)}, & 0.66 \le x \le 0.90 \\[6pt]
\frac{4}{5} - \frac{1}{5} \frac{(x - a_7)}{(a_8 - a_7)}, & 0.90 \le x \le 1.25 \\[6pt]
\frac{3}{5} - \frac{1}{5} \frac{(x - a_8)}{(a_9 - a_8)}, & 1.25 \le x \le 1.83 \\[6pt]
\frac{2}{5} - \frac{1}{5} \frac{(x - a_9)}{(a_{10} - a_9)}, & 1.83 \le x \le 3.00 \\[6pt]
0, & elsewhere
\end{cases}
$$

6. Distance Measurement Functions for Decagonal Fuzzy Number

There are many distance measurement functions were proposed in the literature, but the vertex method is an effective and simple method to calculate the distance between any two fuzzy numbers. Therefore, few important properties of the vertex method for decagonal fuzzy number are described as follows:

Definition 6.1. *Let $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$ be the decagonal Fuzzy Numbers. Then, the vertex method can be defined to calculate the distance between them as*

$$d(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{10}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + \cdots + (a_{10} - b_{10})^2]}$$

Definition 6.2. *Let $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$ be the decagonal Fuzzy Numbers. Then, the fuzzy number is $\tilde{A}$ closer to $\tilde{B}$ when $d(\tilde{A}, \tilde{B})$ tends to 0.*

Some important properties of the vertex method are described as below,

Theorem 6.3. *If $\tilde{A}$ and $\tilde{B}$ are the decagonal Fuzzy Numbers, then the distance measurement $d(\tilde{A}, \tilde{B})$ is identical to theEuclidean distance.*

Proof. Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$ are two real numbers, then $a_1 = a_2 = a_3 = \ldots = a_{10} = a$; $b_1 = b_2 = b_3 = \ldots = b_{10} = b$. The distance measurement $d(\tilde{A}, \tilde{B})$ can be calculated as

$$\begin{aligned} d(\tilde{A}, \tilde{B}) &= \sqrt{\frac{1}{10}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + \cdots + (a_{10} - b_{10})^2]} \\ &= \sqrt{\frac{1}{10}[(a - b)^2 + (a - b)^2 + (a - b)^2 + \cdots + (a - b)^2]} \\ &= \sqrt{(a - b)^2} \\ &= |a - b| \end{aligned}$$

$\square$

Theorem 6.4. *Two decagonal Fuzzy Numbers $\tilde{A}$ and $\tilde{B}$ are identical if and only if $(\tilde{A}, \tilde{B}) = 0$.*

Proof. Let $\tilde{A}$ and $\tilde{B}$ be decagonal Fuzzy Numbers. Since it is identical, $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3, \ldots, a_{10} = b_{10}$. The distance between $\tilde{A}$ and $\tilde{B}$ is

$$\begin{aligned} d(\tilde{A}, \tilde{B}) &= \sqrt{\frac{1}{10}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + \cdots + (a_{10} - b_{10})^2]} \\ &= \sqrt{\frac{1}{10}[(0)^2 + (0)^2 + (0)^2 + \cdots + (0)^2]} \\ &= 0 \end{aligned}$$

Converse, If $d(\tilde{A}, \tilde{B}) = 0$, then

$$d(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{10}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + \cdots + (a_{10} - b_{10})^2]} = 0$$

$$\Rightarrow (a - b)^2 + (a - b)^2 + (a - b)^2 + \cdots + (a - b)^2 = 0$$

$$\Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3, \ldots, a_{10} = b_{10}$$

Therefore, $\tilde{A}$ and $\tilde{B}$ are identical.

$\square$

7. Conclusion

In this paper, decagonal fuzzy number and its arithmetic operations have been proposed. Also, linguistic value of (DFN) and vertex method to find the distance between two (DFN) are introduced. Therefore, decision making techniques such as DEMATEL, TOPSIS and VIKOR techniques can be extended by representing linguistic variable into decagonal fuzzy number under uncertain linguistic environment will be the further research.

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